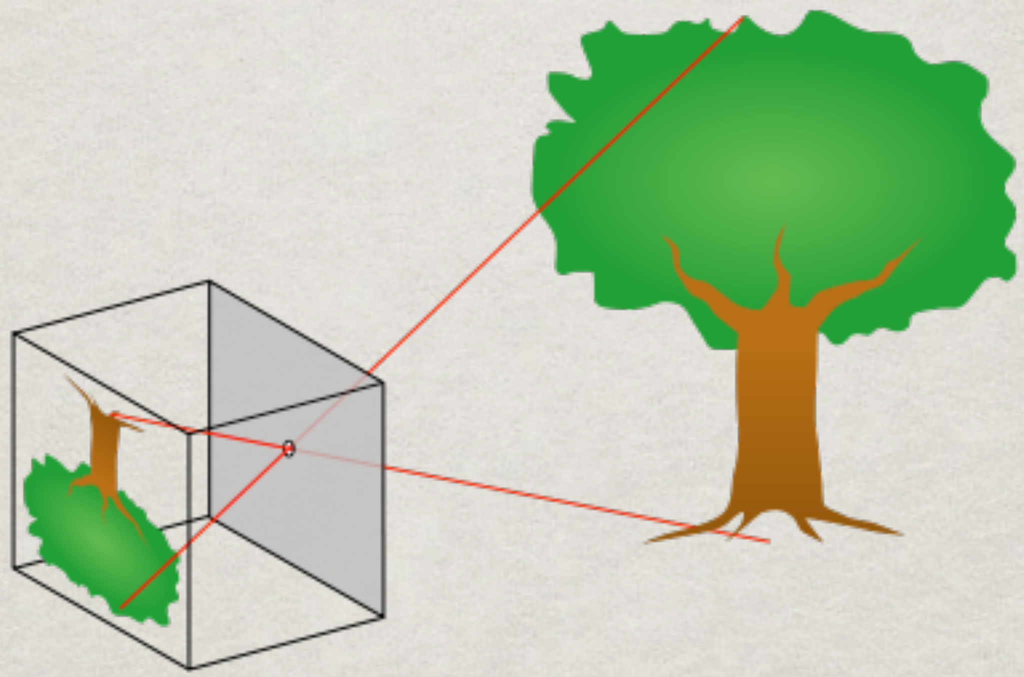
Geometry for computer Vision

Lecture 4a: Calibrated and oriented Epipolar Geometry

Lecture 4a: calibrated and oriented EPG

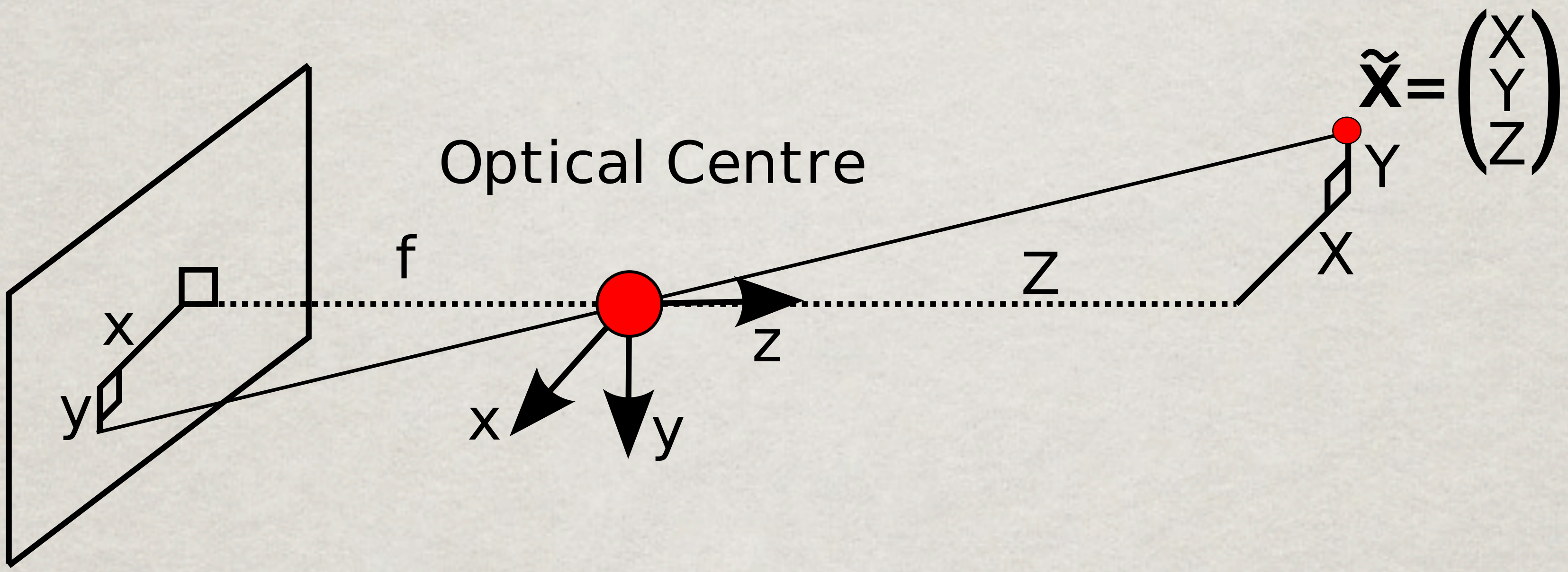
- Extrinsic and intrinsic camera parameters
- Zhang's camera calibration
- Calibrated epipolar geometry
- Oriented epipolar geometry
- Discussion of the paper:
  Mendonça and Cippolla, A *Simple Technique for Self-Calibration,* CVPR99

The Pin-Hole Camera



- A brightly illuminated scene will be projected onto a wall opposite of the pin-hole.
- The image is rotated 180°.

The Pin-Hole Camera



- From similar triangles we get:

$$x = f\frac{X}{Z} \qquad y = f\frac{Y}{Z}$$

$$\gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The Pin-Hole Camera

$$\gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- More generally, we write:

$$\gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & s & c_x \\ 0 & fa & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

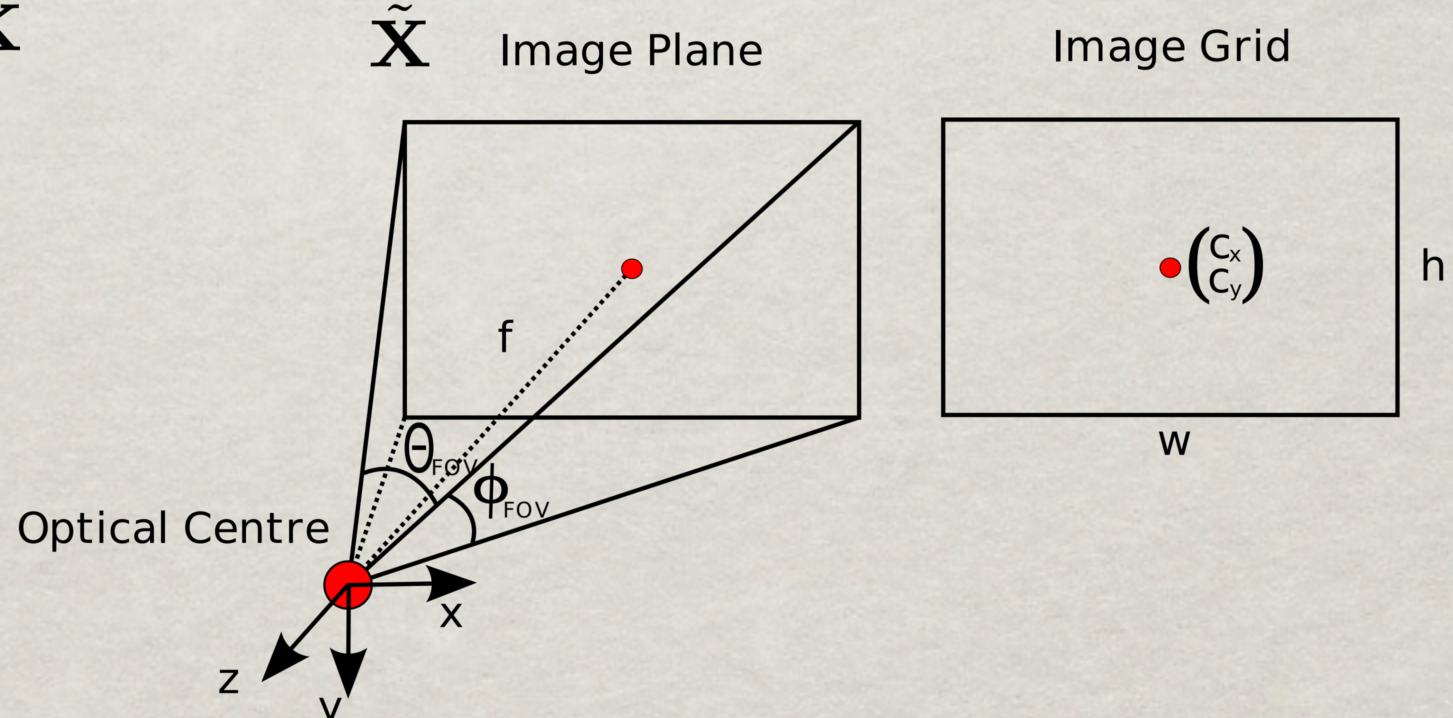
f-focal length, s-skew, a-aspect ratio,
c-projection of optical centre

The Pin-Hole Camera

$$\underbrace{\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} f & s & c_x \\ 0 & fa & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\tilde{\mathbf{X}}}$$

$$\mathbf{x} \sim \mathbf{K}\tilde{\mathbf{X}}$$

- Motivation:



f-focal length, s-skew, a-aspect ratio,
c-projection of optical centre

The Pin-Hole Camera

- For a general position of the world coordinate system (WCS) we have:

$$\mathbf{x} \sim \mathbf{K} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}}_{[\mathbf{R}|\mathbf{t}]} \underbrace{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}}_{\mathbf{X}}$$

The Pin-Hole Camera

- For a general position of the world coordinate system (WCS) we have:

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- $\mathbf{K}$ contains the *intrinsic* parameters
- $[\mathbf{R} \mid \mathbf{t}]$ contain the *extrinsic* parameters

Normalised coordinates

- Metric points transformed to the camera's coordinate system are called *normalised image coordinates*

$$\hat{\mathbf{x}} \sim [\mathbf{R}|\mathbf{t}]\,\mathbf{X}$$

- In contrast to regular image coordinates

$$\mathbf{x} \sim \mathbf{K}\,[\mathbf{R}|\mathbf{t}]\,\mathbf{X} \qquad \mathbf{x} = \mathbf{K}\hat{\mathbf{x}}$$

- **K** contains the *intrinsic* parameters

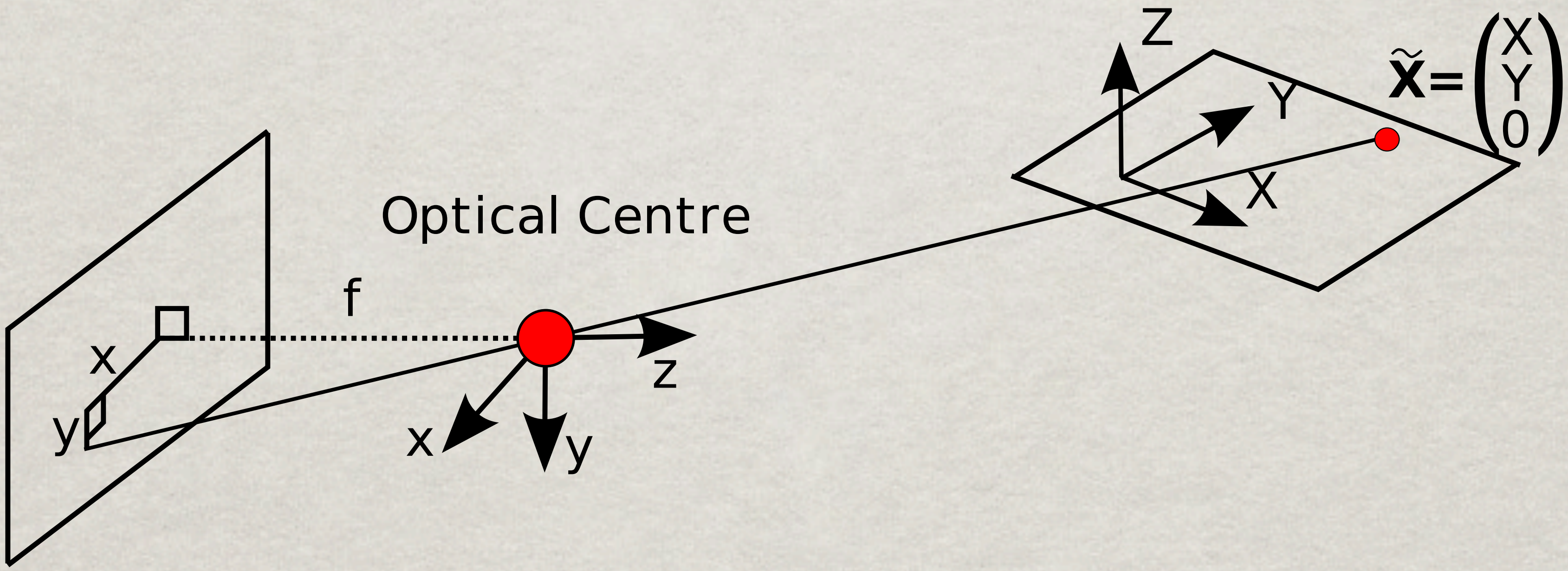
- [**R** | **t**] contain the *extrinsic* parameters

Camera calibration

- Zhang's camera calibration (*A flexible new technique for camera calibration*, PAMI 2000)
- In OpenCV, and in Matlab toolbox
- Finds **K** from 3 or more photos of a planar calibration target
- OpenCV also finds radial distorsion (omitted here).

Camera calibration

- We now imagine a world coordinate system fixed to the planar target



$$\gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera calibration

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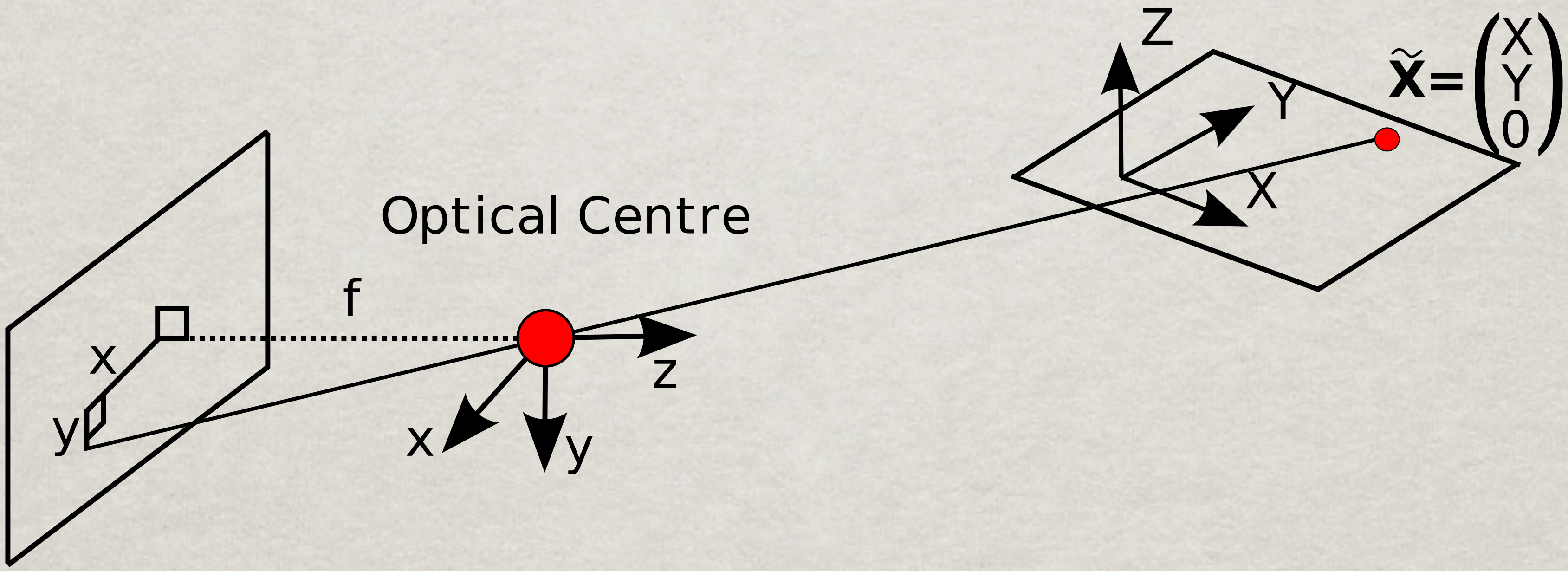

$$\gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Camera calibration

- If we estimate a homography between the image and the model plane (lecture 3) we know **H**

$$\mathbf{H} = [\mathbf{h}_1 \;\; \mathbf{h}_2 \;\; \mathbf{h}_3] = \mathbf{K}\,[\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \mathbf{t}]$$

- We also know that

$$\mathbf{r}_1^T\mathbf{r}_2 = 0 \quad \text{and} \quad \mathbf{r}_1^T\mathbf{r}_1 = \mathbf{r}_2^T\mathbf{r}_2$$

Camera calibration

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- We also know that

$$\mathbf{r}_1^T \mathbf{r}_2 = 0 \quad \text{and} \quad \mathbf{r}_1^T \mathbf{r}_1 = \mathbf{r}_2^T \mathbf{r}_2$$

$$\Rightarrow \mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 = 0$$

$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2$$

Camera calibration

- For a $\mathbf{K}$ of the form $\mathbf{K} = \begin{bmatrix} \alpha & \gamma & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$
- It can be shown that (use e.g. Maple)

$$\mathbf{K}^{-T}\mathbf{K}^{-1} = \mathbf{B} = \begin{bmatrix} \frac{1}{\alpha^2} & -\frac{\gamma}{\alpha^2\beta} & \frac{v_0\gamma - u_0\beta}{\alpha^2\beta} \\ -\frac{\gamma}{\alpha^2\beta} & \frac{\gamma^2}{\alpha^2\beta^2} + \frac{1}{\beta^2} & -\frac{\gamma(v_0\gamma - u_0\beta)}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} \\ \frac{v_0\gamma - u_0\beta}{\alpha^2\beta} & -\frac{\gamma(v_0\gamma - u_0\beta)}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} & \frac{(v_0\gamma - u_0\beta)^2}{\alpha^2\beta^2} + \frac{v_0^2}{\beta^2} + 1 \end{bmatrix}$$

Camera calibration

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- Remember our constraints

  $\mathbf{h}_1^T\mathbf{B}\mathbf{h}_2 = 0$ and $\mathbf{h}_1^T\mathbf{B}\mathbf{h}_1 - \mathbf{h}_2^T\mathbf{B}\mathbf{h}_2 = 0$

Camera calibration

- As $\mathbf{B}$ is symmetric $\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_4 \\ b_2 & b_3 & b_5 \\ b_4 & b_5 & b_6 \end{bmatrix}$

Camera calibration

- As $\mathbf{B}$ is symmetric $\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_4 \\ b_2 & b_3 & b_5 \\ b_4 & b_5 & b_6 \end{bmatrix}$
- If we now define $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}^T$
- The constraints can be written as

$$\begin{bmatrix} \mathbf{v}_{12}^T \\ (\mathbf{v}_{11} - \mathbf{v}_{22})^T \end{bmatrix} \mathbf{b} = 0$$

$$\mathbf{v}_{ij} = [h_{i1}h_{j1},\ h_{i1}h_{j2} + h_{i2}h_{j1},\ h_{i2}h_{j2},\ h_{i3}h_{j1} + h_{i1}h_{j3},\ h_{i3}h_{j2} + h_{i2}h_{j3},\ h_{i3}h_{j3}]^T$$

Camera calibration

- Each view of the plane gives us two rows in the system:

$$\mathbf{V}\mathbf{b} = 0$$

- As **b** has 6 unknowns, we need 3 views of the plane.

- Two views can also work if we require $\gamma = 0$

Camera calibration

- Once **b** has been estimated, we can extract the parameters in **K** according to

$$v_0 = (b_2 b_4 - b_1 b_5)/(b_1 b_3 - b_2^2)$$
$$\lambda = b_6 - (b_3^2 + v_0(b_2 b_4 - b_1 b_5)/b_1$$
$$\alpha = \sqrt{\lambda/b_1}$$
$$\beta = \sqrt{\lambda b_1/(b_1 b_3 - b_2^2)}$$
$$\gamma = -b_2\alpha^2\beta/\lambda$$
$$u_0 = \gamma v_0 \alpha - b_4\alpha^2/\lambda$$

Camera calibration

- Once **b** has been estimated, we can extract the parameters in **K** according to

$$
\begin{aligned}
v_0 &= (b_2 b_4 - b_1 b_5)/(b_1 b_3 - b_2^2) \\
\lambda &= b_6 - (b_3^2 + v_0(b_2 b_4 - b_1 b_5)/b_1 \\
\alpha &= \sqrt{\lambda / b_1} \\
\beta &= \sqrt{\lambda b_1/(b_1 b_3 - b_2^2)} \\
\gamma &= -b_2 \alpha^2 \beta / \lambda \\
u_0 &= \gamma v_0 \alpha - b_4 \alpha^2 / \lambda
\end{aligned}
$$

- The book instead suggests Cholesky factorisation

Camera calibration

- Once **K** is computed we can also find the extrinsic camera parameters **R**,**t** for each image:

$$\mathbf{r}_1 = \lambda \mathbf{K}^{-1}\mathbf{h}_1 \quad \mathbf{r}_2 = \lambda \mathbf{K}^{-1}\mathbf{h}_2 \quad \mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{R} = [\mathbf{r}_1 \ \ \mathbf{r}_2 \ \ \mathbf{r}_3] \quad \mathbf{t} = \lambda \mathbf{K}^{-1}\mathbf{h}_3$$

$$(\lambda = 1/||\mathbf{K}^{-1}\mathbf{h}_1|| = 1/||\mathbf{K}^{-1}\mathbf{h}_2||)$$

Camera calibration

- Once **K** is computed we can also find the extrinsic camera parameters **R**,**t** for each image:

$$\mathbf{r}_1 = \lambda \mathbf{K}^{-1}\mathbf{h}_1 \quad \mathbf{r}_2 = \lambda \mathbf{K}^{-1}\mathbf{h}_2 \quad \mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \mathbf{r}_3] \quad \mathbf{t} = \lambda \mathbf{K}^{-1}\mathbf{h}_3$$

- Finally, **K**, $\mathbf{R}_i$, $\mathbf{t}_i$ are refined using ML (minimising the cost function)

$$\arg\min \sum_{i=1}^{n} \sum_{j=1}^{m} ||\mathbf{x}_{ij} - \hat{\mathbf{x}}(\mathbf{K}, \mathbf{R}_i, \mathbf{t}_i, \mathbf{X}_j)||^2$$

Camera calibration

- So what about the initial homographies?

$$\mathbf{H} = \mathbf{K}\left[\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \mathbf{t}\right]$$

- Assign each point a WCS value $\mathbf{X} = [x \; y \; 0]^T$

Camera calibration

- So what about the initial homographies?

$$\mathbf{H} = \mathbf{K}\left[\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \mathbf{t}\right]$$

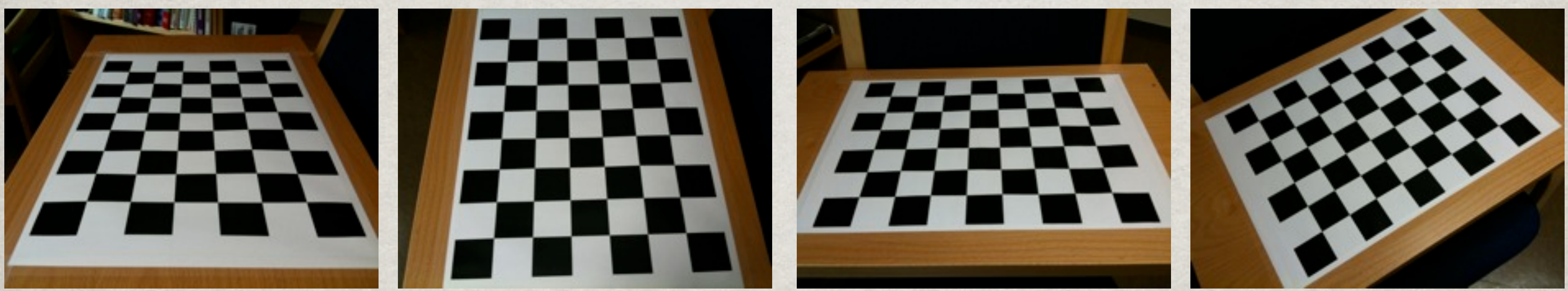
- Assign each point a WCS value $\mathbf{X} = [x\ y\ 0]^T$ Do we need to know which point is the upper left one on the checker-board? Why not?

Camera calibration

- Can we use any combination images of the calibration plane?

Camera calibration

- Can we use any combination images of the calibration plane?



- The constraints used: $\mathbf{r}_1^T\mathbf{r}_2 = 0$ and $\mathbf{r}_1^T\mathbf{r}_1 = \mathbf{r}_2^T\mathbf{r}_2$ have to be linearly independent.

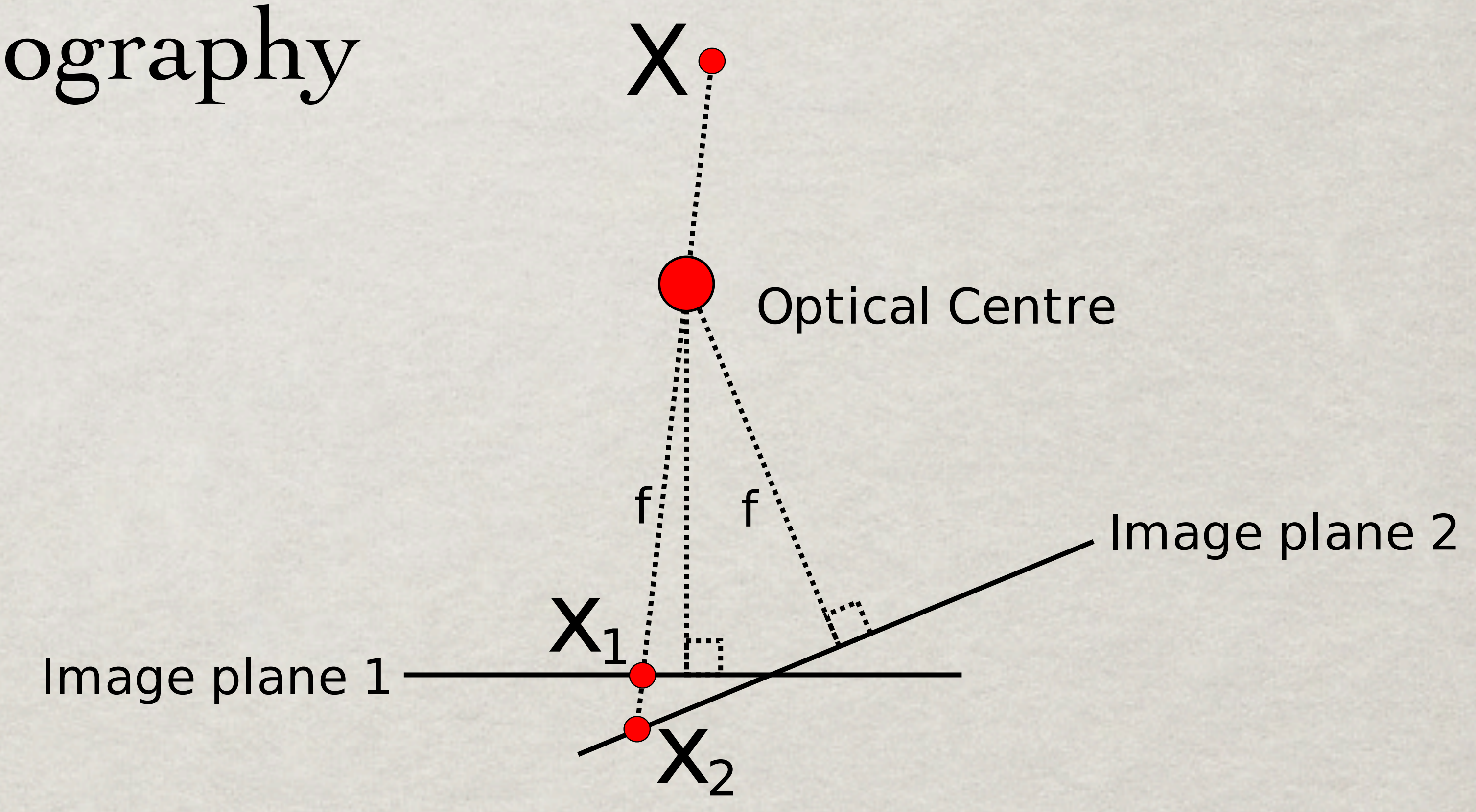
- $\Rightarrow$ Planes must not be parallel!

Calibrated epipolar geometry

- Rotational homography

$$\lambda \mathbf{x}_1 = \mathbf{K}\mathbf{X}$$

$$\lambda \mathbf{x}_2 = \mathbf{K}\mathbf{R}\mathbf{X}$$



Calibrated epipolar geometry

- Rotational homography

$$\lambda \mathbf{x}_1 = \mathbf{K}\mathbf{X}$$

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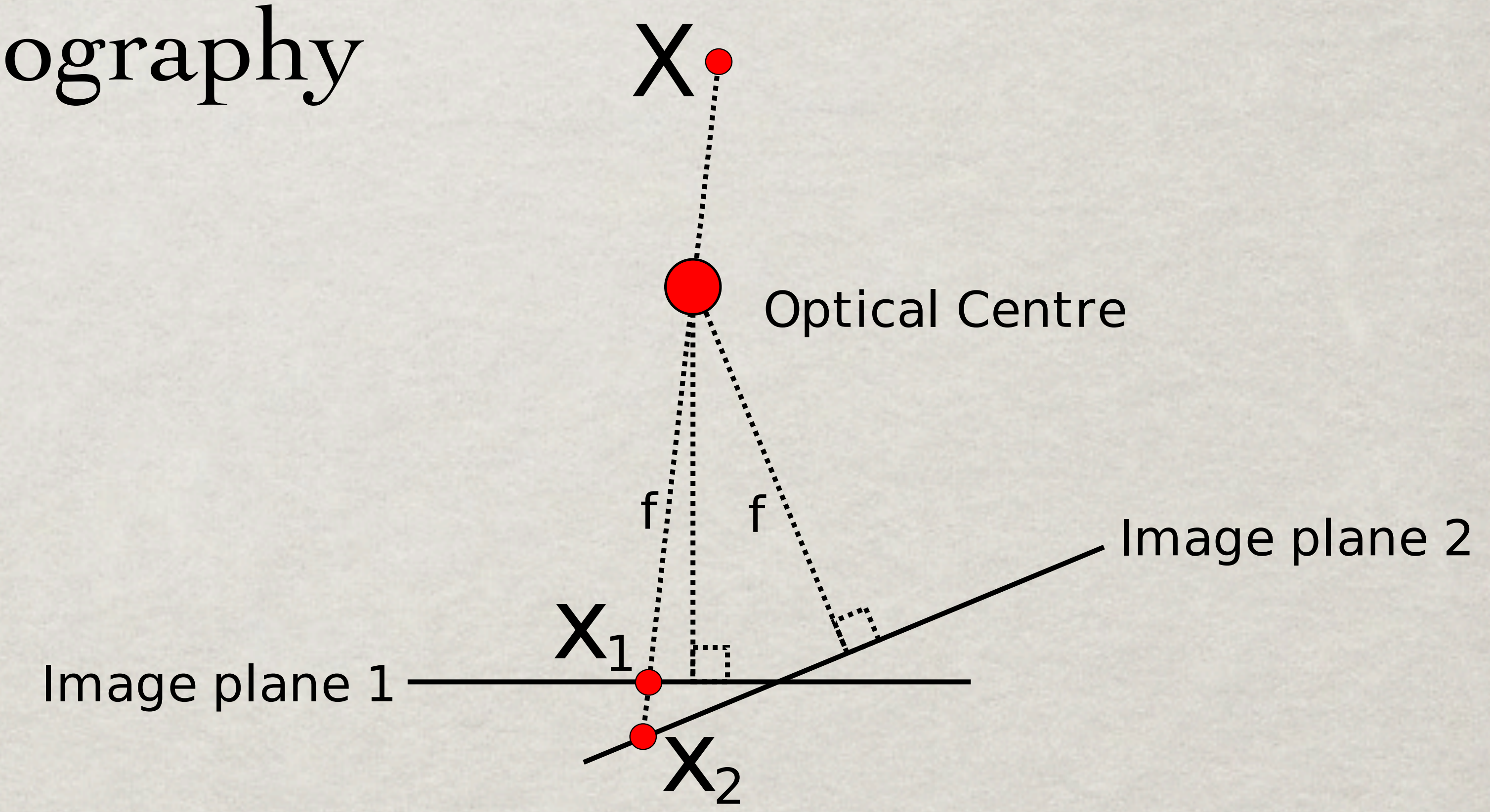


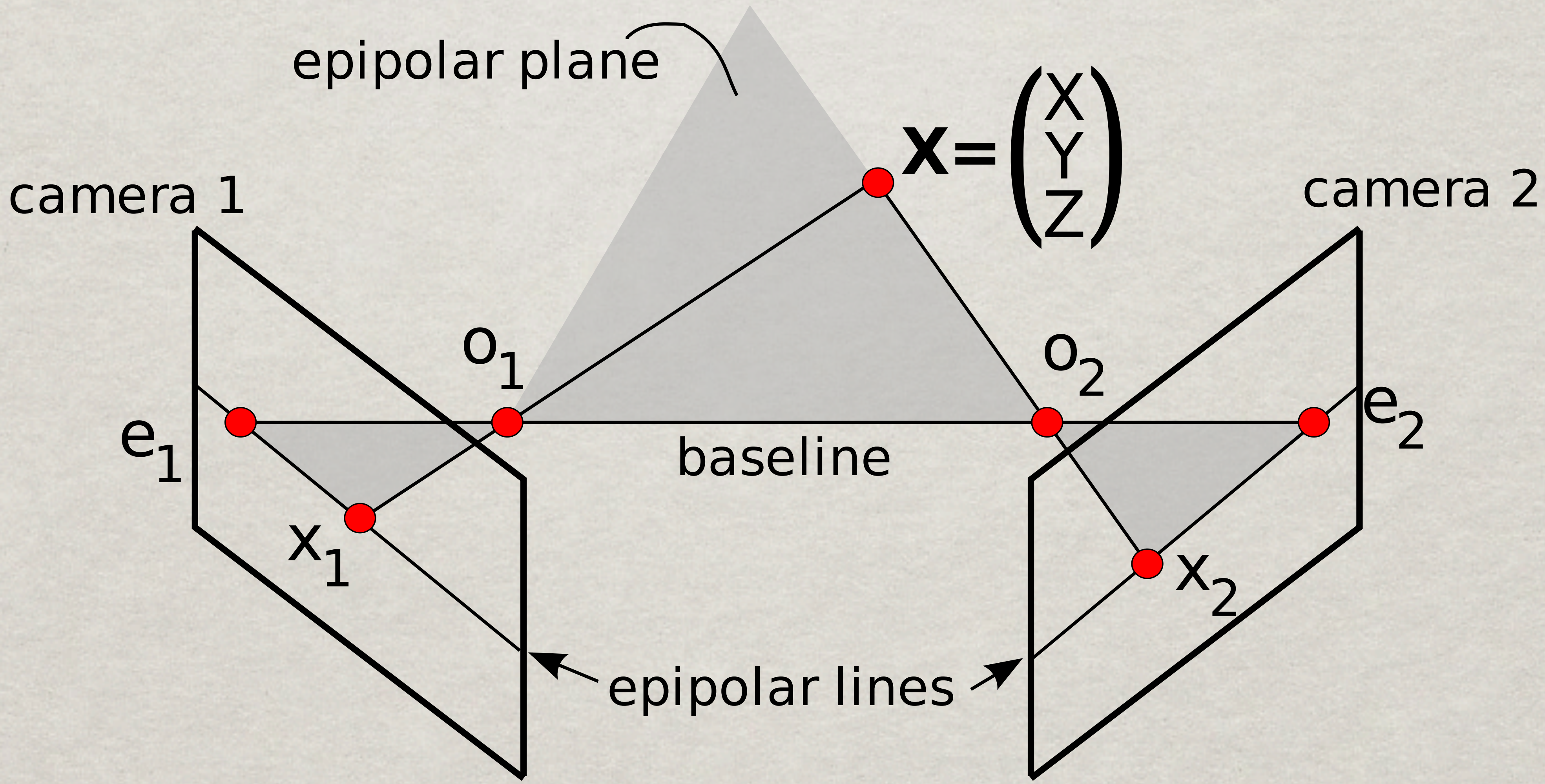
$$\lambda \mathbf{x}_2 = \mathbf{K}\mathbf{R}\mathbf{K}^{-1}\mathbf{x}_1$$

$$\mathbf{H} = \mathbf{K}\mathbf{R}\mathbf{K}^{-1}$$

Can be efficiently computed using the Procrustes algorithm (le 7)

Calibrated epipolar geometry

- Recall the epipolar constraint $\mathbf{x}_1^T \mathbf{F} \mathbf{x}_2 = 0$



Calibrated epipolar geometry

- Recall the epipolar constraint $\mathbf{x}_1^T \mathbf{F} \mathbf{x}_2 = 0$
- ...and the normalised image coordinates

$$\mathbf{x} = \mathbf{K}\hat{\mathbf{x}}$$

- We can instead express the epipolar constraint in normalised coordinates

$$\hat{\mathbf{x}}_1^T \mathbf{K}_1^T \mathbf{F} \mathbf{K}_2 \hat{\mathbf{x}}_2 = 0 \quad \text{or} \quad \hat{\mathbf{x}}_1^T \mathbf{E} \hat{\mathbf{x}}_2 = 0$$

- The matrix **E** is called the essential matrix. It has some interesting properties...

Calibrated epipolar geometry

- In lecture 2 we saw that for cameras $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$\mathbf{F} = [\mathbf{e}_{12}]_\times \mathbf{P}_1 \mathbf{P}_2^+ \qquad \mathbf{e}_{12} = \mathbf{P}_1 \mathbf{O}_2$$

- Now, if $\mathbf{P}_2 = \mathbf{K}_2 [\mathbf{I}|\mathbf{0}]$ and $\mathbf{P}_1 = \mathbf{K}_1 [\mathbf{R}|\mathbf{t}]$

- We get $\mathbf{P}_2^+ = \begin{bmatrix} \mathbf{K}_2^{-1} \\ \mathbf{0}^T \end{bmatrix}$ and

$$\mathbf{F} = [\mathbf{K}_1 \mathbf{t}]_\times \mathbf{K}_1 \mathbf{R} \mathbf{K}_2^{-1}$$

Calibrated epipolar geometry

- Using the cross-product-commutator rule: (A4.3)

$$[\mathbf{b}]_\times \mathbf{A} = \det(\mathbf{A})\mathbf{A}^{-T}\left[\mathbf{A}^{-1}\mathbf{b}\right]_\times$$

- on $\mathbf{F} = [\mathbf{K}_1\mathbf{t}]_\times \mathbf{K}_1\mathbf{R}\mathbf{K}_2^{-1}$

- ...we may express $\mathbf{F}$ as either of

$$\mathbf{F} = \mathbf{K}_1^{-T}[\mathbf{t}]_\times \mathbf{R}\mathbf{K}_2^{-1} \qquad \mathbf{F} = \mathbf{K}_1^{-T}\mathbf{R}\left[\mathbf{R}^T\mathbf{t}\right]_\times \mathbf{K}_2^{-1}$$

$$\mathbf{F} = \mathbf{K}_1^{-T}\mathbf{R}[\mathbf{t}_2]_\times \mathbf{K}_2^{-1}$$

Calibrated epipolar geometry

- This gives us the essential matrix expressions:

$$\mathbf{E} = [\mathbf{t}]_\times \mathbf{R} = \mathbf{R}\left[\mathbf{R}^T\mathbf{t}\right]_\times$$

- **E** has only 5 dof (3 from **R**, 2 from **t**)
recall that **F** has *7*

- A necessary and sufficient condition on **E** is that it has the singular values [a,a,0]
(see 9.6.1 in the book for proof)

Calibrated epipolar geometry

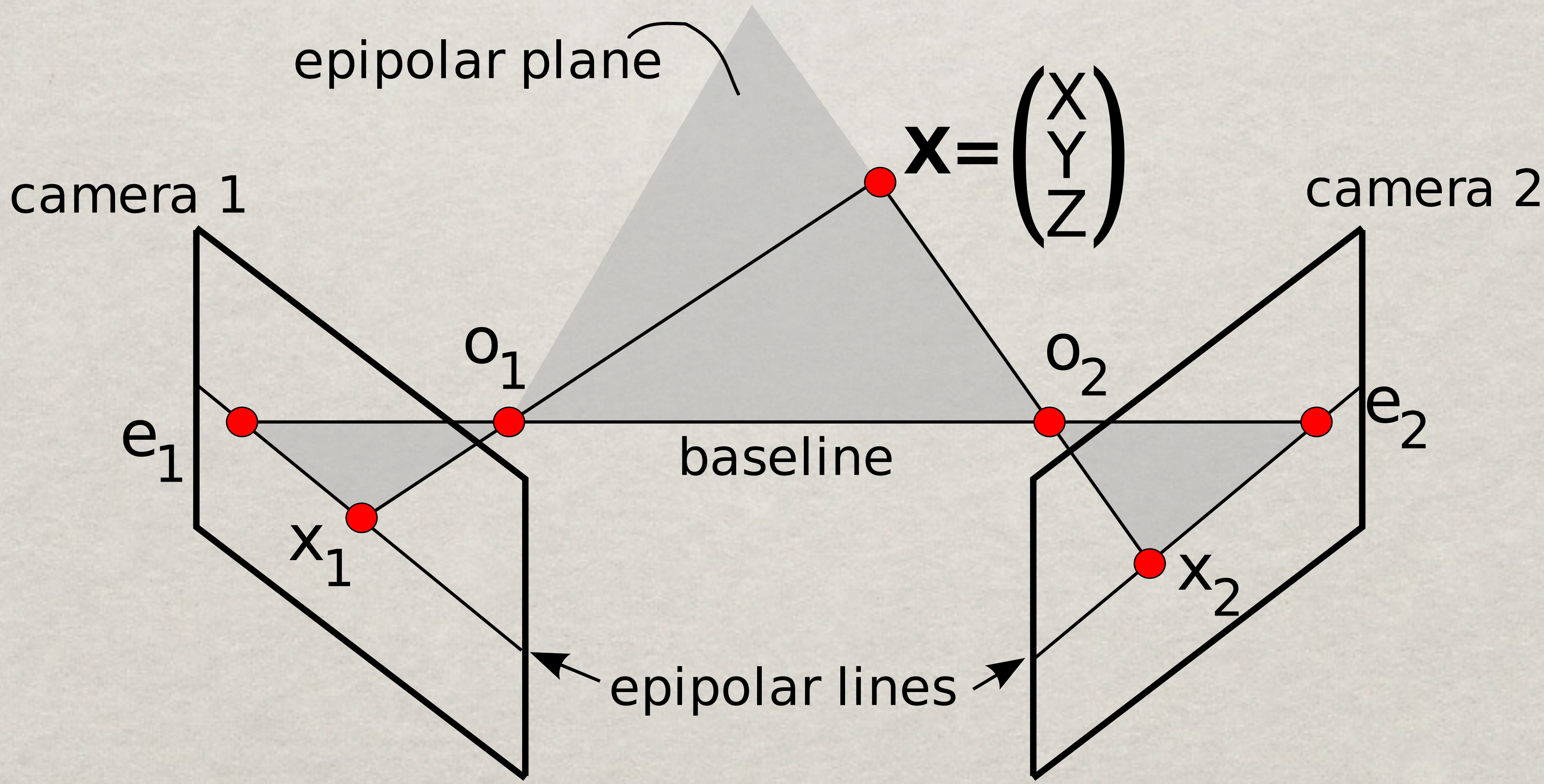
- This gives us the essential matrix expressions:

$$\mathbf{E} = [\mathbf{t}]_\times \mathbf{R} = \mathbf{R}\left[\mathbf{R}^T\mathbf{t}\right]_\times$$

- We can extract **R** and **t** (up to scale) from **E** if we also make use of one point correspondence (a 3D point known to be in front of both cameras). See 9.6.2 in the book.

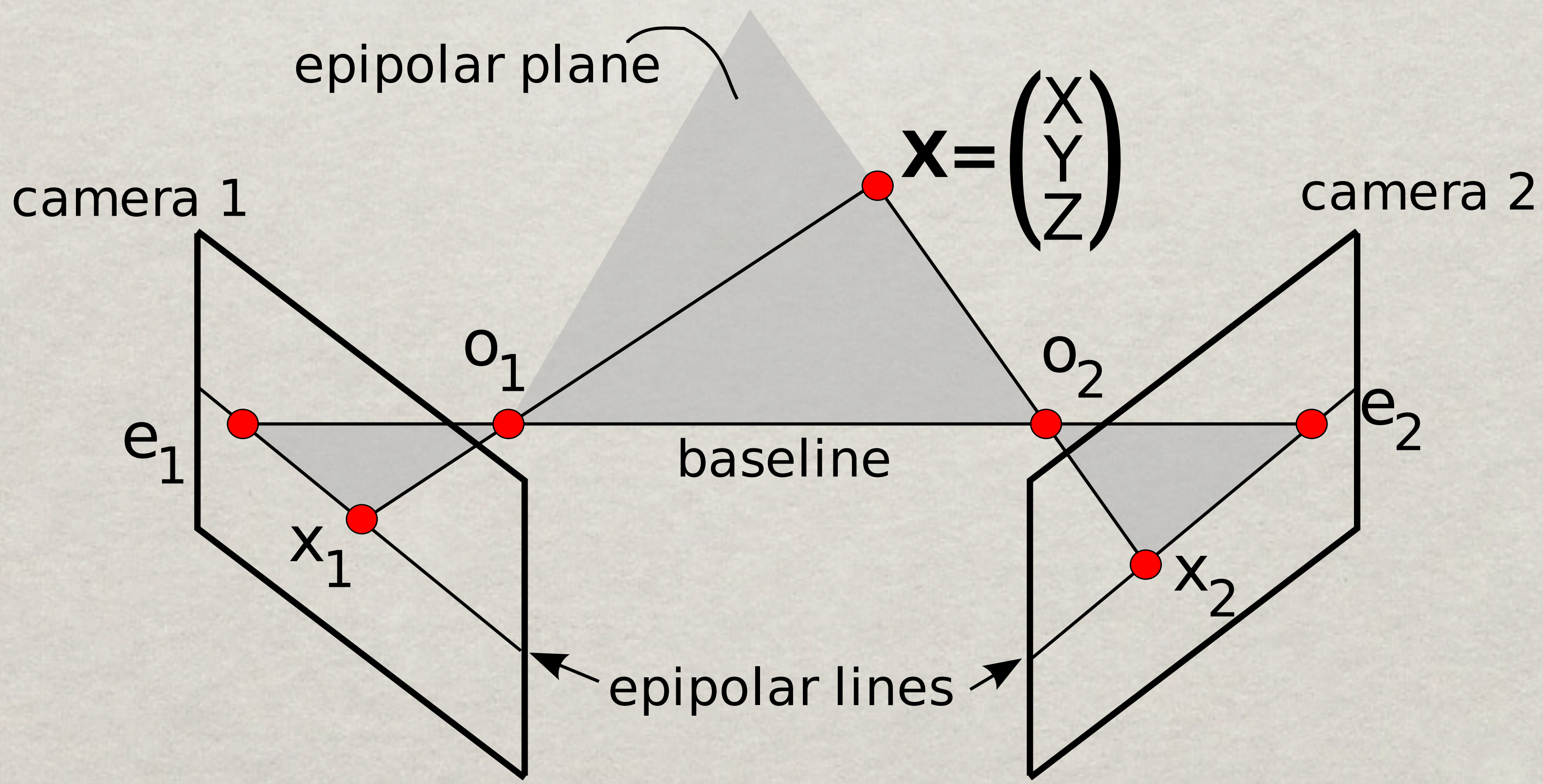
Oriented epipolar geometry

- The regular epipolar constraint $\mathbf{x}_1^T \mathbf{F} \mathbf{x}_2 = 0$ ignores the knowledge that points are in front of the camera.



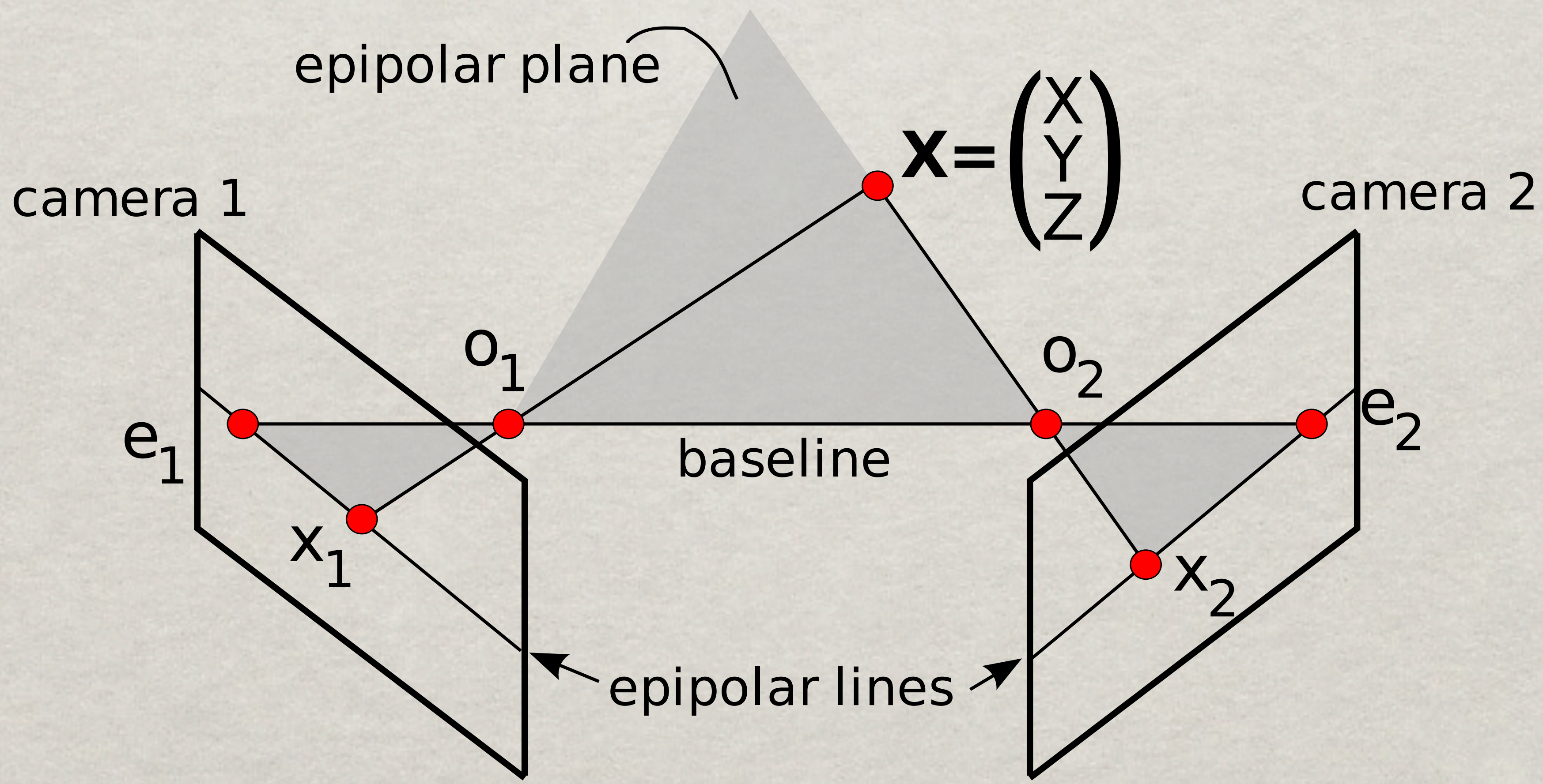
Oriented epipolar geometry

- The *oriented epipolar constraint* properly distinguishes points in front and behind of the camera $\quad \lambda \mathbf{e}_1 \times \mathbf{x}_1 = \mathbf{F}\mathbf{x}_2 \,, \quad \lambda \in \mathbb{R}^+$



Oriented epipolar geometry

- The *oriented epipolar constraint* compares oriented lines $\lambda \mathbf{e}_1 \times \mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$
- Sign of **F** needs to be determined



Oriented epipolar geometry

- The *oriented epipolar constraint* compares oriented lines $\lambda\mathbf{e}_1 \times \mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$
- Sign of **F** needs to be determined
- A point $\lambda\,[x_1\ x_2\ 1]^T$ is said to be in front of the camera if $\lambda > 0$ and behind the camera otherwise.
- Use a trusted correspondence (e.g. one used to estimate **F**) to determine sign

Oriented epipolar geometry

- The *oriented epipolar constraint* compares oriented lines $\lambda \mathbf{e}_1 \times \mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$
- 1. Ensure correct sign of $\mathbf{F}$
- 2. Compare the lines by checking the sign of the scalar product of the *line normals*

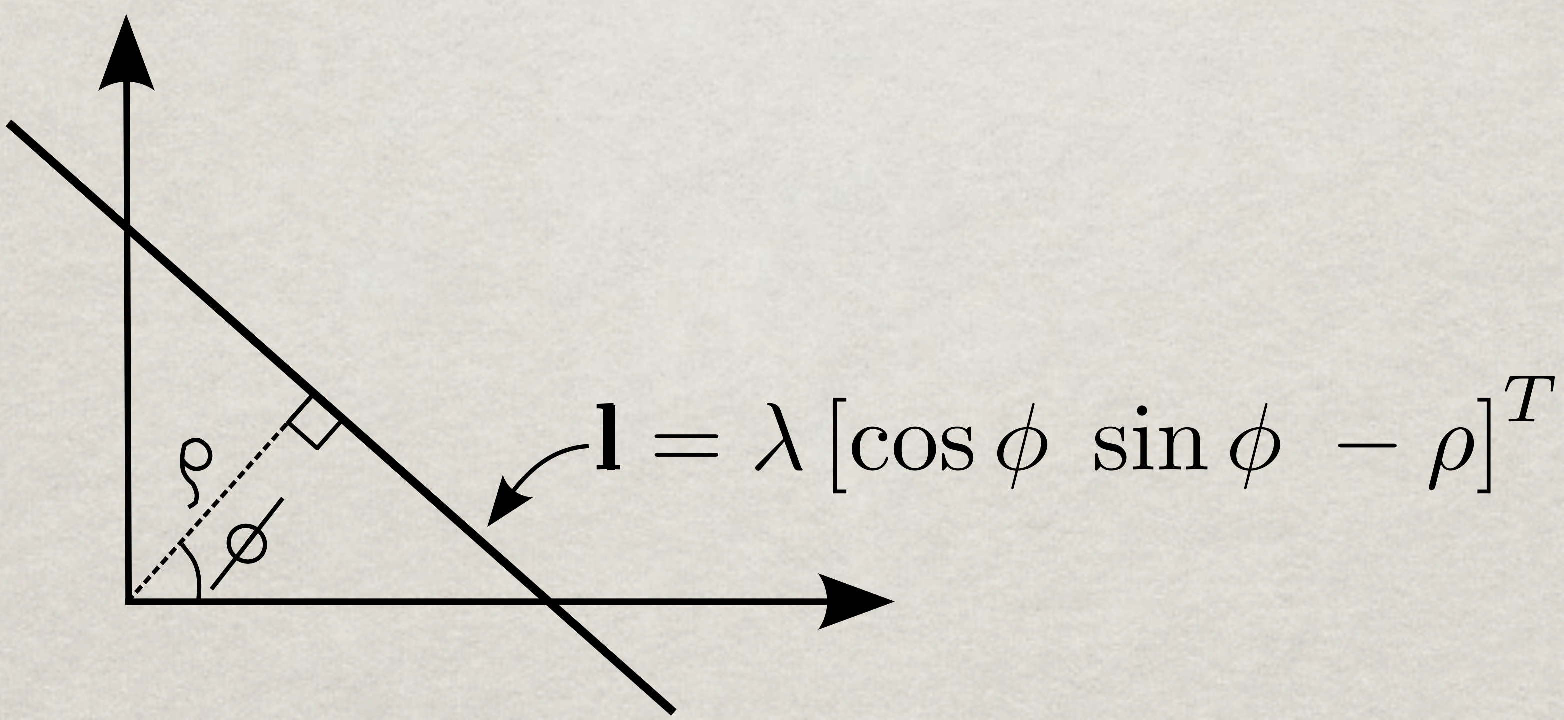
$$\mathbf{l}_1 = \lambda \left[\cos\phi_1 \ \sin\phi_1 \ -\rho_1\right]^T$$

$$\mathbf{l}_2 = \lambda \left[\cos\phi_2 \ \sin\phi_2 \ -\rho_2\right]^T$$

(elements 1 and 2 only) Why?

Oriented epipolar geometry

- What if the points are noisy?



- Small amounts of noise in $\mathbf{x}_1$ or $\mathbf{x}_2$ may cause ρ in $\lambda\mathbf{e}_1 \times \mathbf{x}_1$ or $\mathbf{F}\mathbf{x}_2$ to change sign!

Oriented epipolar geometry

- Usage:

- The oriented epipolar constraint can be used to quickly reject a hypothesized **F** inside a RANSAC loop.

- See: Chum, Werner and Matas, *Epipolar Geometry Estimation via RANSAC benefits from the Oriented Epipolar Constraint*, ICPR04

Discussion

- Discussion of the paper:
  Mendonça and Cippolla, A *Simple Technique for Self-Calibration,* CVPR99

FOR NEXT WEEK...

- A selection from chapters 15 and 16 (see email).
- David Nistér, *An Efficient Solution to the Five-Point Relative Pose Problem*, PAMI04