Formelsamling med 1D och 2D Fouriertransformer för kursen Medicinska Bilder

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1 Definitioner, Egenskaper och Samband

DFT och IDFT, 1D och 2D:

$$F[k] = \sum_{n=0}^{N-1} f[n] \cdot e^{-j\frac{2\pi}{N}nk}, \quad f[n] = \frac{1}{N}\sum_{k=0}^{N-1} F[k] \cdot e^{j\frac{2\pi}{N}nk}$$

$$F[k,l] = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f[n,m] \cdot e^{-j2\pi(nk/N+ml/M)},$$

$$f[n,m] = \frac{1}{NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F[k,l] \cdot e^{j2\pi(nk/N+ml/M)}$$

Symmetrisk DFT och IDFT, 1D och 2D:

$$F[k] = \sum_{n=-N/2}^{N/2-1} f[n] \cdot e^{-j\frac{2\pi}{N}nk}, \quad f[n] = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} F[k] \cdot e^{j\frac{2\pi}{N}nk}$$

$$F[k,l] = \sum_{n=-N/2}^{N/2-1}\sum_{m=-M/2}^{M/2-1} f[n,m] \cdot e^{-j2\pi(nk/N+ml/M)},$$

$$f[n,m] = \frac{1}{NM}\sum_{k=-N/2}^{N/2-1}\sum_{l=-M/2}^{M/2-1} F[k,l] \cdot e^{j2\pi(nk/N+ml/M)}$$

Faltning med en skiftad dirac-puls, 1D och 2D:

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(t-t')\delta(t'-t_0)\,dt' = x(t-t_0)$$

$$f(x,y) * \delta(x-x_0, y-y_0) = \iint_{-\infty}^{\infty} f(x-x', y-y')\delta(x'-x_0, y'-y_0)\,dx'dy' = f(x-x_0, y-y_0)$$

Parseval's formel, 1D och 2D:

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df$$

$$\iint_{-\infty}^{\infty} f(x,y)g^*(x,y)\,dxdy = \iint_{-\infty}^{\infty} F(u,v)G^*(u,v)\,dudv$$

2 1-D Tidskontinuerliga Fouriertransformer vinkelfrekvens, ω

	Signaldomän, $t \in \mathbf{R}$	Fourierdomän, $\omega \in \mathbf{R}$
Definition:	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
Linjäritet:	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Tidsskift:	$x(t-a)$	$e^{-j\omega a}X(\omega)$
Frekvensskift:	$e^{jat}x(t)$	$X(\omega - a)$
Dualitet:	$X(t)$	$2\pi \cdot x(-\omega)$
Skalning:	$x(at)$	$(1/\|a\|) \cdot X(\omega/a)$
Faltning:	$(x * h)(t)$	$X(\omega) \cdot H(\omega)$
Multiplikation:	$x(t) \cdot h(t)$	$(1/(2\pi)) \cdot (X * H)(\omega)$
Derivering:	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Reell signal:	$x(t)$ reell	$X(-\omega) = X^*(\omega)$
Diracpuls:	$\delta(t)$	1
Impulståg:	$\frac{1}{\Delta}\text{III}\left(\frac{t}{\Delta}\right) = \sum_k \delta(t - k\Delta)$	$\text{III}\left(\frac{\Delta\omega}{2\pi}\right) = \frac{2\pi}{\Delta}\sum_n \delta\left(\omega - \frac{n2\pi}{\Delta}\right)$
Enhetssteg:	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{j\omega} + \pi\delta(\omega)$
Rektangelpuls:	$\Pi(t) = \begin{cases} 1, & \|t\| \leq 0.5 \\ 0, & \|t\| > 0.5 \end{cases}$	$\text{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$
Sincfunktion :	$\text{sinc}(t)$	$\Pi(\omega/(2\pi))$
Triangelpuls :	$\Lambda(t) = \begin{cases} 1 - \|t\|, & \|t\| \leq 1 \\ 0, & \|t\| > 1 \end{cases}$	$\text{sinc}^2(\omega/(2\pi))$
Cosinusvåg:	$\cos\omega_0 t$	$\pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$
2 Dirac-er:	$\left(\delta(t - \Delta) + \delta(t + \Delta)\right)/2$	$\cos(\Delta\omega)$
Sinusvåg:	$\sin\omega_0 t$	$j\pi\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$
2 Dirac-er:	$\left(-\delta(t - \Delta) + \delta(t + \Delta)\right)/2$	$j\sin(\Delta\omega)$
Konstant:	1	$2\pi\delta(\omega)$
Gauss:	$e^{-\pi t^2}$	$e^{-\pi(\omega/2\pi)^2}$
Diverse:	$e^{-at}u(t)$	$1/(a + j\omega)$
	$te^{-at}u(t)$	$1/(a + j\omega)^2$
	$e^{-a\|t\|}$	$2a/(a^2 + \omega^2)$
	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$

3 1-D Tidskontinuerliga Fouriertransformer frekvens, f

	Signaldomän, $t \in \mathbf{R}$	Fourierdomän, $f \in \mathbf{R}$
Definition:	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$
Linjäritet:	$ax_1(t) + bx_2(t)$	$aX_1(f) + bX_2(f)$
Tidsskift:	$x(t-a)$	$e^{-j2\pi fa} X(f)$
Frekvensskift:	$e^{j2\pi at} x(t)$	$X(f-a)$
Dualitet:	$X(t)$	$x(-f)$
Skalning:	$x(at)$	$(1/\lvert a\rvert) \cdot X(f/a)$
Faltning:	$(x * h)(t)$	$X(f) \cdot H(f)$
Multiplikation:	$x(t) \cdot h(t)$	$(X * H)(f)$
Derivering:	$\frac{d}{dt} x(t)$	$j2\pi f X(f)$
Reell signal:	$x(t)$ reell	$X(-f) = X^*(f)$
Diracpuls:	$\delta(t)$	1
Impulståg:	$\frac{1}{\Delta} \text{III}\left(\frac{t}{\Delta}\right) = \sum_k \delta(t - k\Delta)$	$\text{III}(\Delta f) = \frac{1}{\Delta} \sum_n \delta\left(f - \frac{n}{\Delta}\right)$
Enhetssteg:	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
Rektangelpuls:	$\Pi(t) = \begin{cases} 1, & \lvert t\rvert \leq 0.5 \\ 0, & \lvert t\rvert > 0.5 \end{cases}$	$\text{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$
Sincfunktion :	$\text{sinc}(t)$	$\Pi(f)$
Triangelpuls :	$\Lambda(t) = \begin{cases} 1 - \lvert t\rvert, & \lvert t\rvert \leq 1 \\ 0, & \lvert t\rvert > 1 \end{cases}$	$\text{sinc}^2(f)$
Cosinusvåg:	$\cos 2\pi f_0 t$	$(\delta(f + f_0) + \delta(f - f_0))/2$
2 Dirac-er:	$(\delta(t - \Delta) + \delta(t + \Delta))/2$	$\cos(2\pi\Delta f)$
Sinusvåg:	$\sin 2\pi f_0 t$	$j(\delta(f + f_0) - \delta(f - f_0))/2$
2 Dirac-er:	$(-\delta(t - \Delta) + \delta(t + \Delta))/2$	$j \sin(2\pi\Delta f)$
Konstant:	1	$\delta(f)$
Gauss:	$e^{-\pi t^2}$	$e^{-\pi f^2}$
Diverse:	$e^{-at} u(t)$	$1/(a + j2\pi f)$
	$te^{-at} u(t)$	$1/(a + j2\pi f)^2$
	$e^{-a\lvert t\rvert}$	$2a/(a^2 + (2\pi f)^2)$
	$e^{-at} \sin(2\pi f_0 t) u(t)$	$\frac{2\pi f_0}{(a + j2\pi f)^2 + (2\pi f_0)^2}$
	$e^{-at} \cos(2\pi f_0 t) u(t)$	$\frac{a + j2\pi f}{(a + j2\pi f)^2 + (2\pi f_0)^2}$

4 2-D Kontinuerliga Fouriertransformer spatiella frekvenser, (u, v)

	Spatialdomän, $x, y \in \mathbf{R}$	Fourierdomän, $u, v \in \mathbf{R}$
Definition:	$f(x,y) = \iint_{-\infty}^{\infty} F(u,v) e^{j2\pi(xu+yv)} du dv$	$F(u,v) = \iint_{-\infty}^{\infty} f(x,y) e^{-j2\pi(xu+yv)} dx dy$
Reell signal:	$f(x,y)$ reell	$F(-u,-v) = F^*(u,v)$
Linjäritet:	$af_1(x,y) + bf_2(x,y)$	$aF_1(u,v) + bF_2(u,v)$
Translation, tid:	$f(x-a, y-b)$	$e^{-j2\pi(au+bv)} F(u,v)$
Translation, frekv:	$e^{j2\pi(ax+by)} f(x,y)$	$F(u-a, v-b)$
Skalning:	$f(ax, by)$	$(1/\lvert ab \rvert) \cdot F(u/a, v/b)$
Faltning:	$(f * g)(x,y)$	$F(u,v) \cdot G(u,v)$
Korrelation:	$(f \Box g)(x,y)$	$F^*(u,v) \cdot G(u,v)$
Multiplikation:	$f(x,y) \cdot g(x,y)$	$(F * G)(u,v)$
Derivering i x:	$\frac{\partial}{\partial x} f(x,y)$	$j2\pi u \cdot F(u,v)$
Derivering i y:	$\frac{\partial}{\partial y} f(x,y)$	$j2\pi v \cdot F(u,v)$
Laplace:	$\nabla^2 f(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(x,y)$	$-4\pi^2(u^2+v^2) \cdot F(u,v)$
Generell skalning:	$f(\mathbf{A}\mathbf{x}),\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$	$\frac{1}{\lvert \det \mathbf{A} \rvert} F((\mathbf{A}^{-1})^T \mathbf{u}),\ \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$
Rotation 1:	$f(\mathbf{R}\mathbf{x}),\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$	$F(\mathbf{R}\mathbf{u}),\ \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$
Rotation 2:	$f(r, \theta + \theta_0)$ $x = r\cos\theta,\ y = r\sin\theta$	$F(\omega, \varphi + \theta_0)$ $u = \omega\cos\varphi,\ v = \omega\sin\varphi$
Separabel funktion:	$f(x,y) = g(x) \cdot h(y)$	$F(u,v) = G(u) \cdot H(v)$
Diracpuls:	$\delta(x,y) = \delta(x) \cdot \delta(y)$	1
Box:	$\Pi(x,y) = \Pi(x) \cdot \Pi(y)$	sinc$(u) \cdot$ sinc(v)
Böjd pyramid:	$\Lambda(x,y) = \Lambda(x) \cdot \Lambda(y)$	sinc$^2(u) \cdot$ sinc$^2(v)$
Gauss:	$e^{-\pi(x^2+y^2)} = e^{-\pi x^2} \cdot e^{-\pi y^2}$	$e^{-\pi(u^2+v^2)} = e^{-\pi u^2} \cdot e^{-\pi v^2}$