

Contents

Preface	11
1 Background and Overview	15
1.1 Euclidean geometry	15
1.2 Perspective	15
1.3 Projective geometry	16
1.4 Photogrammetry	17
1.5 Computer vision	18
I Representations	19
2 Cartesian Representations	21
2.1 Points in \mathbb{E}^2	22
2.2 Lines in \mathbb{E}^2	22
2.2.1 Slope and Intercept	23
2.2.2 Hesse Normal Form	24
2.2.3 Parametric Representations of a Line	25
2.3 Points, planes and lines in \mathbb{E}^3	26
2.4 Basic geometric operations	27
2.4.1 The line that intersects two points in \mathbb{E}^2	27
2.4.2 The point of intersection between two lines in \mathbb{E}^2	28
2.4.3 Distance between a point and a line in \mathbb{E}^2	29
2.5 Before We Continue	30
3 Homogeneous Representations in 2D	31
3.1 Homogeneous coordinates of 2D points	31
3.1.1 P-normalization	32
3.2 Dual homogeneous coordinates of 2D lines	34
3.2.1 D-normalization	35
3.3 Relations between points and lines	36
3.3.1 The line that intersects two points	36
3.3.2 The point of intersection between two lines	37
3.3.3 Points on a line, or not	37
3.3.4 Lines through a point, or not	39
3.4 Operations on normalized homogeneous coordinates	39
3.4.1 Distance between two points	39
3.4.2 Distance between a point and a line	40
3.4.3 Area of a triangle	42
3.4.4 Concluding remarks	44
3.5 Points and lines at infinity	45
3.5.1 Points at infinity	45
3.5.2 The line at infinity	47

3.5.3	Projective geometry and the projective plane	48
3.6	Parametric representations of lines	49
3.7	Plücker coordinates	50
3.8	Duality Principle in the Projective Plane	52
4	Transformations in 2D	53
4.1	How to characterize a transformation	53
4.1.1	Degrees of freedom	53
4.2	Rigid transformations	54
4.3	Similarity transformations	58
4.4	Affine transformations	59
4.4.1	Non-uniform scaling	60
4.4.2	Reflections	60
4.4.3	Shearing	62
4.4.4	Decomposition of an affine transformation	63
4.5	Dual transformations	63
4.5.1	Geometric characterization of affine transformations	64
4.6	Projections	65
4.7	Concluding remarks	65
5	Homogeneous Representations in 3D	67
5.1	Homogeneous coordinates of 3D points	67
5.2	Dual homogeneous coordinates of 3D planes	68
5.2.1	Parametric representation of a plane	69
5.3	Homogeneous representations of 3D lines	70
5.3.1	Parametric representation of a 3D line	70
5.3.2	Plücker coordinates of 3D lines	71
5.3.3	L-normalization	72
5.3.4	Dual Plücker coordinates	73
5.3.5	The duality mapping	74
5.3.6	Internal constraint and degrees of freedom	74
5.3.7	DL-normalization	75
5.4	Incidence relations between points, planes, and lines	75
5.4.1	The intersection of a line with a plane	76
5.4.2	The plane that intersects a line and a point	76
5.4.3	The point that intersects three planes	77
5.4.4	The plane that intersects three points	78
5.4.5	Points on a plane, or not	79
5.4.6	Planes through a point, or not	80
5.4.7	The intersection of two lines	80
5.5	Distances	81
5.5.1	Distance between two points	81
5.5.2	Signed distance between a point and a plane	81
5.5.3	Distance between a point and a line	81
5.6	The duality principle in \mathbb{E}^3	82
6	Transformations in 3D	83
6.1	Degrees of freedom	83
6.2	Rigid transformations	83
6.3	Similarity transformations	84
6.4	Affine transformations	84
6.5	Dual transformations	86
6.6	Line transformations	86

7 Homographies	87
7.1 Homographies in 2D	87
7.1.1 Applications of homographies	90
7.1.2 Points at infinity revisited	90
7.2 Homographies in 3D	92
7.3 Canonical set of homogeneous coordinates	92
7.3.1 Canonical set homogeneous coordinates in 2D	92
7.3.2 Canonical set of homogeneous coordinates in 3D	93
7.3.3 Concluding remarks	93
8 The Pinhole Camera	95
8.1 Central projection	95
8.1.1 Camera obscura	97
8.2 Camera centered coordinate systems	98
8.2.1 The normalized camera	101
8.3 The general pinhole camera	105
8.3.1 Camera transformations	105
8.3.2 The camera at infinity	106
8.3.3 Internal and external camera parameters	107
8.3.4 Camera resectioning	110
8.4 The digital image	112
8.5 The geometry of a pinhole camera	115
8.5.1 Projection lines	115
8.5.2 The image of a line	116
8.5.3 Projection planes	117
8.5.4 Camera projection constraints	118
8.5.5 Geometric interpretation of \mathbf{C}	119
8.5.6 Field of view	119
9 Prelude to Two-View Geometry	121
9.1 Correspondences	121
9.1.1 Interest points	122
9.1.2 The correspondence problem	124
9.2 Planar homography	126
9.2.1 Simple derivation of a planar homography	126
9.2.2 The calibrated case	127
9.2.3 Solving for the plane and relative camera pose	127
9.2.4 Applications	130
9.3 Rotational homography	131
9.3.1 Derivation of a rotational homography	131
9.3.2 A constraint on rotational homographies	133
9.3.3 Solving for \mathbf{R} and \mathbf{K}	134
9.3.4 Before we continue	135
9.3.5 Applications	136
10 Epipolar Geometry	137
10.1 Stereo Cameras	139
10.1.1 Epipolar points	140
10.1.2 Epipolar line	141
10.1.3 Epipolar plane	142
10.2 The fundamental matrix and the epipolar constraint	143
10.2.1 The fundamental matrix	143
10.2.2 The epipolar constraint	144
10.2.3 Symmetry	144
10.2.4 The epipolar constraint in practice	146

10.2.5	Transfer	146
10.3	More properties of the fundamental matrix	147
10.3.1	Invariance to homography transformations of 3D space	147
10.3.2	Transformations of the image coordinates	148
10.3.3	Parameterization of \mathbf{F}	149
10.3.4	Camera matrices from \mathbf{F}	149
10.3.5	Factorization of \mathbf{F}	151
10.3.6	Summary of properties related to \mathbf{F}	152
10.4	Triangulation	153
10.4.1	Ambiguities of the world coordinate system	154
10.5	Calibrated epipolar geometry	155
10.5.1	Normalized stereo cameras	155
10.5.2	The essential matrix	156
10.5.3	Camera poses from \mathbf{E}	157
10.5.4	Transformations of the image coordinates	159
10.5.5	Minimal parameterization of \mathbf{E}	161
10.5.6	Summary of properties related to \mathbf{E}	161
11	Representations of 3D Rotations	163
11.1	Rotation matrices	164
11.1.1	Representations of $SO(3)$	165
11.2	Axis-angle representation	166
11.2.1	Rodrigues' rotation formula	167
11.2.2	Axis-angle from $SO(3)$	169
11.2.3	Summary	170
11.3	$so(3)$	170
11.3.1	Using the matrix exponential function	170
11.3.2	The Cayley transform	172
11.3.3	Summary	174
11.4	Unit quaternions	174
11.4.1	The quaternionic embedding of $SO(3)$	175
11.4.2	\mathbf{R} from unit quaternion	176
11.4.3	Unit quaternion from \mathbf{R}	177
11.4.4	Relation to the Cayley representation	178
11.4.5	Summary	178
11.5	Three-angle representations	179
11.5.1	Finding the Euler angles for \mathbf{R}	180
11.5.2	Summary	181
11.6	Which representation do I choose?	181
11.7	Related topics	183
11.7.1	Uniformly sampled random rotations	183
11.7.2	Twisted rotations	183
11.7.3	Singularities and gimbal lock	187
II	Estimation	189
12	Introduction to Estimation in Geometry	191
12.1	Least squares formulation	191
12.2	Total least squares formulation	194
12.3	What causes the errors? (Part I)	196
12.3.1	Measurement errors	196
12.3.2	Model errors	197
12.3.3	Outliers and inliers	199
12.4	Geometric errors	200

12.4.1	Examples of geometric errors	201
12.4.2	Before we continue	204
12.5	Algebraic errors	204
12.5.1	Introductory example	205
12.5.2	Model parameters, data matrix and residual	205
12.5.3	Internal constraints	206
12.5.4	The size of \mathbf{A}	206
12.5.5	General solution strategy for the over-determined case	208
12.5.6	The inhomogeneous method	209
12.5.7	The homogeneous method	212
12.6	Probability based estimation	213
12.6.1	Back to points and a line	213
12.6.2	Maximum a posteriori estimation	216
12.6.3	Maximum likelihood estimation	216
13	Estimation of Transformations	219
13.1	Homography estimation	219
13.1.1	Geometric errors	220
13.1.2	Direct Linear Transformation (DLT)	220
13.1.3	The correspondence problem	223
13.1.4	Minimal case estimation	223
13.1.5	Degeneracies	224
13.2	The homogeneous method revisited	225
13.2.1	SVD profile	225
13.2.2	Data normalization	228
13.2.3	SVD profile, again	235
13.2.4	Closing remarks	237
13.3	Camera matrix estimation	239
14	Non-linear Estimation	241
14.1	Non-linear estimation techniques	241
14.1.1	Initial solutions	242
14.1.2	Parameterizations	242
14.2	Re-parameterization	245
14.2.1	An example	246
14.3	The residual	247
14.3.1	Re-mapping of the residual	247
14.3.2	Choosing residual	248
14.4	An example: estimation of a line	250
14.4.1	Using a homogeneous representation as parameters	250
14.4.2	Minimal parameterizations	252
14.5	An example: estimation of a homography	254
14.6	Practical issues	256
14.6.1	Computing the Jacobian	256
14.6.2	Sparse Jacobian	256
15	Estimation Involving Rotations	259
15.1	Estimation of 3D rotations	260
15.1.1	Using OPP	260
15.1.2	Using SOPP	261
15.1.3	Using quaternions	261
15.1.4	OK, so which algorithm do I choose?	263
15.2	Estimation of rigid transformations	263
15.2.1	Iterative Closest Point (ICP)	264
15.3	Non-linear least squares estimation involving rotations	266

15.3.1	Quaternions	266
15.3.2	Axis-angle representation	267
15.3.3	Euler angles	269
15.4	The Perspective n -Point Problem (PnP)	270
15.4.1	Algebraic estimation	270
15.4.2	Geometric minimization	271
15.4.3	Minimal case estimation ($P3P$)	272
16	Estimation for Two-View Geometry	275
16.1	Triangulation	275
16.1.1	The mid-point method	275
16.1.2	Algebraic methods	277
16.1.3	Optimal triangulation	277
16.1.4	Closing remarks	280
16.2	Estimation of \mathbf{F}	282
16.2.1	The 8-point algorithm	282
16.2.2	The 7-point algorithm	285
16.2.3	Degeneracies	286
16.2.4	Minimization of geometric errors	287
16.3	Estimation of \mathbf{E}	290
16.3.1	Algebraic estimation	290
16.3.2	Minimal case estimation of \mathbf{E}	290
16.4	Estimation of ω from a rotational homography	291
16.4.1	Algebraic estimation	292
17	Robust Estimation	295
17.1	What causes the errors? (Part II)	296
17.1.1	Inliers and outliers	296
17.2	Robust errors	297
17.3	RANSAC	298
17.3.1	An indeterministic approach	299
17.3.2	Robust estimation of a 2D line	300
17.3.3	Probabilities and number of trials	302
17.3.4	Variants of the basic RANSAC algorithm	302
17.4	Revisiting the correspondence problem	303
17.4.1	RANSAC and the correspondence problem	304
17.4.2	Preprocessing of data	305
17.4.3	Closing remarks	305
17.5	Robust estimation in practice	306
17.5.1	Robust estimation of \mathbf{H}	307
17.5.2	Robust estimation of \mathbf{F}	307
17.5.3	Robust estimation of \mathbf{E}	308
17.5.4	Robust PnP	310
III	Applications	313
18	Camera Calibration	315
18.1	Automatic camera calibration	316
18.1.1	Mathematical foundation	316
18.2	Lens distortion	317
18.2.1	Radial lens distortion	319
18.2.2	General distortion models	322
18.2.3	General coordinate systems	323
18.2.4	OK, so which distortion model do I choose?	323

18.2.5	Calibration of lens distortion	324
18.2.6	Compensating for the lens distortion	324
18.3	Zhang’s calibration method	325
18.3.1	Geometric errors	328
18.3.2	Lens distortion	328
18.3.3	Summary	328
19	Image Mosaics	331
19.1	Introduction	331
19.2	Using homographies	333
19.2.1	A mosaic from two images	333
19.2.2	Blending	338
19.2.3	A mosaic from several images	340
19.3	Using spherical coordinates for panoramas	340
19.3.1	A panorama from two images	345
19.4	Further reading	346
20	Rectified Stereo	349
20.1	Rectified stereo cameras	349
20.1.1	Preliminary results	350
20.1.2	Fully rectified stereo	351
20.2	Rectified epipolar geometry	353
20.2.1	Fundamental matrix	354
20.2.2	Triangulation	354
20.3	Synthetic rectification	357
20.3.1	Preliminary results	358
20.3.2	Hartley’s rectification method	359
21	Structure from Motion	365
21.1	Simplifying assumptions	365
21.2	Reconstruction from normalized cameras	367
21.2.1	Accumulative reconstruction from triplets of normalized cameras	368
21.2.2	Analysis	369
21.3	Bundle adjustment	369
21.4	Incremental bundle adjustment	370
21.4.1	Bookkeeping	371
21.4.2	Initialization and first bundle adjustment	372
21.4.3	Adding an image	373
21.5	Practical issues	378
21.5.1	Parameterization of the camera poses	378
21.5.2	Flexible scheme for adding cameras	381
21.5.3	Image coordinate system	381
21.5.4	The residual vector \mathbf{r}	381
21.5.5	Sparsity of \mathbf{J}	382
21.5.6	The Schur complement trick	382
21.5.7	Scale ambiguity	384
21.5.8	The first camera pose	384
21.6	Further reading	385
	Bibliography	387
	Index	391
	List of Algorithms	396

