TSBB15
Computer Vision

Lecture 10

Recap. of TSBB06, Maximum Likelihood and RANSAC

Prelude to Project 2

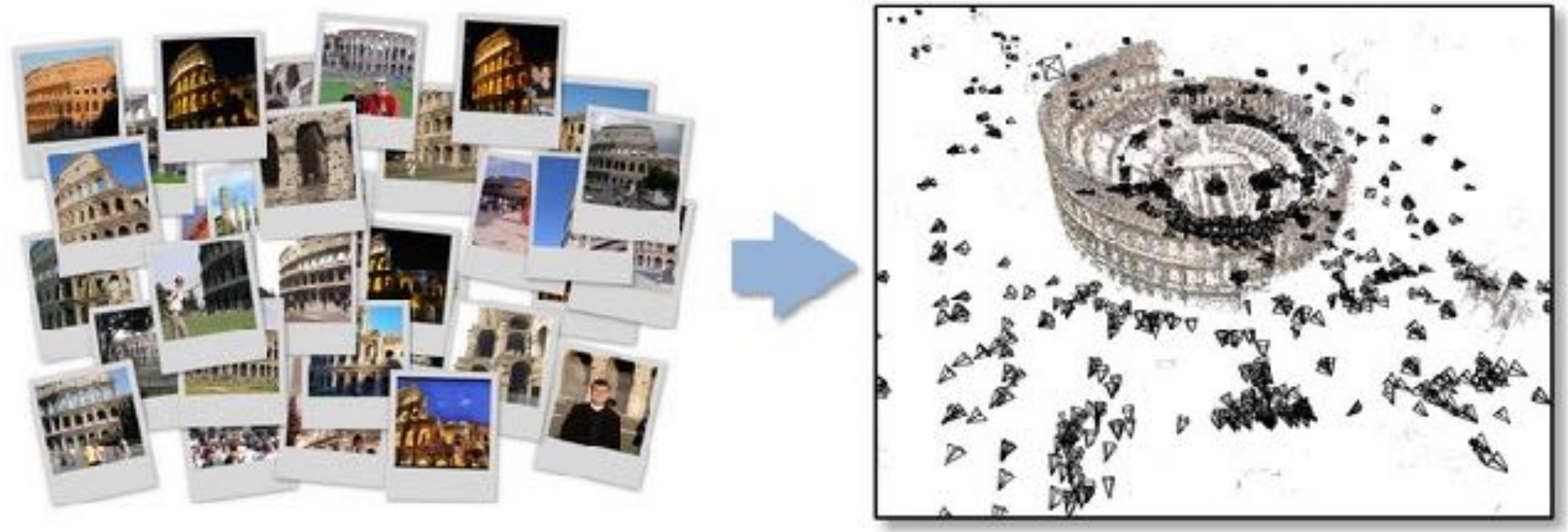


Image credit: Bundler home page http://www.cs.cornell.edu/~snavely/bundler/

- Project 2: Start with two views, and successively add more.
- Finally perform simultaneous refinement of 3D point positions and camera positions (Bundle adjustment).

Prelude to Project 2

Highlights of differences compared to Project 1:

- Less programming
- More math skills
- More ways to get lost in 3D
  Unit testing (also of external functions) even more important

Recap: 2D homogeneous coordinates

- A 2D point (y_1, y_2) is given a homogeneous representation as: $\mathbf{y} \in \mathbb{R}^3$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}$$

Canonical form of homogeneous coordinates for a 2D point

- Any scalar multiple of this **v** is also a homogeneous representation of (y_1, y_2)
- We will use **y** to refer to the 2D point as well as its homogeneous coordinates!

Recap: 2D homogeneous coordinates

- A 2D line, at distance *d* from the origin, and with ($\cos \alpha$, $\sin \alpha$) as a normal vector is given a (dual) homogeneous representation as $\mathbf{l} \in \mathbb{R}^3$:

$$\mathbf{l} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \\ -\rho \end{pmatrix}$$

Canonical form of homogeneous coordinates for a 2D line

- Any scalar multiple of this **l** is also a homogeneous representation of the same line
- We will use **l** to refer to the 2D line as well as its homogeneous coordinates!

Recap: 2D homogeneous coordinates

- From these homogeneous representations follow:
- Point **y** lies on line **l** $\Leftrightarrow \; \mathbf{y} \cdot \mathbf{l} = 0$
- Point **y** intersects lines $\mathbf{l}_1$ and $\mathbf{l}_2$ $\Leftrightarrow \; \mathbf{y} = \mathbf{l}_1 \times \mathbf{l}_2$
- Line l intersects points $\mathbf{y}_1$ and $\mathbf{y}_2$ $\Leftrightarrow \; \mathbf{l} = \mathbf{y}_1 \times \mathbf{y}_2$
- The distance from point **y** to line **l** is given by $|\mathbf{y} \cdot \mathbf{l}|$ *if they both are in canonical form*

Recap: 2D homogeneous coordinates

- Rigid transformations: translation + rotation
- Scaling
- Affine transformations
- Projective transformations (homographies)

Can be represented as linear mappings of the homogeneous coordinates:

$$\mathbf{y}' = \mathbf{H}\mathbf{y} \qquad \mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$

⇦ Dual transformations of lines!

Recap: Cross product operator

The cross product between vectors **a** and **b**:

$$\mathbf{a} \times \mathbf{b}$$

can sometimes be written more conveniently as a 3×3 matrix $[\mathbf{a}]_\times$ applied to **b**:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

Recap: 3D homogeneous coordinates

- Homogeneous coordinates for 2D points and lines can be extended in a straight-forward way to homogeneous coordinates for 3D points **x** and planes **p**
- Also 3D lines can be given a homogeneous representation
  - Parameter form: $t\,\mathbf{x}_1 + (t - 1)\,\mathbf{x}_2$
  - Plücker coordinates: $\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^{\mathrm{T}} - \mathbf{x}_2\mathbf{x}_1^{\mathrm{T}}$

Recap: Pinhole camera

- The pinhole camera maps 3D points to a 2D image:

$$\mathbf{y} \sim \mathbf{C}\mathbf{x}$$

- **C** is the 3×4 camera (projection) matrix
- Does not represent geometric distortion from the camera lens

Recap: Pinhole camera

- The camera matrix can be decomposed as

$$\mathbf{C} = \begin{pmatrix} f_1 & \gamma & c_1 \\ 0 & f_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Intrinsic camera calibration matrix
3×3

Normalized camera projection matrix
3×4

Extrinsic camera calibration matrix
(a rigid transformation)
4×4

Recap: Pinhole camera

In a more compact form:

$\mathbf{C} = \mathbf{K} [\, \mathbf{R} \mid \mathbf{t} \,]$ $[\, \mathbf{R} \mid \mathbf{t} \,]$ is 3×4

K = **intrinsic calibration** matrix (3×3)

- Constant in this project
- Camera calibration

R, **t** = **extrinsic calibration**

- The camera pose for each image in the sequence

Recap: Camera center

The pinhole camera projects 3D point onto the image plane through its center **n**:

The camera centre $\mathbf{n} \in \mathbb{R}^4$ satisfies **C n = 0**

If **C** = **K** [**R** | **t**] ⇒

The 3D coordinates of **n** is $\mathbf{t}' = -\mathbf{R}^T\mathbf{t}$

Alternatively: if **C** = **K** [**R** | –**R t**] ⇒

The 3D coordinates of **n** is **t**

A practical issue

The numerical values of a specific camera matrix always **refer to a specific coordinate system** in the image and in 3D space

There may be more than one coordinate system that is used for particular problem:

- A camera centred coordinate system
- Another camera coordinate system
- The world coordinate system

We need to know which coordinate system it refers to in order to use it practically

Maybe transform it to a standard coordinate system

Recap: Equivalent cameras

- Two cameras $\mathbf{C}_1$ and $\mathbf{C}_2$ with identical camera center $\mathbf{n}$: $\mathbf{C}_1\,\mathbf{n} = \mathbf{C}_2\,\mathbf{n} = 0$ are called **equivalent**
- The images of two equivalent cameras can always be made identical by means of a homography transformation

  Disregarding that images are of finite size
- Cameras that are not equivalent are called **distinct**

Recap: Estimation

Many problems where we estimate some type of geometric object can be formulated as

$$\mathbf{A}\,\mathbf{x} = \mathbf{0}$$

x is a representation of the unknown geometric object that we want to determine

- Usually in terms of homogeneous coordinates
- Or, at least, in terms of a projective element

A is data dependent

Recap: Estimation

In practice, the data in **A** includes noise

The equation **A x** = **0** is not satisfied exactly

Solution (homogeneous method):

minimise: $$\epsilon(\mathbf{x}) = \|\mathbf{A}\mathbf{x}\| \text{ subject to } \|\mathbf{x}\| = 1$$

This approach minimises an **algebraic error**

Optimal for Gaussian residuals, i.e. $\mathbf{A}\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \sigma\mathbf{I})$

This is **not** the case in general…

Recap: Estimation

- Algebraic errors lead to linear solution methods (e.g. using SVD), they are simple to use
- The actual data noise, however, occurs in the Euclidean geometric space (2D or 3D)
- Geometric errors are almost always non-linear functions of the free parameters that we optimise
  - No closed form solutions exist
  - Iterative minimisation methods must be used
  - Good initial solutions are critical (use linear estimators!)
  - Use optimisation tools, e.g., in **`scipy.optimize`**
  - To be continued...

Recap: Epipolar geometry

A 3D point **x** is viewed by two distinct cameras **C**$_1$ and **C**$_2$

$$\mathbf{y}_1 \sim \mathbf{C}_1\mathbf{x}$$

$$\mathbf{y}_2 \sim \mathbf{C}_2\mathbf{x}$$

y$_1$ and **y**$_2$ are *corresponding points*

Either:

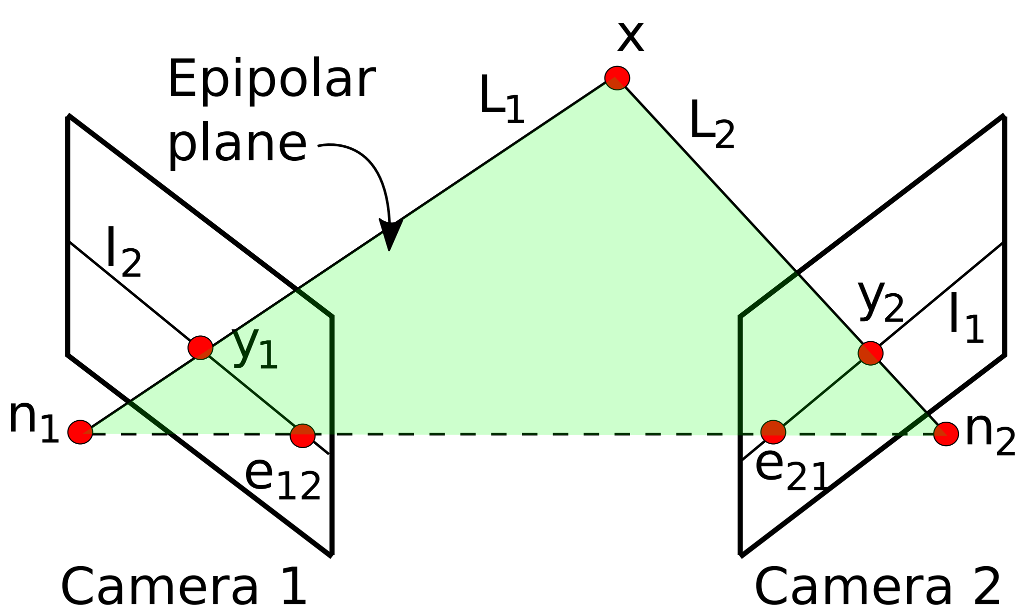
Two physically distinct cameras

The same camera that is moving over time

Recap: Epipolar geometry

- A 3D point **x** is viewed by two distinct cameras $\mathbf{C}_1$ and $\mathbf{C}_2$

$$\mathbf{y}_1 \sim \mathbf{C}_1\mathbf{x} \qquad \mathbf{y}_2 \sim \mathbf{C}_2\mathbf{x}$$



Recap: Epipolar points

The image of camera center $\mathbf{n}_1$ in camera 2 is called the **epipole** (or *epipolar point)* $\mathbf{e}_{21}$:

$$\mathbf{e}_{21} = \mathbf{C}_2\mathbf{n}_1$$

Correspondingly, the image of $\mathbf{n}_2$ in camera 1 defines the epipole $\mathbf{e}_{12}$:

$$\mathbf{e}_{12} = \mathbf{C}_1\mathbf{n}_2$$

Recap: Epipolar constraint

In this case, it follows that there exists a 3×3 matrix **F**, the **fundamental matrix**, such that

$$\mathbf{y}_1^T \mathbf{F} \mathbf{y}_2 = 0$$

The epipolar constraint between *corresponding image points* $\mathbf{y}_1$ and $\mathbf{y}_2$

F depends only on $\mathbf{C}_1$ and $\mathbf{C}_2$:

$$\mathbf{F} = [\mathbf{e}_{12}]_\times \mathbf{C}_1 \mathbf{C}_2^+$$

Pseudo-inverse

Calibrated case: **F** is determined from known $\mathbf{C}_1$ and $\mathbf{C}_2$

Recap: epipolar lines

F maps $\mathbf{y}_2$ to a line $\mathbf{l}_1 = \mathbf{F}\mathbf{y}_2$ in the first view

$\mathbf{l}_1$ is an **epipolar line**:

it goes through the epipole, $\mathbf{e}_{12} \cdot \mathbf{l}_1 = 0$

If $\mathbf{y}_1$ corresponds to $\mathbf{y}_2$:

then $\mathbf{y}_1$ lies on $\mathbf{l}_1$: $\mathbf{y}_1 \cdot \mathbf{l}_1 = 0$

Similarly for the other epipolar line $\mathbf{l}_2 = \mathbf{F}^T \mathbf{y}_1$

Recap: 8-point algorithm

F can also easily be estimated from a set of 8 (or more) corresponding image points:

- Form a data matrix **A**
- Determine right null vector **f** of **A** (SVD)
- Reshape **f** to **F**
- Enforce internal constraint det **F** = 0
- Important to use Hartley normalisation to increase accuracy!

Recap: triangulation

Given two corresponding image points $\mathbf{y}_1$ and $\mathbf{y}_2$ we want to determine the 3D point $\mathbf{x}$

Mid-point method

Algebraic method

Probabilistic method (”optimal” triangulation)

In general, different methods give slightly different results if $\mathbf{y}_1$ and $\mathbf{y}_2$ are noisy

BREAK

Maximum Likelihood

- For normally distributed noise we can often define a maximum likelihood solution to estimation problems
- Consider a direct observation **x**:

$$x \sim \mathcal{N}(\mu, \sigma)$$

- with the observation **likelihood**:

$$Pr(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5(x-\mu)^2/\sigma^2}$$

Maximum Likelihood

- If we have several **independent** observations:

$$x_1, \ldots, x_N \sim \mathcal{N}(\mu, \sigma)$$

- Their joint likelihood becomes:

$$Pr(x_1|\mu,\sigma) \cdot \ldots \cdot Pr(x_N|\mu,\sigma) = \prod_{n=1}^{N} Pr(x_n|\mu,\sigma)$$

- We can now consider the estimation problem as one of finding the parameters that maximise this joint likelihood.

Maximum Likelihood

- Maximise the joint likelihood:

$$\{\mu^*, \sigma^*\} = \arg\max_{\mu,\sigma} \prod_{n=1}^{N} Pr(x_n|\mu, \sigma)$$

- Take the negative log of this expression:

$$\{\mu^*, \sigma^*\} = \arg\min_{\mu,\sigma} J(\mu, \sigma)$$

- For:

$$J(\mu, \sigma) = \frac{1}{2}\sum_{n=1}^{N}(x_n - \mu)^2/\sigma^2 - \tfrac{1}{2}\log(2\pi\sigma^2)$$

Maximum Likelihood

- Setting partial derivatives to zero gives us estimates:

$$\frac{\partial J(\mu,\sigma)}{\partial \mu} = 0 \quad \Rightarrow \quad \mu = \frac{1}{N}\sum_{n=1}^{N} x_i$$

- and

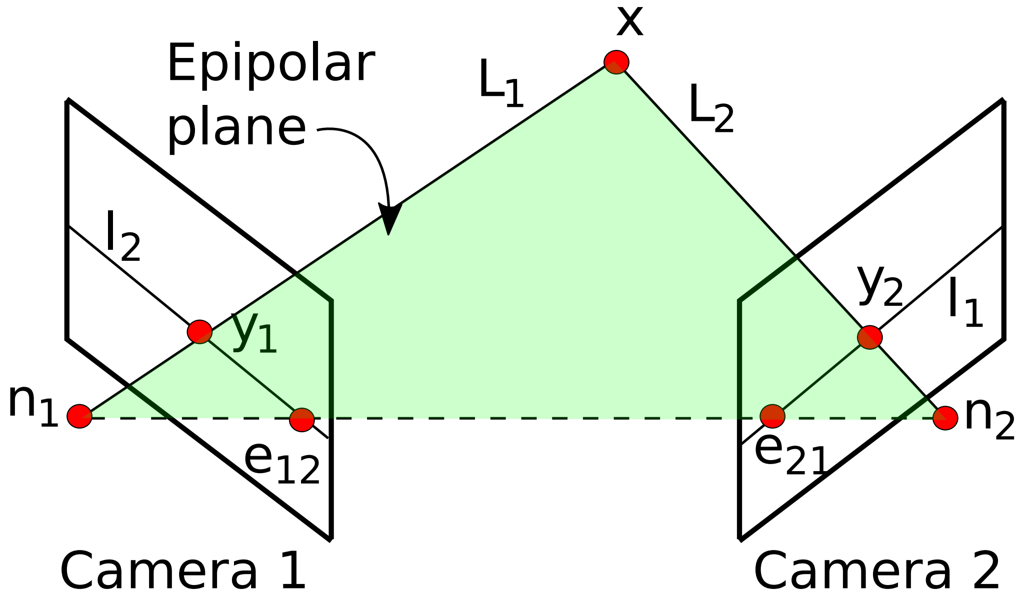
$$\frac{\partial J(\mu,\sigma)}{\partial \sigma} = 0 \quad \Rightarrow \quad \sigma^2 = \frac{1}{N}\sum_{n=1}^{N} (x_i - \mu)^2$$

True μ!

Optimal triangulation

- Given two cameras and two observations of the same 3D point:

$$\mathbf{y}_1 \sim \mathbf{C}_1\mathbf{x} \quad \text{and} \quad \mathbf{y}_2 \sim \mathbf{C}_2\mathbf{x}$$



Optimal triangulation

- Given two cameras and two observations of the same 3D point:

$$\mathbf{y}_1 \sim \mathbf{C}_1\mathbf{x} \quad \text{and} \quad \mathbf{y}_2 \sim \mathbf{C}_2\mathbf{x}$$

- Gives us the joint likelihood to maximize:

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} Pr(\mathbf{y}_1|\mathbf{C}_1\mathbf{x})Pr(\mathbf{y}_2|\mathbf{C}_2\mathbf{x})$$

- Negative log gives us a least squares problem:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} d^2(\mathbf{y}_1, \mathbf{C}_1\mathbf{x}) + d^2(\mathbf{y}_2, \mathbf{C}_2\mathbf{x})$$

Optimal triangulation

- A least squares problem:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} d^2(\mathbf{y}_1, \mathbf{C}_1\mathbf{x}) + d^2(\mathbf{y}_2, \mathbf{C}_2\mathbf{x})$$

- Each 3D point **x** defines an epipolar plane, which defines two epipolar lines.
- We can look for an epipolar line in one image, and transfer it to the other image.
- This leads to a 6th degree polynomial.
- For the best line pair, find the closest auxiliary points ($\mathbf{y}_1$'=$\mathbf{C}_1\mathbf{x}$ and $\mathbf{y}_2$'=$\mathbf{C}_2\mathbf{x}$)
- Triangulate these, using linear triangulation (OK as these satisfy the EG perfectly, and have zero reprojection error)

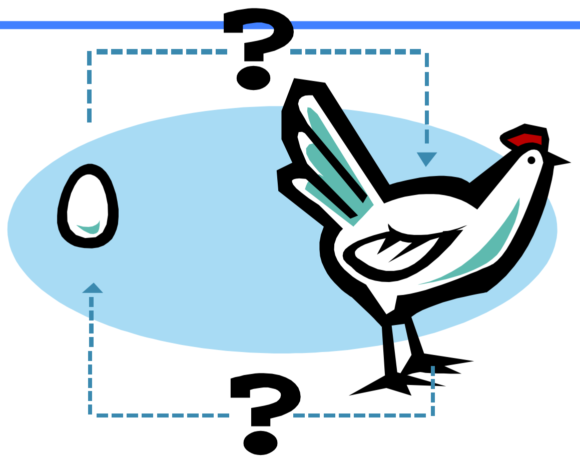
Optimal triangulation

- You will use optimal triangulation in CE3.
- Source code is provided.
- Full algorithm is described in IREG 16.3
- More details on the derivation are given there.

Robust estimation

- We often have two categories of errors:
  - **Measurement noise**: we cannot determine numerical values of the data with high accuracy
  - **Outliers**: all measurements in the dataset cannot be fitted to a consistent model
- We refer to data that can be fitted to the model as **inliers**
- **Robust estimation:** to determine a model from a dataset that contains significant amount of outliers

A chicken and egg problem



We need corresponding points to estimate **F**

Point correspondences can be verified if we know **F**

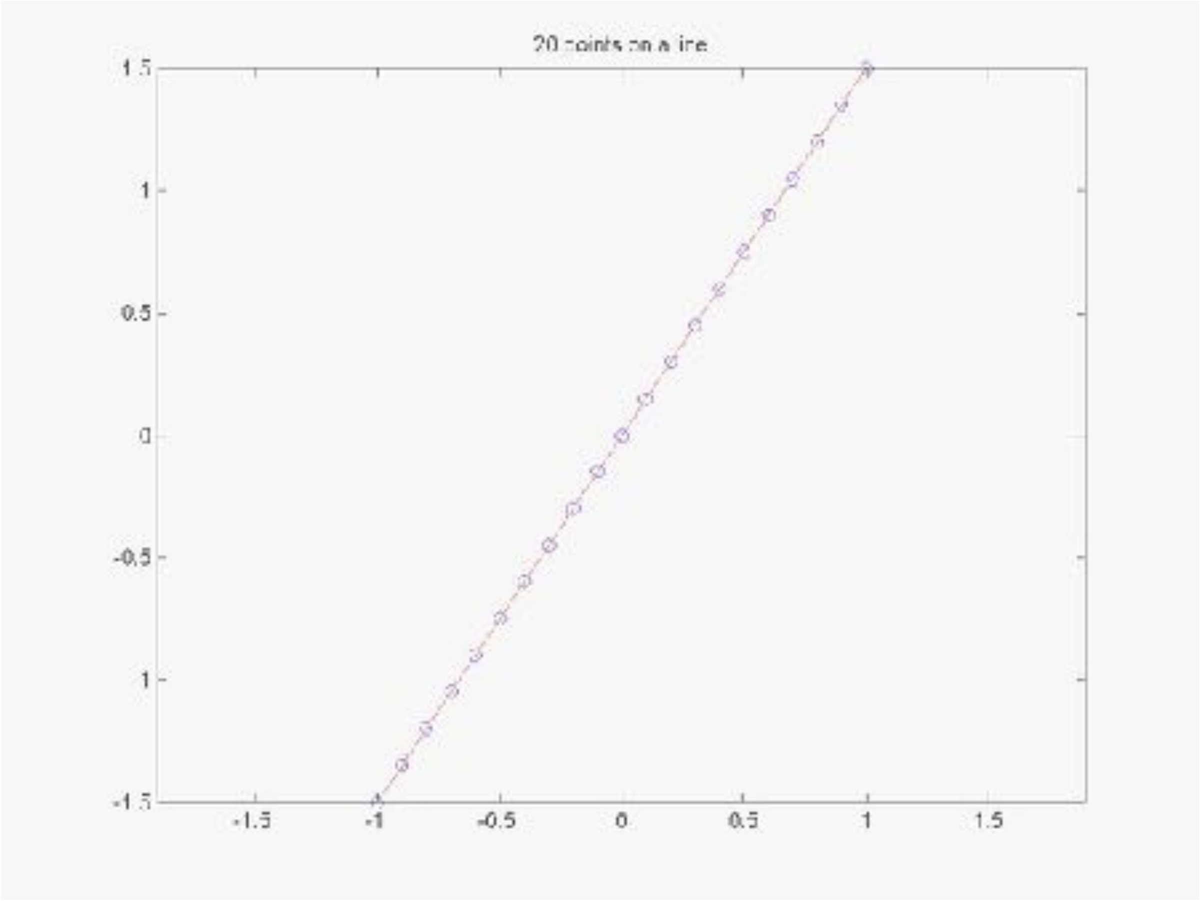
Can we determine **F** and verify correspondences at the same time?

Robust estimation

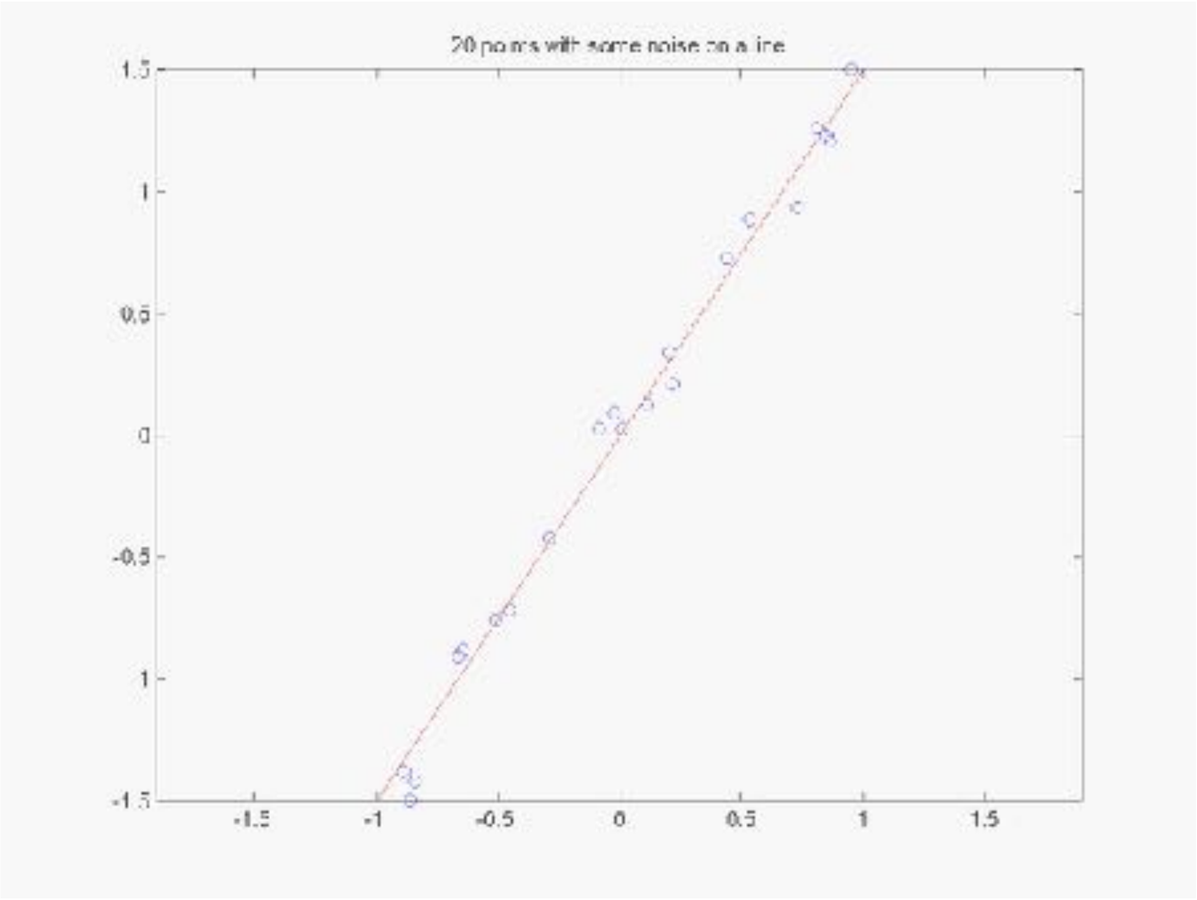
Model estimation from measured data where some of the data items are incorrect. E.g.

Data	Model
Points in 2D or in 3D	A 2D line, A 3D plane
Lines in 2D or planes in 3D	Point of intersection in 2D or 3D
Corresponding points in 2 views	Homography
Corresponding points in 2 views	Fundamental matrix

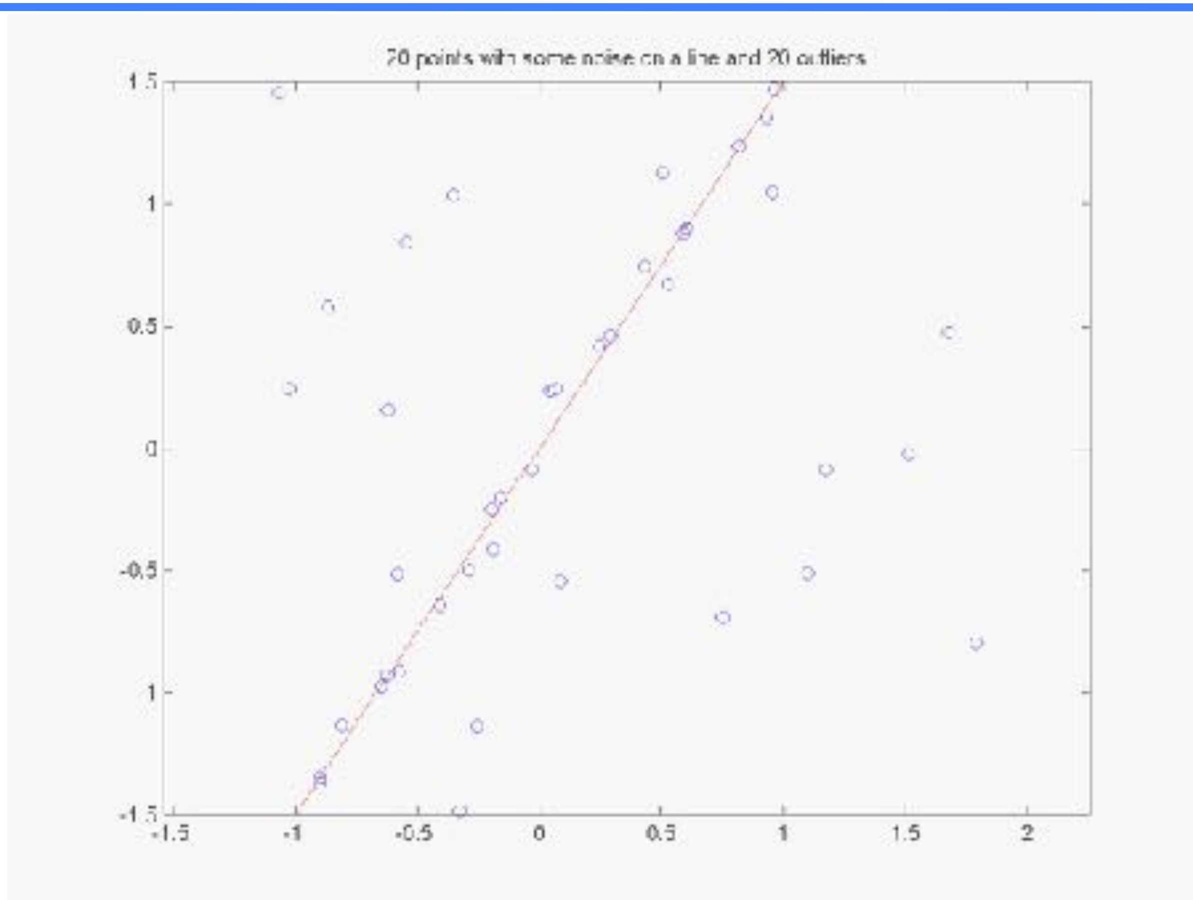
Example: estimation of a line from points



Example: estimation of a line from points



Example: estimation of a line from points



Observations

- We need (in this case!) a minimum of 2 points to determine a line
- Given such a line **l**, we can determine how well any other point **y** fits the line **l**
  - For example: distance between **y** and **l**
- If we pick 2 **random** points from the dataset:
  - We can easily determine a line **l**
  - **l** is the correct line with some probability p_{LINE}
  - p_{LINE} is related to the chance of picking only inliers
  - p_{LINE} is larger the fewer points that are used to determine **l**
  - In general: if **l** is supported by the data there are more additional points that lie on it.

Probabilities

Let S be a set of $|S|=M$ points in total and M_0 of them are inliers

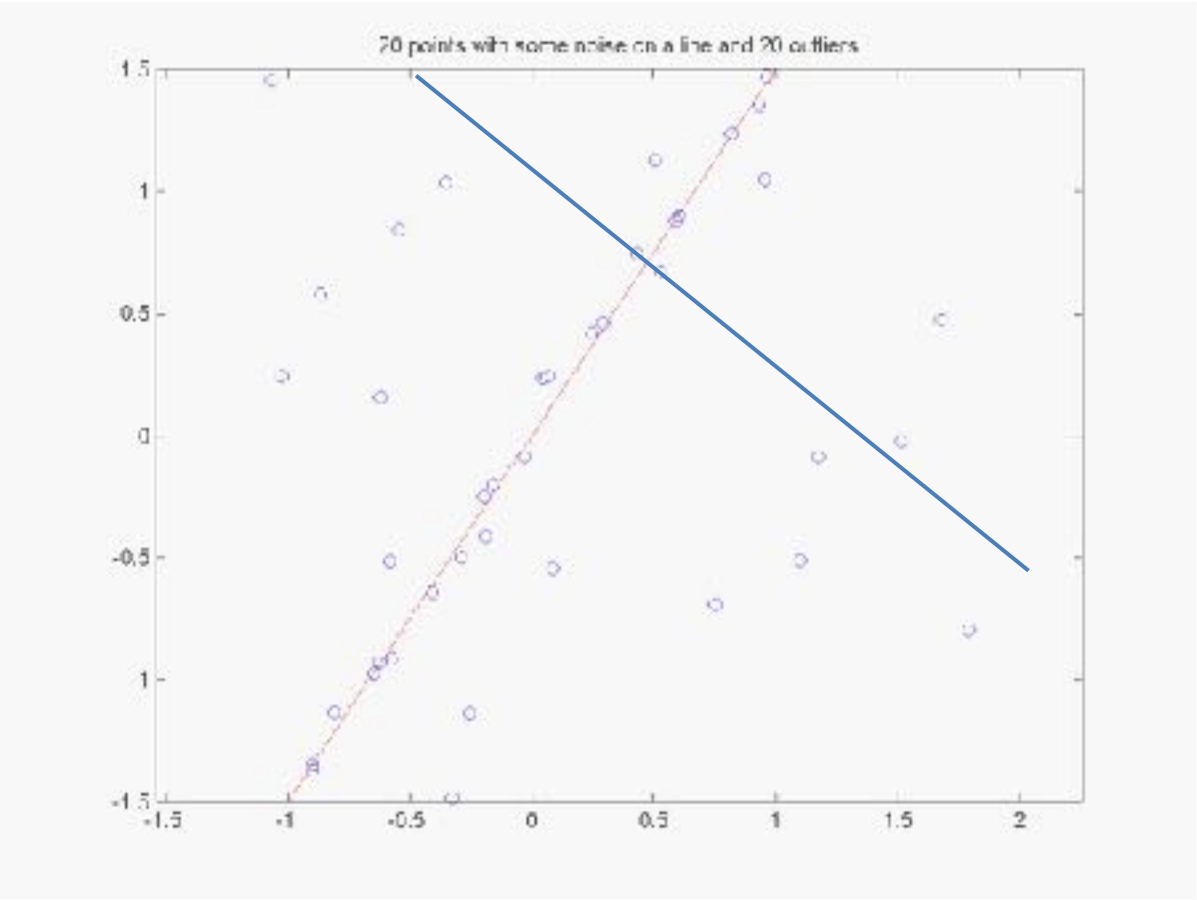
There are $M\,(M-1)$ ways to draw 2 distinct points in a certain order, ($\approx M^2$ if M is large)

This means $p_{\text{LINE}} \approx p^2_{\text{POINT}} = (M_0/M)^2$

This is in practice **an upper bound**:

Even if we pick two inliers, the corresponding line may not be supported by the remaining inliers

Line estimated from 2 inliers



Basic iteration

1. Draw 2 random points from *S*
2. Fit a line **l** to the points
3. Determine how many other points² in *S* that support the line **l** within some error bound.

   ² These form the **consensus set** *C*
4. If *C* is sufficiently large, then the found line is probably OK. Keep it

Basic algorithm

- Iterate *r* times
  1. Draw 2 random points from S
  2. Fit a line **l** to the points
  3. Form the consensus set *C*, together and
     Count the number of points in *C*
     (or p(*S*|**l**) average likelihood of the dataset given the line)
  4. If the consensus set is sufficiently large, then the found line is OK. In particular, keep **l** if *N* or p(*S*|**l**) is the best one this far.
- Each iteration increases p_{SUCCESS} = the probability that the correct line has been found
- We need to iterate sufficiently many time to raise p_{SUCCESS} to a useful level

RANSAC

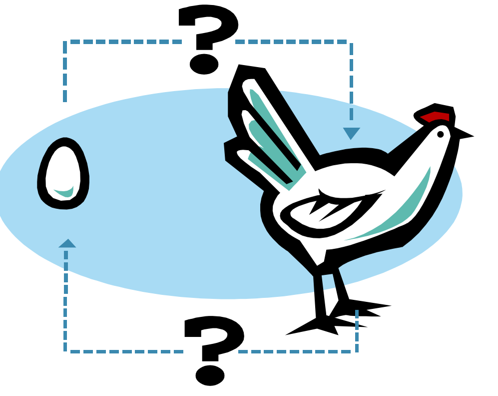
- This algorithm is called RANSAC
  - RANdom SAmple Consensus
- Published by Fischler & Bolles in 1981

  "Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography". *Comm. of the ACM* **24**: 381–395.
- Several extensions / variations in the literature

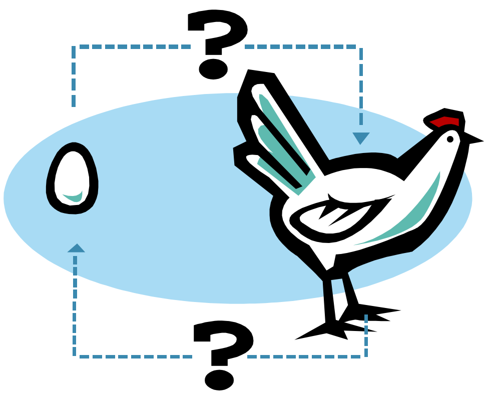
  Preemptive RANSAC, MLESAC, PROSAC, …

RANSAC for **F**

- For estimation of the fundamental matrix **F**, points that satisfy an epipolar constraint for some **F** are **assumed to be inliers**
- The points that violate the epipolar constraint are **outliers**



Chicken and egg revisited



- Let there be two views with a point set P_1 in one view and P_2 in the other view
- Drawing correspondences randomly from the two point sets is futile: Pr(correct)=min(1/|P_1|,1/|P_2|).
- We need to find a subset $\mathcal{S} \subset \mathcal{P}_1 \times \mathcal{P}_2$ of likely correspondences, or **tentative** correspondences
- Typically done using some heuristic

RANSAC for **F**

Pick 8 random correspondences from *S*

We do not know if they really correspond, but this can be tested:

1. Use the 8-point algorithm to estimate **F**
2. Check how well **F** matches each pair in *S*
3. Collect those that fit well into the consensus set *C*
4. If *C* is sufficiently large: **F** is OK: keep **F** and *C*

Iterate *r* times

Probability of success

Let w be the fraction of inliers in S

In each iteration we pick N points that are all inliers with probability w^N (approximately)

The probability of not all N points are inliers is then given by $1 - w^N$

The probability of not all N points are inliers in r iterations is $(1 - w^N)^r$

The probability that in iteration r, at least once, all N points are inliers: $p = 1 - (1 - w^N)^r$

Solve for r:

$$r = \frac{\log(1-p)}{\log(1-w^N)}$$

Minimising geometric errors

- The 8-point algorithm uses an algebraic error. We would like to also estimate **F** using ML.
- This requires us to use an error that corresponds to a likelihood.
- We know that image coordinates from detection are well modelled as normally distributed. Via the negative log we can then obtain a least-squares problem.
- Thus, we can look for a geometric error, but will all geometric errors lead to independent normally distributed likelihoods?

Geometric errors for **F**

Given a set of *N* corresponding image points $\{\mathbf{y}_{1k}, \mathbf{y}_{2k}\}$, $k = 1, \ldots, N$

We can formulate a geometric error as

$$\epsilon_1 = \sum_{k=1}^{N} d_{PL}(\mathbf{y}_{1k}, \mathbf{F}\mathbf{y}_{2k})^2 + d_{PL}(\mathbf{y}_{2k}, \mathbf{F}^T\mathbf{y}_{1k})^2$$

Epipolar lines!

where d_{PL} is the Euclidean distance between a point and a line

Referred to as an L_2 error

Geometric errors for F

This error does not take into account that the epipolar lines also, are perturbed by noise

As an alternative, we want to find

A fundamental matrix **F**

A set of auxiliary points $\{\mathbf{y}'_{1k}, \mathbf{y}'_{2k}\}$

where $(\mathbf{y}'_{1k})^T \mathbf{F}\, \mathbf{y}'_{2k} = 0$ (exactly!) for all k

such that

$$\epsilon = \sum_{k=1}^{N} d_{PP}(\mathbf{y}_{1k}, \mathbf{y}'_{2k})^2 + d_{PP}(\mathbf{y}_{2k}, \mathbf{y}'_{1k})^2$$

is minimised

d_{PP} is the Euclidean point-to-point distance

Re-parameterisation of **F**

- This means that we want to optimise over combinations of **F** *and* the auxiliary points that always satisfy the epipolar constraint!
  - Cannot be done directly in a simple way
- Instead:
  - Let **F** be determined from $\mathbf{C}_1$ and $\mathbf{C}_2$
  - Let the auxiliary points be the images of some *virtual* 3D points projected through $\mathbf{C}_1$ and $\mathbf{C}_2$
  - The auxiliary points and **F** are then always consistent
  - Vary the 3D points and $\mathbf{C}_1$ and $\mathbf{C}_2$

Cameras from **F**

Each camera matrix has 11 degrees of freedom: two cameras have 22 DOF

F has 7 DOF

- There are many combinations of two cameras that produce the same **F**

Cameras from **F**

Choose $\mathbf{C}_1 = [\ \mathbf{I} \mid \mathbf{0}\ ]$ (set WCS=CCS for camera 1)

A convenient choice of $\mathbf{C}_2$ is then given by

$$\mathbf{C}_2 = [[\mathbf{e}_{21}]_\times \mathbf{F}^T | \mathbf{e}_{21}]$$ (why?)

Note that $\mathbf{e}_{21}$ is given by $\mathbf{F}\ \mathbf{e}_{21} = \mathbf{0}$

$\mathbf{C}'_1 = \mathbf{C}_1\ \mathbf{H}$ and $\mathbf{C}'_2 = \mathbf{C}_2\ \mathbf{H}$, for arbitrary 3D homography **H**, also gives the same **F** (why?)

Gold Standard estimation of **F**

- Determine an initial $\mathbf{F}_0$ using a linear method
- From $\mathbf{F}_0$: determine initial $\mathbf{C}_1$ and $\mathbf{C}_2$
- From $\{\mathbf{y}_{1k}, \mathbf{y}_{2k}\}$ and $\mathbf{C}_1$, $\mathbf{C}_2$, triangulate virtual 3D points $\mathbf{x}_k$
    - Relative to some arbitrary 3D coordinate system
    - Projective triangulation
- Re-project the 3D points through $\mathbf{C}_1$ and $\mathbf{C}_2$ to the auxiliary points $\{\mathbf{y}'_{1k}, \mathbf{y}'_{2k}\}$
    - These are consistent with $\mathbf{F}_0$
- Minimise ε over the 3D points $\mathbf{x}_k$ and $\mathbf{C}_2$ ($\mathbf{C}_1$ is fixed)
    - Use a non-linear optimisation tool, e.g., **lsqnonlin**
    - ε is referred to as an L_2 **reprojection error**
- The resulting $\mathbf{C}_2$ (+ $\mathbf{C}_1$) gives $\mathbf{F}_{\text{optimal}}$ that
    - Minimises a geometric error
    - Maximises the likelihood of the correspondences under Gaussian noise
    - This is referred to as **Gold Standard estimation** of **F**
- Computer Exercise 3

Summary

- The math introduced here and in the next two lectures builds on **TSBB06 Multidimensional Signal Analysis**
- The **Maximum Likelihood**(ML) trick is to look for models that **make observations likely**.
- **Optimal Triangulation**, and the **Gold Standard method** to find **F** are two examples of ML. More examples will follow…
- The **RANSAC algorithm** is an important tool for handling data with **outliers**. Other tools are clustering and mode finding (LE6).