Variational Methods

Computer Vision, Lecture 15
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Optimization: Overview

Function		**Output (codomain / target set)**	
	Set	Continuous	Discrete
Input (domain of definition)	Continuous	Lecture 15	Lecture 15
	Discrete	Lecture 13	Lecture 13

ex: diffusion — ex: level-set segmentering

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Repetition: Vector Analysis

- Nabla operator $\nabla = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$
- On a scalar function $\nabla f = \mathrm{grad} f = \begin{bmatrix} \partial_x f \\ \partial_y f \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$
- On a vector field $\langle \nabla | \mathbf{f} \rangle = \nabla^T \mathbf{f} = \mathrm{div}\, \mathbf{f} = \partial_x f_1 + \partial_y f_2$
- Laplace operator $\Delta = \nabla^2 = \langle \nabla | \nabla \rangle = \mathrm{div}\,\mathrm{grad} = \partial_x^2 + \partial_y^2$ $\quad (\partial_x^2 f = f_{xx} \neq f_x^2)$
- Green's first identity $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\int_\Omega (f \Delta g + \langle \nabla f | \nabla g \rangle)\, d\mathbf{x} = \oint_{\partial \Omega} f \langle \nabla g | \mathbf{n} \rangle\, dS = 0$$

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Revisit: Diffusion

- Lecture on image enhancement:

$$f_s = \frac{\partial}{\partial s} f = \mathrm{div}(\mathbf{D}(\nabla f)\, \nabla f) = \langle \nabla | \mathbf{D}(\nabla f)\, \nabla f \rangle$$

- Consider scalar diffusivities $\mathbf{D}(\nabla f) \mapsto d(\nabla f)$
- Can diffusion be related to the iterations in an optimization process?

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Evolution Equation

- diffusion is an evolution process starting from the original image: **initial value problem (IVP)**
- discrete steps: gradient descent steps (forward Newton scheme) on a **boundary value problem (BVP)**
- BVP is obtained by **variational calculus** from a continuous objective function

Variational Methods

- Minimize the local integral of a Lagrange function $L(f, f_x, f_y, x, y)$

$$\varepsilon(f) = \int_\Omega L(f, \nabla f, \mathbf{x})\, d\mathbf{x}$$

- gives the Euler-Lagrange equation on Ω

$$L_f - \operatorname{div} L_{\nabla f} = L_f - \partial_x L_{f_x} - \partial_y L_{f_y} = 0 \qquad \forall x, y$$

- if we require $\langle \nabla f | \mathbf{n} \rangle = 0 \quad \text{on } \partial\Omega$

Insight: EL Equation

- for all test functions *g*, the **Gâteaux derivative**

$$\langle \delta\varepsilon(f), g \rangle = \frac{d\varepsilon(f + \eta g)}{d\eta}\Big|_{\eta=0} = \lim_{\eta\to 0} \frac{\varepsilon(f+\eta g) - \varepsilon(f)}{\eta}$$

must vanish (scalar product in function space)
- Inserting the Lagrangian gives

$$\begin{aligned}\langle \delta\varepsilon(f), g \rangle &= \int_\Omega \lim_{\eta\to 0} \frac{L(f+\eta g, \nabla(f+\eta g), \mathbf{x}) - L(f, \nabla f, \mathbf{x})}{\eta}\, d\mathbf{x} \\ &= \langle L_f(f, \nabla f, \cdot), g \rangle + \langle L_{\nabla f}(f, \nabla f, \cdot), \nabla g \rangle\end{aligned}$$

- Note $h(\mathbf{y}) = h(\mathbf{a}) + (\mathbf{y} - \mathbf{a})^T \nabla h(\mathbf{a}) + \mathcal{O}(|\mathbf{y} - \mathbf{a}|^2)$

Insight: EL Equation

- use homogenity of Green's first identity

$$\langle L_{\nabla f}, \nabla g \rangle + \langle \operatorname{div} L_{\nabla f}, g \rangle = \oint_{\partial\Omega} (L_{\nabla f}^T \mathbf{n}) g \, dS = 0$$

to rewrite $\langle L_{\nabla f}, \nabla g \rangle = -\langle \operatorname{div} L_{\nabla f}, g \rangle$

- thus $\langle \delta\varepsilon(f), g \rangle = \langle L_f - \operatorname{div} L_{\nabla f}, g \rangle$
- and we obtain the necessary condition (for all **x**)

$$L_f - \operatorname{div} L_{\nabla f} = 0$$

Linear Regularization

- Minimizing $\varepsilon(f) = \frac{1}{2}\int_\Omega f_x^2 + f_y^2 \, dx\, dy$

  i.e. no data term $L(f, f_x, f_y, x, y) = L(f_x, f_y, x, y)$
- Gives the Euler-Lagrange equation
  (note: $L_f = 0,\ L_{f_x} = f_x,\ Lf_y = f_y$)
  $$(\partial_x f_x + \partial_y f_y) = \Delta f = 0$$
- Such that gradient descent gives $f^{(s+1)} = f^{(s)} + \alpha\Delta f^{(s)}$
  or continuous formulation $f_s = \mathrm{div}(\nabla f) = \Delta f$
- Converges towards trivial solution

Non-Linear Regularization

- Minimizing $\varepsilon(f) = \int_\Omega \Psi(|\nabla f|)\, dx\, dy$

  special case: $\Psi() = \mathrm{Id}() \Rightarrow \Psi'() = 1$
- Gives the Euler-Lagrange equation

$$\partial_x \frac{\Psi'(|\nabla f|)}{|\nabla f|} f_x + \partial_y \frac{\Psi'(|\nabla f|)}{|\nabla f|} f_y = \mathrm{div}\left(\frac{\Psi'(|\nabla f|)}{|\nabla f|}\nabla f\right) = 0$$

- Such that gradient descent gives
  $$f^{(s+1)} = f^{(s)} + \alpha\,\mathrm{div}\left(\frac{\Psi'(|\nabla f^{(s)}|)}{|\nabla f^{(s)}|}\nabla f^{(s)}\right)$$

Interpretation

- Diffusion is an evolution over "time" s
- Starts at the measured image (IVP)
- Converges towards DC signal
- Critical parameter 1: "stopping time"
- Critical parameter 2: α
- Several examples in the enhancement lecture

Beyond Diffusion

- In what follows: add data term to minimization problem
- Converges towards non-trivial solution
- IVP with standard forward Euler scheme

Linear Restauration

- Minimizing

$$\varepsilon(f) = \frac{1}{2}\int_\Omega \underbrace{(f-f_0)^2 + \lambda(f_x^2 + f_y^2)}_{L(f, f_x, f_y, x, y)}\, dx\, dy$$

- Gives the Euler-Lagrange equation

$$\underbrace{f - f_0}_{L_f} - \underbrace{\lambda\Delta f}_{\mathrm{div}(L_{f_x}, L_{f_y})} = 0$$

- Such that gradient descent gives

$$\begin{aligned} f^{(s+1)} &= f^{(s)} - \alpha(f^{(s)} - f_0 - \lambda\Delta f^{(s)}) \\ &= (1-\alpha)f^{(s)} + \alpha(f_0 + \lambda\Delta f^{(s)}) \end{aligned}$$

Non-Linear Restauration

- Minimizing

$$\varepsilon(f) = \int_\Omega \frac{1}{2}(f-f_0)^2 + \lambda\Psi(|\nabla f|)\, dx\, dy$$

- Gives the Euler-Lagrange equation

$$f - f_0 - \lambda\,\mathrm{div}\left(\frac{\Psi'(|\nabla f|)}{|\nabla f|}\nabla f\right) = 0$$

- Such that gradient descent gives

$$\begin{aligned} f^{(s+1)} &= f^{(s)} - \alpha\left(f^{(s)} - f_0 - \lambda\,\mathrm{div}\left(\frac{\Psi'(|\nabla f^{(s)}|)}{|\nabla f^{(s)}|}\nabla f^{(s)}\right)\right) \\ &= (1-\alpha)f^{(s)} + \alpha(f_0 + \lambda \mathrm{div}(\ldots)) \end{aligned}$$

Special Case: TV/ROF

- Minimizing

$$\varepsilon(f) = \int_\Omega \frac{1}{2}(f-f_0)^2 + \lambda|\nabla f|\, dx\, dy$$

- Gives the Euler-Lagrange equation

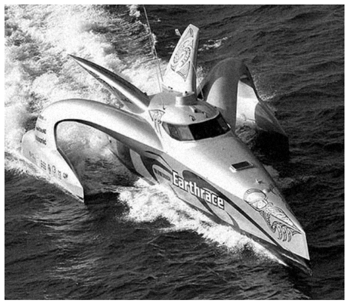
$$f - f_0 - \lambda\,\mathrm{div}\left(\frac{1}{|\nabla f|}\nabla f\right) = 0$$

- Such that gradient descent gives

$$f^{(s+1)} = f^{(s)} - \alpha\left(f^{(s)} - f_0 - \lambda\,\mathrm{div}\left(\frac{1}{|\nabla f^{(s)}|}\nabla f^{(s)}\right)\right)$$

Example (lecture 13)

- Paramters: α =0.0005, λ=0.5, noise(0,0.001)



Explicit vs Implicit

- All gradients so far are based on the previous estimate: the time discretization leads to an **explicit scheme** (least calculations, easiest)
- If the gradients are based on the new estimate, we obtain an **implicit scheme** (always stable, large time steps)
- If the gradients are based on both, we obtain the **Crank-Nicolson scheme** (always stable, small time steps)

Interpretation

- Restauration is an IVP
- Uses the measured image as input in each iteration
- Converges towards non-trivial solution
- Critical parameter 1: "meta" parameter λ
- Critical parameter 2: α

The Data Term

- Data term can be used to describe the measurement model
- Leads to non-trivial iterations

Deblurring

- Minimizing

$$\varepsilon(f) = \frac{1}{2}\int_\Omega (g * f - f_0)^2 + \lambda(f_x^2 + f_y^2)\,dx\,dy$$

- Gives the Euler-Lagrange equation

$$g(-\cdot) * (g * f - f_0) - \lambda \Delta f = 0$$

- Such that gradient descent gives

$$f^{(s+1)} = f^{(s)} - \alpha(g(-\cdot) * (g * f^{(s)} - f_0) - \lambda \Delta f^{(s)})$$

Comments

- *g*: point spread function (PSF)
- *g*(-*x*): correlation operator / adjoint operator
- even symmetry PSF: self adjoint
- definition of adjoint operator $\langle x|Ay\rangle = \langle A^* x|y\rangle$
- Example from lecture 13

Demonstration

Indirect Measurements

- Similar to target tracking, where observations might be different from states
- We observe image information but apply the variational framework to estimate other fields
- Two examples here: optical flow and segmentation (binary partition)

Optical Flow $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

- Minimizing
$$\varepsilon(\mathbf{f}) = \frac{1}{2}\int_\Omega \overbrace{(\langle \mathbf{f}|\nabla g\rangle + g_t)}^{\text{BCCE}}{}^2 + \lambda(|\nabla f_1|^2 + |\nabla f_2|^2)\,dx\,dy$$
- Gives the Euler-Lagrange equation (HS!)
$$(\langle \mathbf{f}|\nabla g\rangle + g_t)\nabla g - \lambda\,\Delta\mathbf{f} = 0$$
- Laplacian is approximately
$$\Delta\mathbf{f} \approx \bar{\mathbf{f}} - \mathbf{f}$$

1	1	1	-	3	·	0	1	0	=	1	-2	1

Optical Flow

- Plugging into the EL-equation gives

$$(\lambda + \nabla g \nabla g^T)\mathbf{f} \;=\; \lambda \bar{\mathbf{f}} - g_t \nabla g$$

- Explicitly solving for **f** results in

$$\begin{aligned}
(\lambda + \nabla g \nabla g^T)\mathbf{f} &= (\lambda + \nabla g \nabla g^T)\bar{\mathbf{f}} - (\nabla g \nabla g^T \bar{\mathbf{f}} + \nabla g g_t) \\
&= (\lambda + \nabla g \nabla g^T)\bar{\mathbf{f}} - \nabla g(\nabla g^T \bar{\mathbf{f}} + g_t) \\
&= (\lambda + \nabla g \nabla g^T)\bar{\mathbf{f}} - \frac{\lambda + \nabla g^T \nabla g}{\lambda + \nabla g^T \nabla g} \nabla g(\nabla g^T \bar{\mathbf{f}} + g_t) \\
&= (\lambda + \nabla g \nabla g^T)\bar{\mathbf{f}} - \frac{\lambda + \nabla g \nabla g^T}{\lambda + \nabla g^T \nabla g} \nabla g(\nabla g^T \bar{\mathbf{f}} + g_t) \\
\mathbf{f} &= \bar{\mathbf{f}} - \frac{1}{\lambda + \nabla g^T \nabla g} \nabla g(\nabla g^T \bar{\mathbf{f}} + g_t)
\end{aligned}$$

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Optical Flow

- Iterating the solution

$$\mathbf{f} = \bar{\mathbf{f}} - \frac{1}{\lambda + \nabla g^T \nabla g} \nabla g(\nabla g^T \bar{\mathbf{f}} + g_t)$$

- Results in the Horn & Schunck iteration

$$\mathbf{f}^{(s+1)} = \bar{\mathbf{f}}^{(s)} - \frac{1}{\lambda + |\nabla g|^2}(\langle \bar{\mathbf{f}}^{(s)} | \nabla g \rangle + g_t)\nabla g$$

- Significant improvement: use median instead of $\bar{\mathbf{f}}$!

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Demonstration

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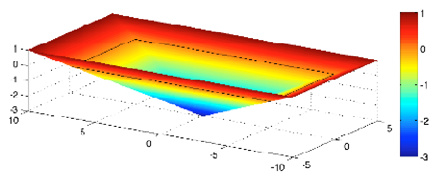
Segmentation / Contours

- Segmentation function (level-set function) to be optimized
- Negative / positive in background / object region
- Contour is the zero-level



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Segmentation / Contours

- Chan-Vese energy minimized of level-set function ϕ

$$E(\phi) = \int_\Omega (H(\phi) - 1) f_2 - H(\phi) f_1 + \lambda |\nabla H(\phi)| \, d\mathbf{x}$$

- H is the (regularized) Heaviside function
- f are weights computed from the image (e.g. squared deviation from certain greyscale)
- EL equation

$$\delta(\phi) \left(f_2 - f_1 + \lambda \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) \right) = 0$$

- Problem: (regularized) delta function δ

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Segmentation / Contours

- Omitting delta-function
- Original solution remains solution
- Corresponds to minimizing

$$E(\phi) = \int_\Omega (f_2 - f_1) \phi + \lambda |\nabla \phi| \, d\mathbf{x}$$

- Non-existence of minimizer (!)

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Segmentation / Contours

- Binary function instead of level-set function
- becomes Ising model (lecture 13)

$$E(\phi) = -\int_{\Omega_2} f_2 \, d\mathbf{x} - \int_{\Omega_1} f_1 \, d\mathbf{x} + \lambda |C|$$

- Hard to solve – use relaxation
  - Binary function replaced by smooth approximation
  - After optimization apply threshold
- Discrete optimization (lecture 13)

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Examples

http://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/CREMERS2/

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Demonstration

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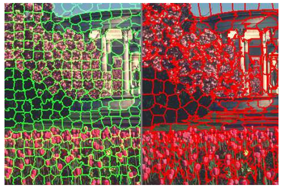
Over-Segmentation / Superpixels

- So far: attempt for semantic segmentation
- Alternative: over-segmentation based on stationarity of image process
  - MSER (lecture 8)
  - Superpixel algorithms – clustering in 5D (*x*,*y*,*R*,*G*,*B*)
  - Left: contour-relaxed superpixels
  - Right: SLIC



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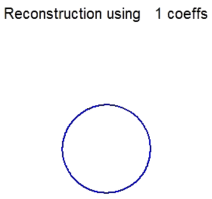
Alternative Contour Methods

- Popular application:
  - Geodesic active contours
  - Snakes
- Contour parametrized as

$$\mathbf{v}(s) = [x(s), y(s)] \qquad s \in [0,1]$$

- Usually approximated as spline
- Option: Fourier descriptors



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Geodesic Active Contours

- Consider a curve moving in time

$$\mathbf{v}(s,t) = [x(s,t), y(s,t)]$$

- let the curve develop according to the inward normal **n** and the curvature *c*

$$\frac{\partial \mathbf{v}}{\partial t} = V(c)\mathbf{n}$$

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Geodesic Active Contours

- Assume level set function $\phi(x, y, t)$ such that $\phi(\mathbf{v}(s,t), t) = 0$
- Negative inside and positive outside gives

$$\mathbf{n} = -\frac{\nabla \phi}{|\nabla \phi|}$$

- Plug in normal into evolution equation gives

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{V(c)\nabla \phi}{|\nabla \phi|}$$

Geodesic Active Contours

- What remains is to re-write l.h.s. of

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{V(c)\nabla \phi}{|\nabla \phi|}$$

- Time derivative of $\phi(\mathbf{v}(s,t), t)$ gives

$$\frac{\partial \phi}{\partial t} + \nabla \phi \frac{\partial \mathbf{v}}{\partial t} = 0$$

- Such that $\dfrac{\partial \phi}{\partial t} = V(c)|\nabla \phi|$
- Level-set equation

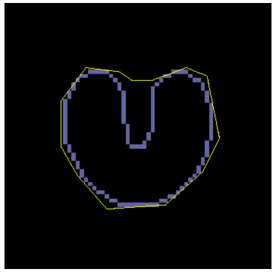
Snake Function

- Energy function consists of typically 3 terms:
  - internal energy
  - image forces
  - external constraint forces

$$\varepsilon(\mathbf{v}(s)) = \int_0^1 E_{\text{int}}(\mathbf{v}(s)) + E_{\text{image}}(\mathbf{v}(s)) + E_{\text{con}}(\mathbf{v}(s))\, ds$$

Limitations

- Initialization close to solution
- Problems at concave regions



http://iacl.ece.jhu.edu/projects/gvf/

GVF Snakes

- Gradient vector flow snakes
- GVF used as external force
- GVF field computation related to optical flow approach

GVF Field

- Minimizing (GVF: **f**)

$$\varepsilon(\mathbf{f}) = \frac{1}{2}\int_\Omega |\mathbf{f} - \nabla g|^2 |\nabla g|^2 + \lambda(|\nabla f_1|^2 + |\nabla f_2|^2)\, dx\, dy$$

- Gives the Euler-Lagrange equations

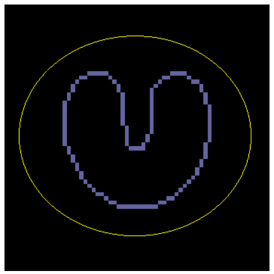
$$(\mathbf{f} - \nabla g)|\nabla g|^2 - \lambda\,\Delta\mathbf{f} = 0$$

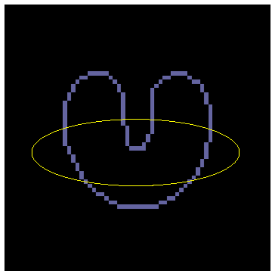
- Such that gradient descent gives

$$\mathbf{f}^{(s+1)} = \mathbf{f}^{(s)} - \alpha\left((\mathbf{f}^{(s)} - \nabla g)|\nabla g|^2 - \lambda\,\Delta\mathbf{f}^{(s)}\right)$$

Examples

- No concavity problem
- No initialization problem





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