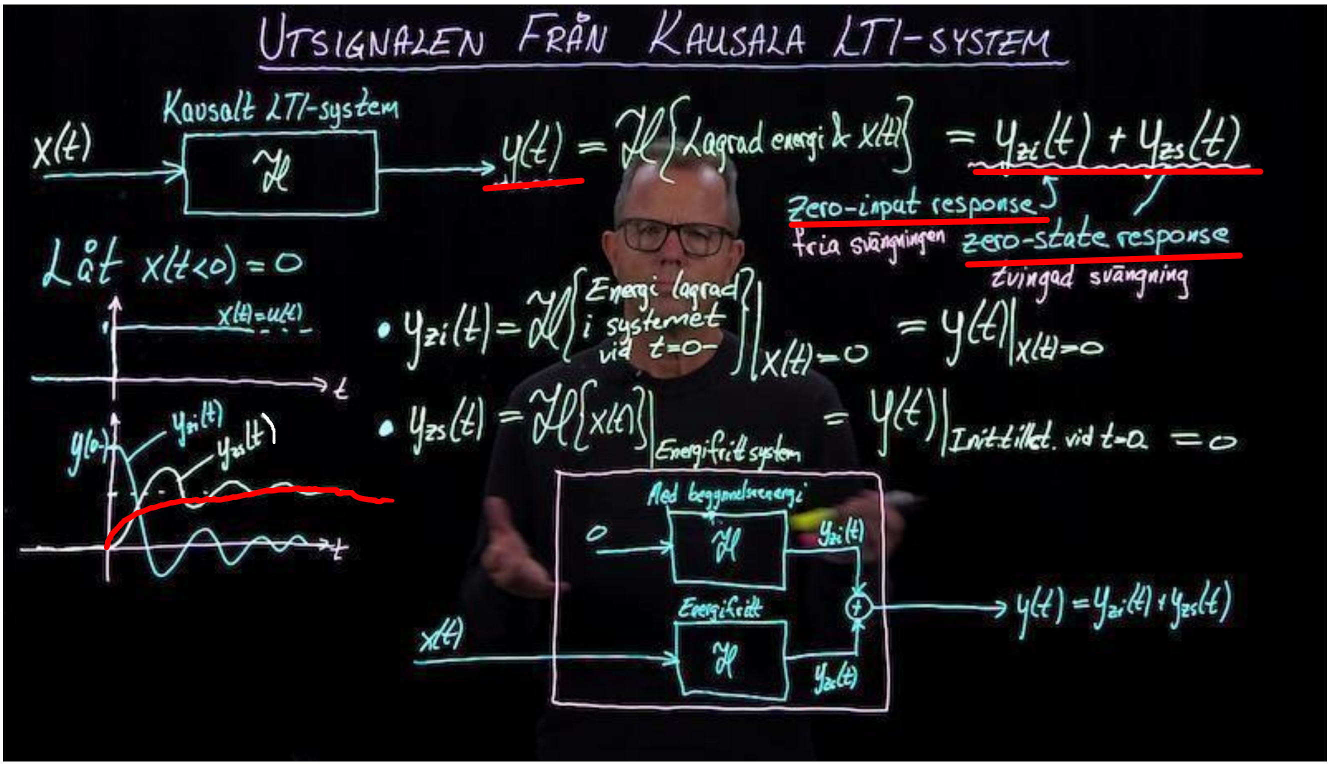
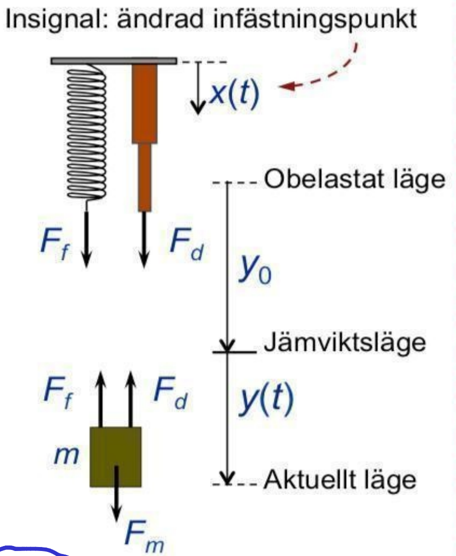
Linjära System – Föreläsning 9: Tidsdomänanalys av LTI-system



VIDEO 1

Systemexempel 1 – Mekaniskt svängningssystem, massa i dämpad fjäder

Svängande dämpad fjäder – frilägg och sätt ut krafter:



Fjäderkraften $F_f = k\cdot y_{tot}(t) = k\cdot(y_0 + y(t) - x(t))$

Dämpkraften $F_d = c\cdot(y_{tot}(t))' = c\cdot(y'(t) - x'(t))$

Tyngdkraften $F_m = m\cdot g$ (g = tyngdaccelerationen)

Newtons 2:a lag: $F_m - F_f - F_d = m\cdot y''(t)$

$\Rightarrow m\cdot y''(t) + c\cdot y'(t) + k\cdot y(t) = m\cdot g - k\cdot y_0 + c\cdot x'(t) + k\cdot x(t)$

Vid vila är $x=0,\ x'=0,\ y=0,\ y'=0,\ y''=0 \ \Rightarrow\ m\cdot g = k\cdot y_0$

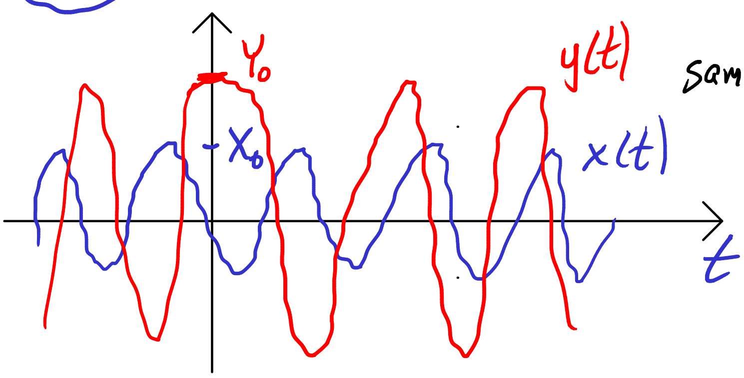
$\Rightarrow$

$$\boxed{m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\cdot y(t) = c\frac{dx(t)}{dt} + k\cdot x(t)}$$

$y_{zi}(t)$

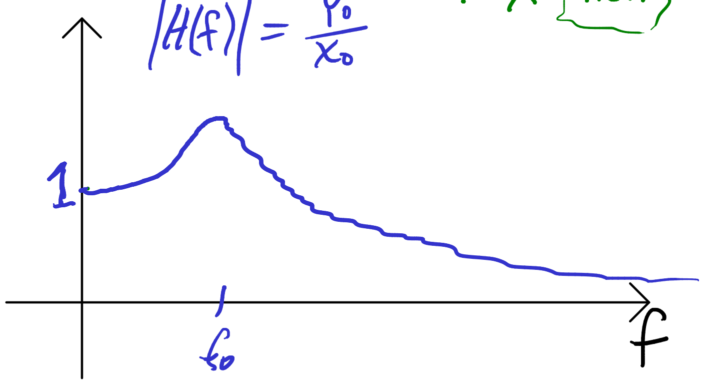
gäller på $\omega = 2\pi f$

$Y = X\cdot H(f)$

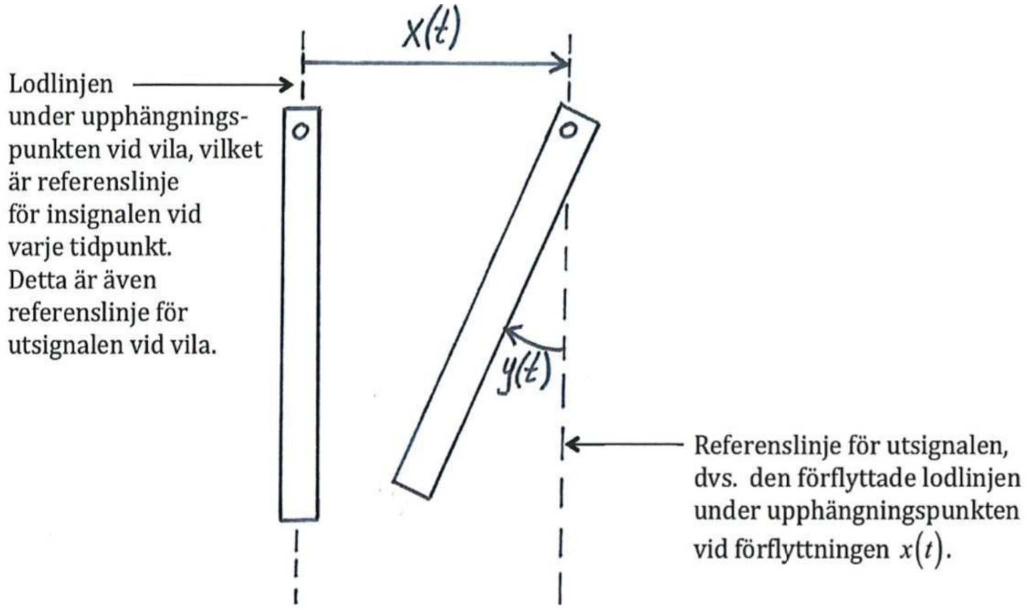


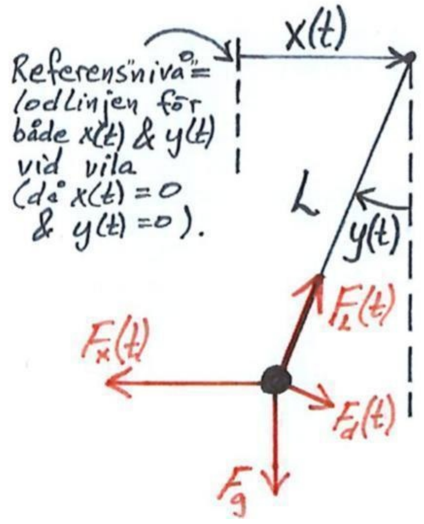
samma f_0

$|H(f)| = \frac{Y_0}{X_0}$



Systemexempel 2 – Mekaniskt svängningssystem, pendlande linjal





Ideal modell

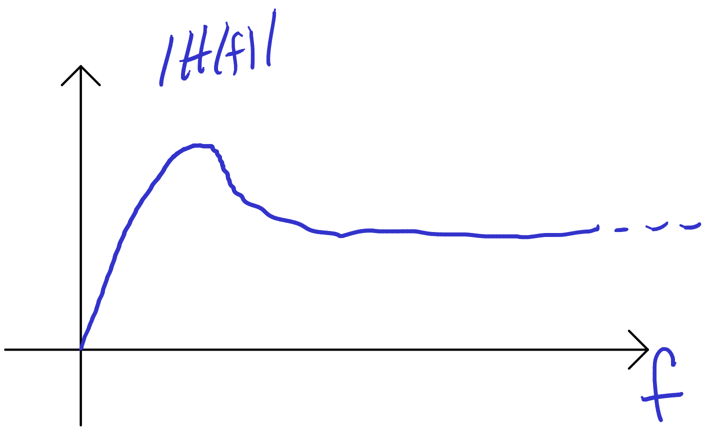
Newtons 2:a lag $\Longrightarrow$ $\quad y'' + \frac{c}{m} y' + \frac{g}{m} \sin(y) = \frac{\cos(y)}{L} \cdot x''$

Icke-linjärt system, p.g.a. sin(y) och cos(y)

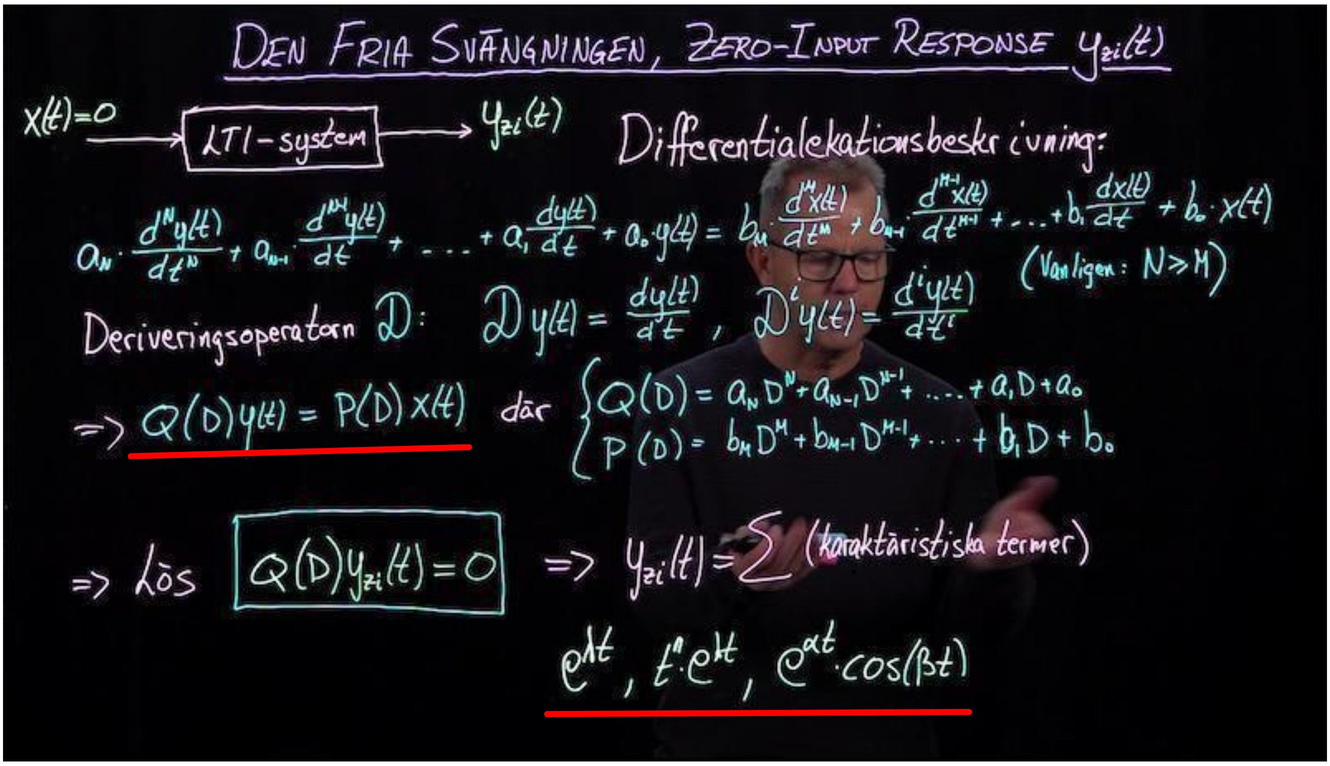
$\Longrightarrow$ Linjärisera, dvs. approximera med linjär modell:

Om vinkeln y är liten $\Longrightarrow$ $\sin(y) \approx y\,,\ \cos(y) \approx 1$

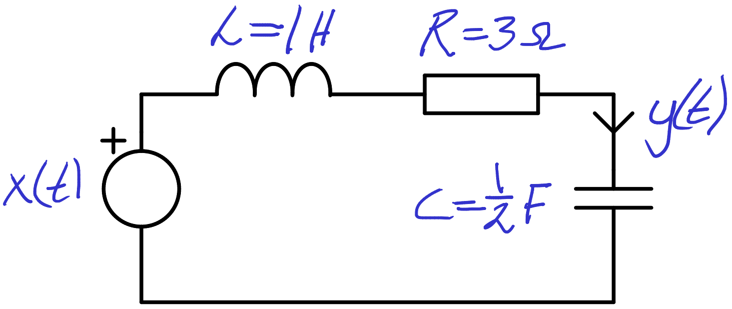
$$\Rightarrow \quad \boxed{\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{g}{L} y(t) = \frac{1}{L}\frac{d^2x(t)}{dt^2}}$$



VIDEO 2



Exempel:



$$(D^2+3D+2)y(t) = Dx(t)$$

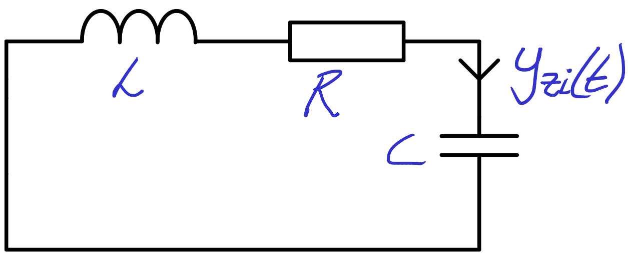
$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$y(0^-) = 0, \quad y'(0^-) = -5$$

I VT2 kommer ni, i En- och flervariabelkursen, att lära er hur man löser (dvs. beräknar y(t)) sådana här differentialekvationer!

Beräkna $y_{zi}(t)$

Ekvivalent krets då x(t)=0 :



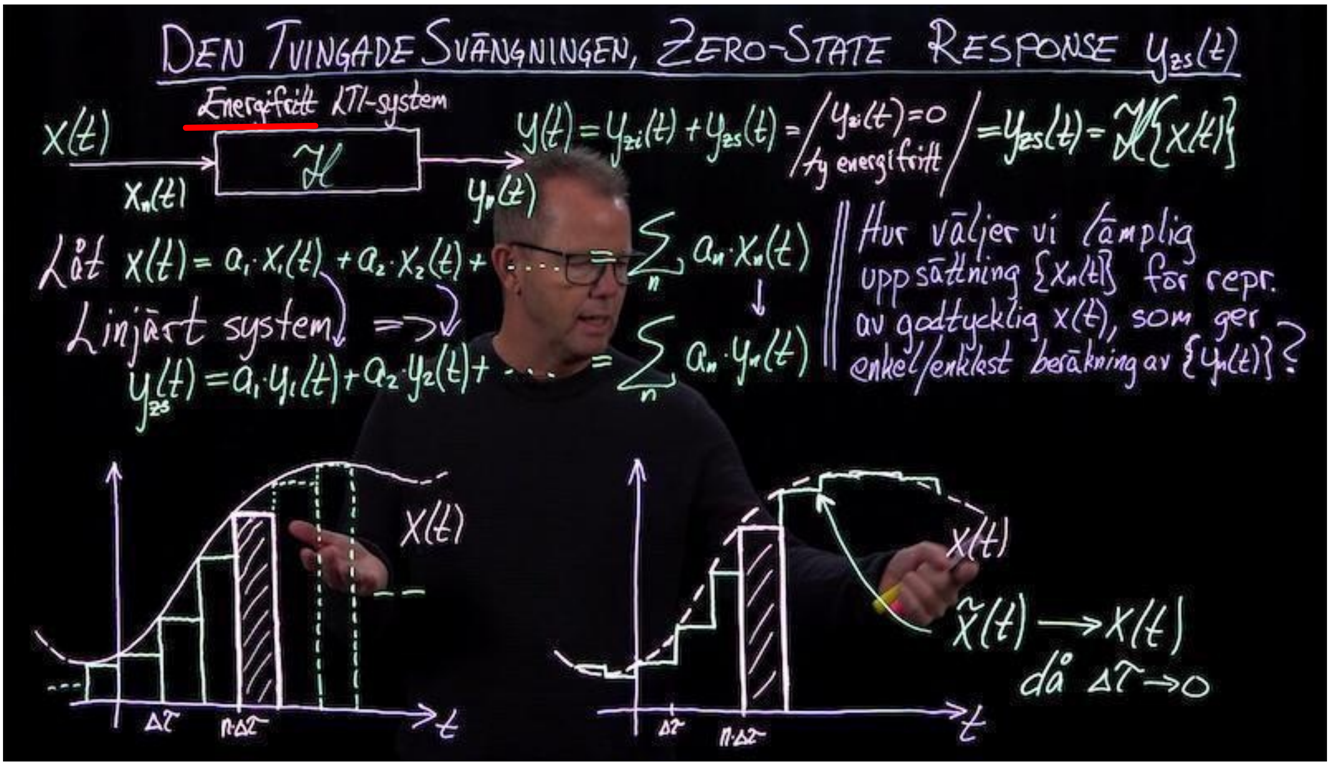
$$(D^2+3D+2)y_{zi}(t) = 0$$

$\lambda^2 + 3\lambda + 2 = 0$ Karaktäristiska ekvationen

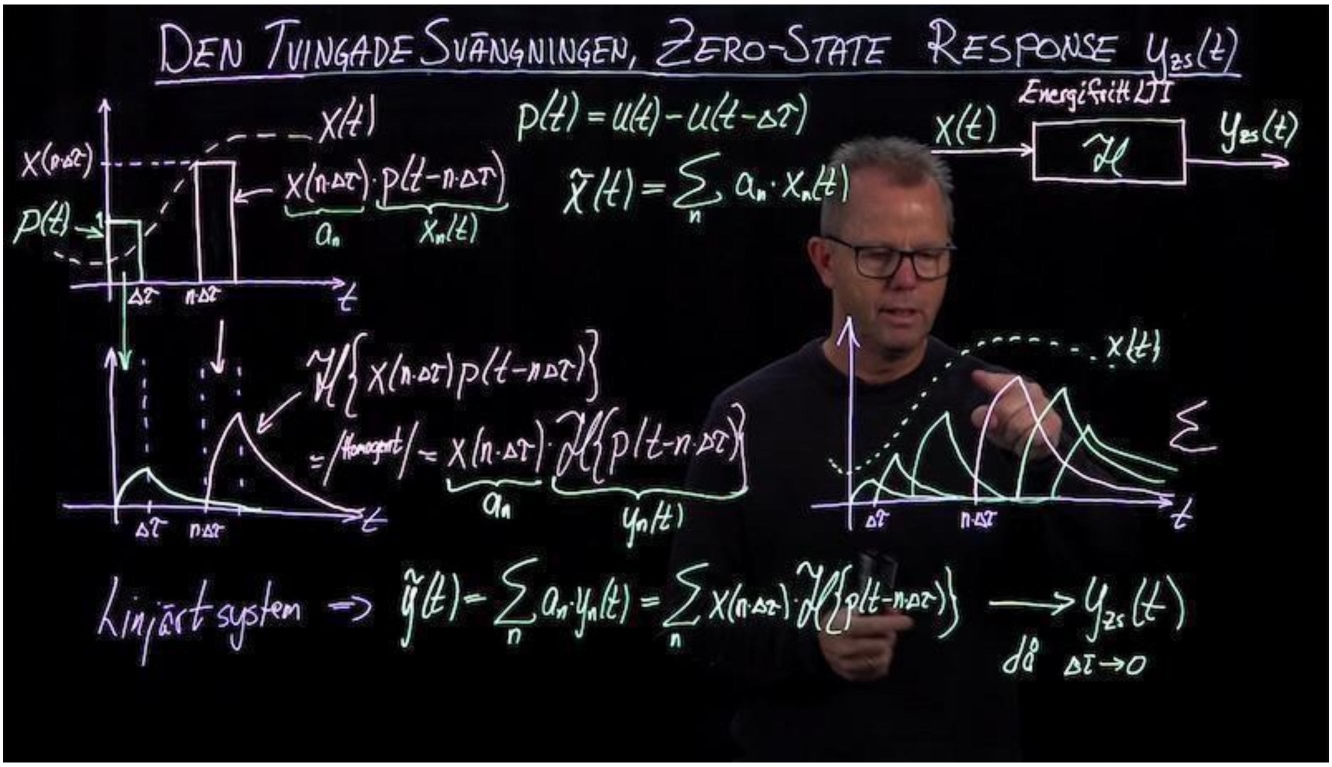
$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = -2 \quad \Rightarrow y_{zi}(t) = K_1 \cdot e^{\lambda_1 t} + K_2 \cdot e^{\lambda_2 t}$

$\boxed{\text{Systemets ordning} = 2}$

$$h(t) = A \cdot e^{\lambda_1 t} + B e^{\lambda_2 t}$$

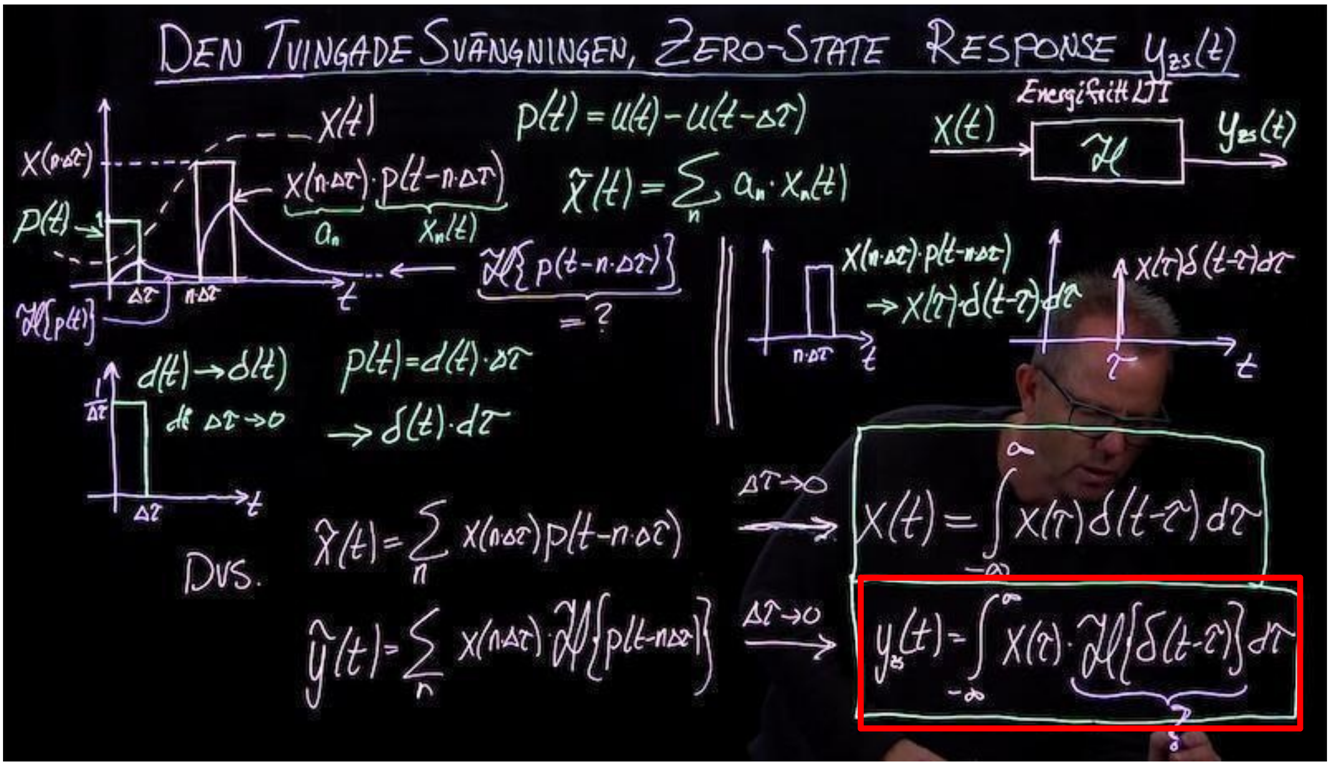


VIDEO 3.1

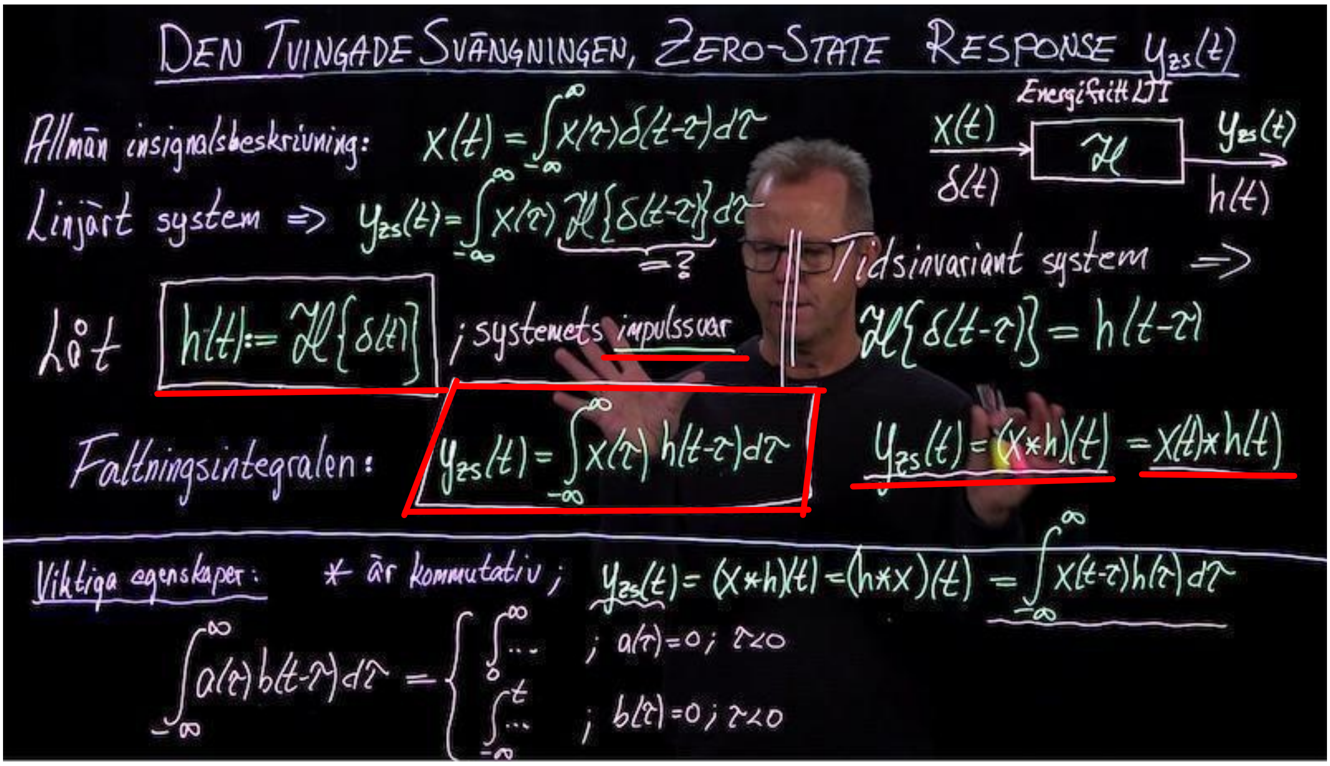


VIDEO 3.2

VIDEO 3.3



VIDEO 3.4



OBS: Kursboken använder λ i stället för τ som integrationsvariabel

VIDEO 4.1

Faltning

Energifritt LTI-system

$$x(t) \longrightarrow \boxed{\mathcal{H}\{\bullet\}} \longrightarrow y(t)$$

$\mathcal{H}$ = systemoperatorn; $y(t) = \mathcal{H}\{x(t)\}$

- Från def. av $\delta(t)$ följer: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$\Rightarrow \underline{y(t)} = \mathcal{H}\{x(t)\} = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\} = /\text{Linjärt}/$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(t-\tau)\}\,d\tau = /\text{Tids-invariant}/ = \underline{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}$$

Faltningsintegralerna konvergerar garanterat om

$\int_{-\infty}^{\infty} |x(t)|dt < \infty$ och $|h(t)| < \infty$ eller $|x(t)| < \infty$ och $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

.

VIDEO 4.2

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = \mathcal{H}\{x(t)\}$$

$x_1(t)$ — $y_1(t)$

$x_2(t)$ — $y_2(t)$

$a_1 x_1(t) + a_2 \cdot x_2(t)$ /Linjärt/ $a_1 \cdot y_1(t) + a_2 \cdot y_2(t)$

$\delta(t)$ — $h(t)$

$\delta(t-\tau)$ /Tidsinvariant/ $h(t-\tau)$

$x(\tau)\cdot\delta(t-\tau)$ — $x(\tau)h(t-\tau)$

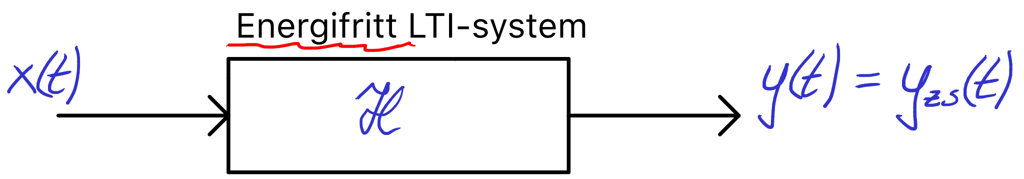
$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ /LTI/ $\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (x * h)(t)}$

Slutsats efter video 3 & 4 ovan:

$y(t) = y_{zi}(t) + y_{zs}(t)$

Utsignalens zero-statekomponent (den tvingade svängningen):

$(y_{zi}(t) = 0)$



där

$$\boxed{y_{zs}(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(\underbrace{t-\tau}_{\lambda})\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau}$$

$t-\lambda \qquad \lambda \qquad d\lambda$

Eller $x(t) * h(t)$

Faltningsintegralen/-erna

där $h(t) = \mathcal{H}\{\delta(t)\}$ är LTI-systemets impulssvar

---

Räkneexempel – faltning

Ett visst energifritt LTI-system har impulssvaret $h(t) = 6e^{-2t}u(t)$.

a) Beräkna systemets utsignal $y(t)$ då dess insignal är

$x(t) = 4\big(u(t) - u(t-3)\big)$

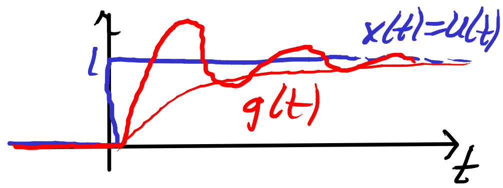
$= y_{zi}(t) + y_{zs}(t)$, $y_{zi}(t) = 0$

b) Beräkna systemets stegsvar $g(t)$

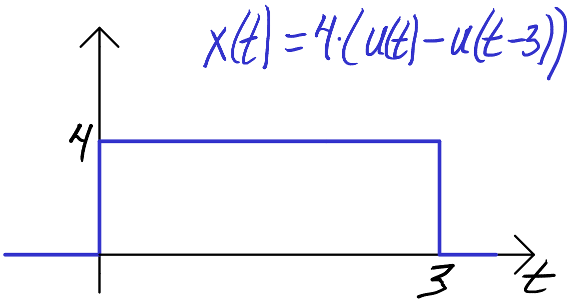
$= y(t)$ då $x(t) = u(t)$

c) Bestäm systemets kausalitetsegenskap

d) Bestäm systemets stabilitetsegenskap

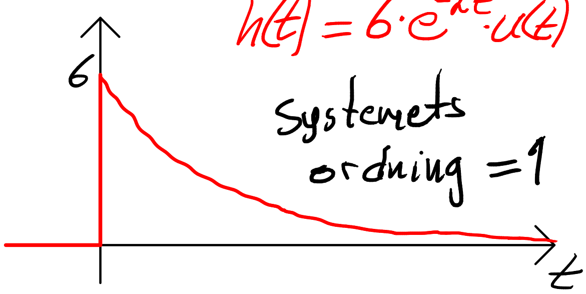


$x(t) = 4\cdot(u(t) - u(t-3))$



$h(t) = 6\cdot e^{-2t}\cdot u(t)$

Systemets ordning $= 1$



a) Beräkna systemets utsignal y(t)

$y(t) = y_{zs}(t)$ ty energifritt system

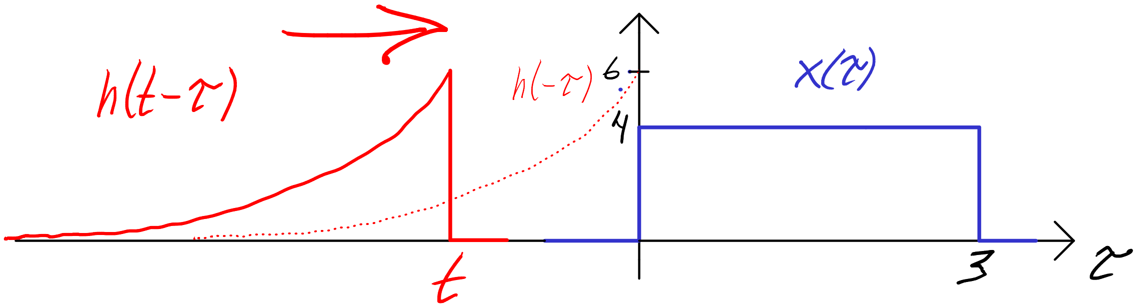
$h(t-\tau)$? ① $h(-\tau)$

② $h(t-\tau) = h(-\tau + t)$

$$\boxed{y_{zs}(t) = (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

"Grafisk faltning":

$t < 0$

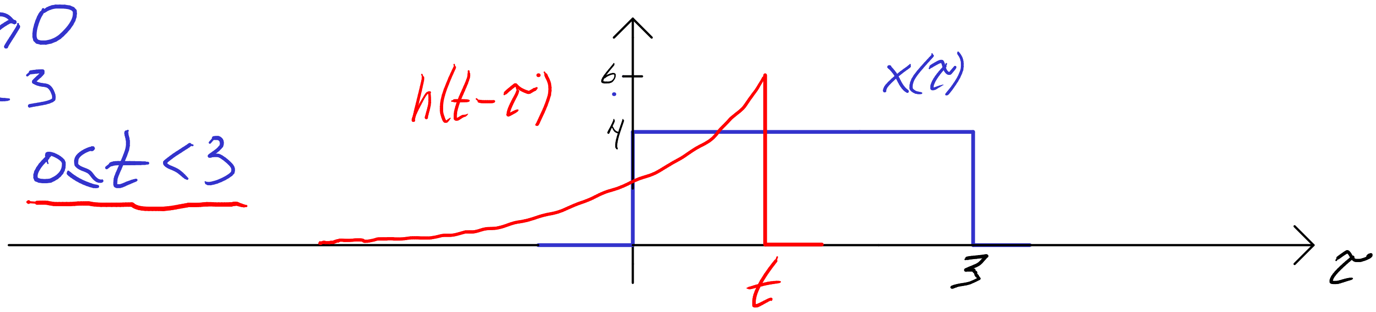


$\Rightarrow x(\tau)h(t-\tau) = 0 \quad \Rightarrow y_{zs}(t) = \int \dots = 0$

$t \geq 0$

$t < 3$

dvs. $0 \leq t < 3$



överlapp, både x & $h \neq 0$

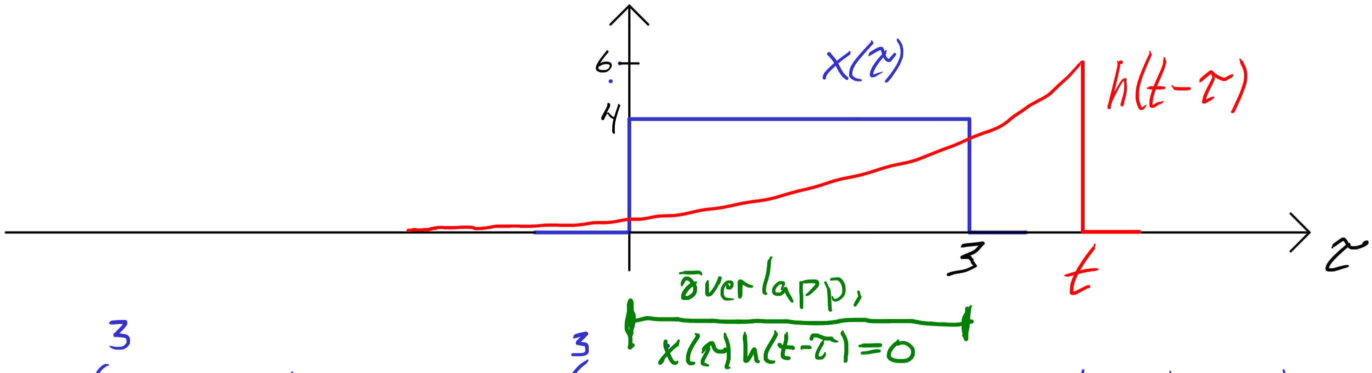
$$y_{zs}(t) = \underbrace{\int_{-\infty}^{0} 0 \cdot h(t-\tau)\,d\tau}_{=0} + \int_{0}^{t} 4 \cdot 6 \cdot e^{-2(t-\tau)}\,d\tau + \underbrace{\int_{t}^{\infty} x(\tau) \cdot 0\,d\tau}_{=0}$$

$$= 24e^{-2t}\int_{0}^{t} e^{2\tau}\,d\tau = \dots = 12(1-e^{-2t})$$

Lasses demo-material under föreläsningen – ingår ej i studentmaterialet:

$h(t-\tau)$

$\underline{t \geq 3}$

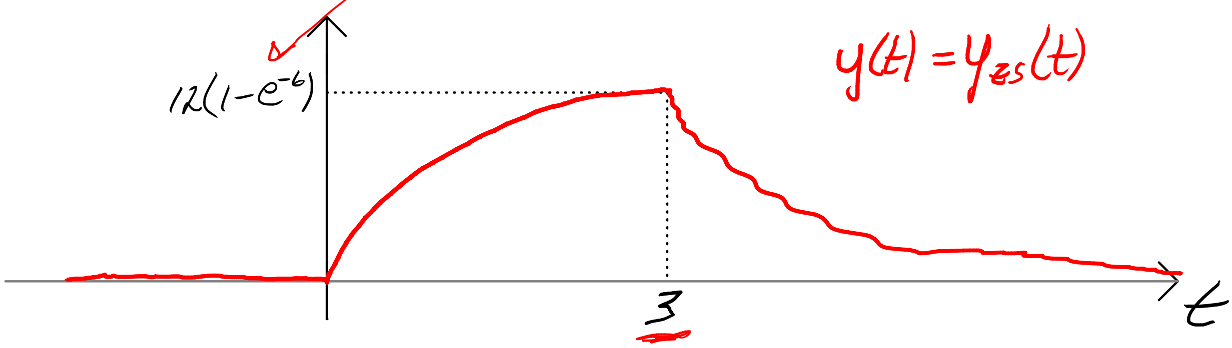


$$\underline{y_{zs}(t)} = \int_0^3 4 \cdot 6 e^{-2(t-\tau)} d\tau = 24 e^{-2t} \int_0^3 e^{2\tau} d\tau = \ldots = \underline{12(e^6-1)e^{-2t}}$$

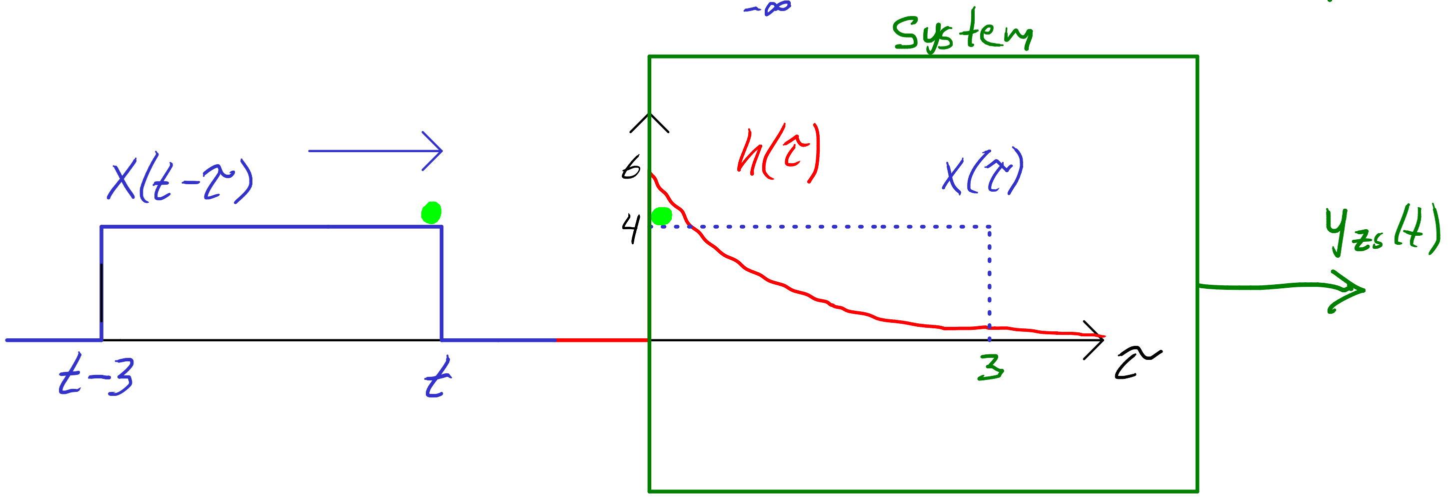
$$\therefore \; \underline{y_{zs}(t)} = \begin{cases} 0; & t < 0 \\ 12(1-e^{-2t}); & 0 \leq t < 3 \\ 12(e^6-1)e^{-2t}; & t \geq 3 \end{cases}$$

$\leftarrow t=3: 12(1-e^{-2\cdot 3})$

$12(e^6-1)e^{-6}$



Testa själv att beräkna $y_{zs}(t)=\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$!



b) Beräkna systemets stegsvar g(t)

$$\underline{g(t)} = \mathcal{H}\{u(t)\} = (u*h)(t) = \int_{\underline{0}\;-\infty}^{\infty\;\underline{t}} \underbrace{u(t-\tau)}\,\underbrace{h(\tau)}\,d\tau$$

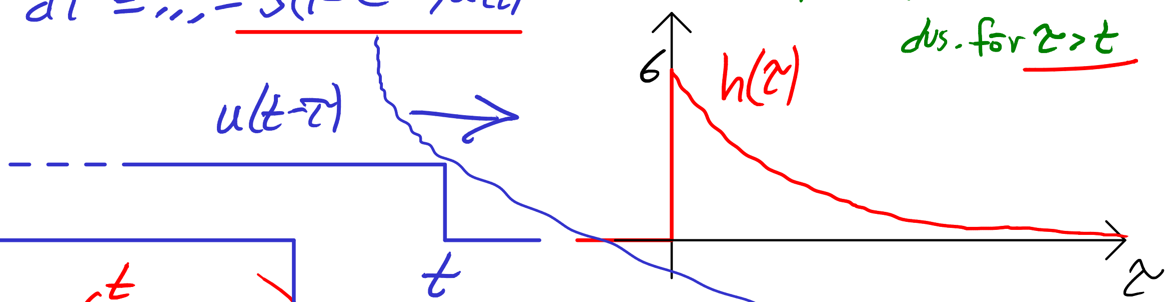
$h(\tau)=0$ för $\underline{\tau<0}$

$u(t)=0$ för $t<0$

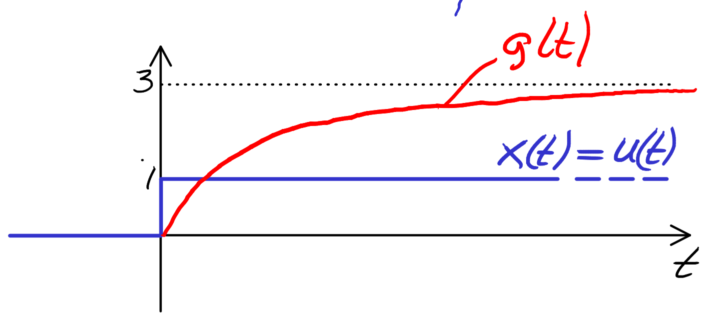
$\Rightarrow u(t-\tau)=0$ för $t-\tau<0$

dvs. för $\underline{\tau>t}$

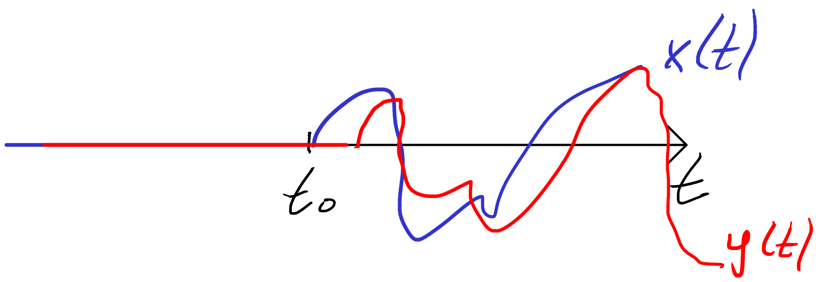
$$= \int_0^t 1\cdot h(\tau)\,d\tau = \ldots = \underline{3(1-e^{-2t})u(t)}$$

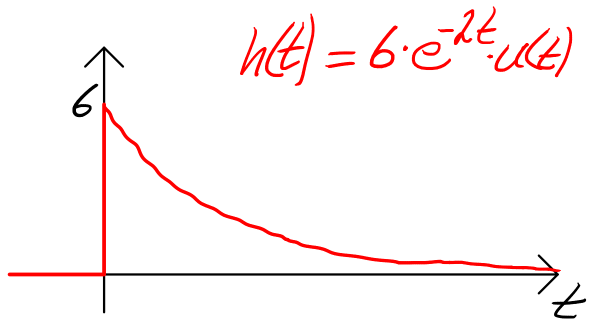


(Allmänt: $\boxed{g(t)=\int_{-\infty}^{t} h(\tau)\,d\tau}$) $\Rightarrow$ $\boxed{h(t)=\dfrac{dg(t)}{dt}}$



c) Bestäm systemets kausalitetsegenskap





Om $x(t)=0$ för $t<t_0$

$\Rightarrow$ $y(t)=0$ för $t<t_0$

Kausalt system

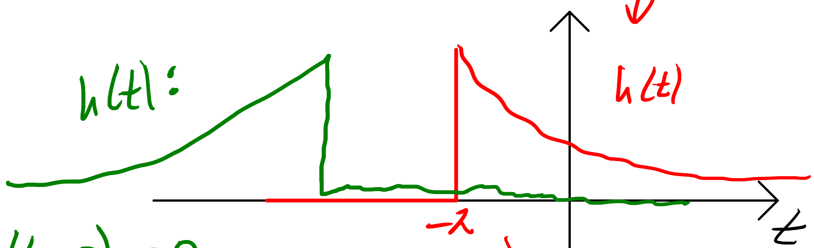
Ex: $x(t)=\delta(t)$; $\delta(t)=0$ för $t<0$

Kausalt system $\Rightarrow$ $y(t)=h(t)=0$ för $t<0$

Icke-kausalt system

Har: $h(t<0)=0$ $\Rightarrow$

Kausalt system



$h(t>0)=0$

$\Rightarrow$ Anti-kausalt system

$h(t<0) \neq 0$

$$y(\underset{0}{t}) = \int_{-\infty}^{\infty} x(\tau)h(\underset{0}{t}-\tau)\,d\tau$$

$$y(0) = \int_{-\infty}^{0} x(\tau)h(0-\tau)\,d\tau + \underbrace{\int_{0+}^{\infty} x(\tau)h(0-\tau)\,d\tau}_{=0 \text{ för kausala system}}$$

$h(-\tau)=0$ för $\tau>0$

$\Rightarrow h(\underset{t}{\tau})=0$ för $\underset{t}{\tau}<0$

---

d) Bestäm systemets stabilitetsegenskap

Stabilitet – för **LTI-system**

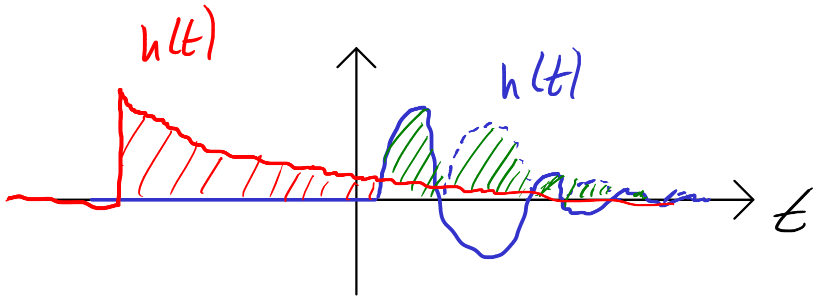
Stabilt system:

Varje begränsad insignal ger upphov till en begränsad utsignal. $\Leftrightarrow$ $\boxed{\int_{-\infty}^{\infty} |h(t)|\,dt < \infty}$

$$|y_{zs}(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau\right| \le \int_{-\infty}^{\infty} \underbrace{|x(t-\tau)|}_{\le M<\infty}\,|h(\tau)|\,d\tau \le M\int_{-\infty}^{\infty} |h(\tau)|\,d\tau \le N < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Exempel på impulssvar h(t) för stabilt system:

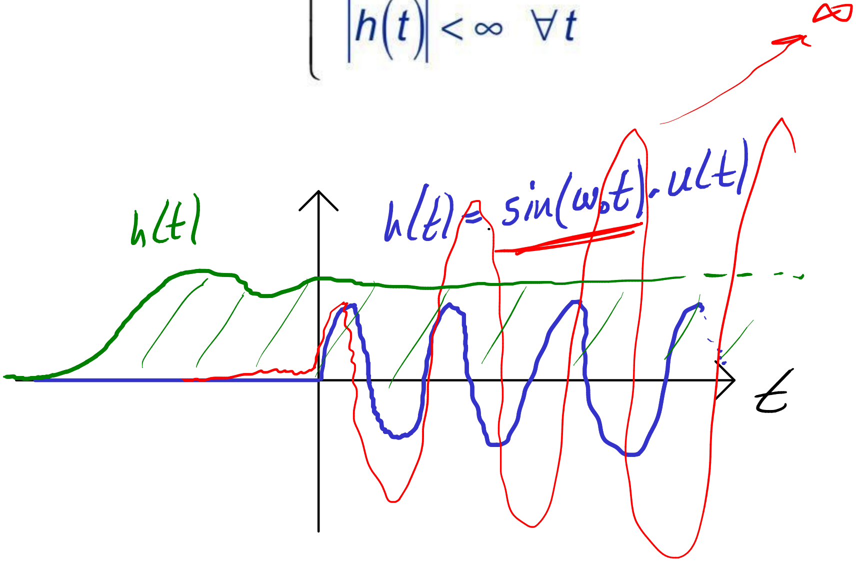


Marginellt stabilt system:

De flesta begränsade insignaler ger upphov till begränsade utsignaler.

$$\Leftrightarrow \begin{cases} \int_{-\infty}^{\infty} |h(t)|\,dt \nless \infty \\ |h(t)| < \infty \quad \forall t \end{cases}$$

Exempel på impulssvar h(t) för marginellt stabilt system:

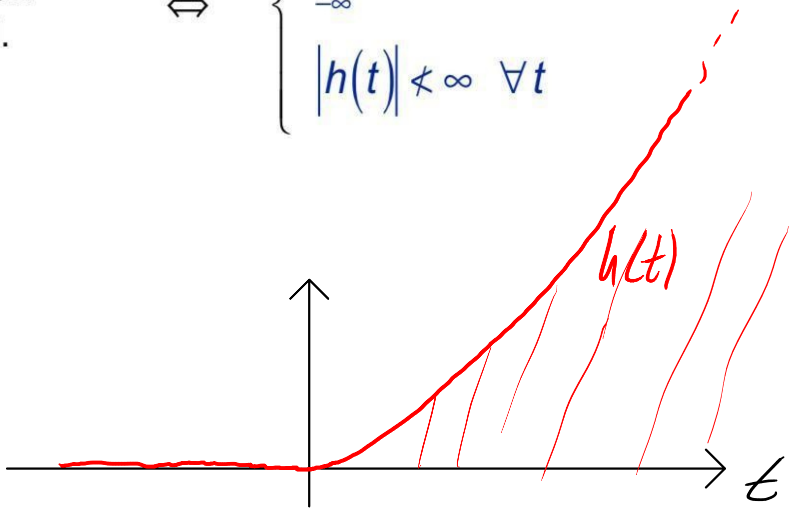


Instabilt system:

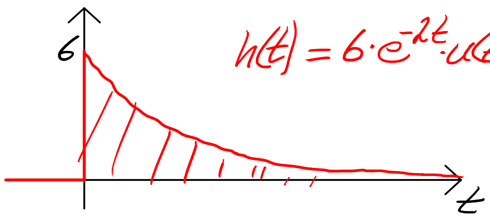
Ingen begränsad nollskild insignal kan ge upphov till en begränsad utsignal.

$$\Leftrightarrow \begin{cases} \int_{-\infty}^{\infty} |h(t)|\,dt \nless \infty \\ |h(t)| \nless \infty \quad \forall t \end{cases}$$

Exempel på impulssvar h(t) för instabilt system:



d) Stabilitetsegenskap för LTI-systemet i räkneuppgiften?



$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} 6 \cdot e^{-2t}\,dt$$

$$= 6\left[\frac{e^{-2t}}{-2}\right]_0^{\infty} = -3\left(\underbrace{e^{-\infty}}_{=0} - \underbrace{e^{0}}_{=1}\right) = 3$$

$\Rightarrow$ Stabilt LTI-system