

UTSIGNALEN FRÅN KAUSALA LTI-SYSTEM

Kausalt LTI-system

$$x(t) \rightarrow \mathcal{H} \rightarrow y(t) = \mathcal{H}\{\text{lagrad energi \& } x(t)\} = \underbrace{y_{zi}(t)}_{\text{Zero-input response}} + \underbrace{y_{zs}(t)}_{\text{zero-state response}}$$

Låt $x(t < 0) = 0$

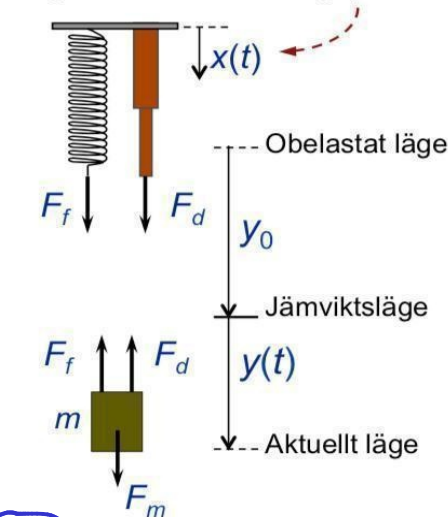
$y_{zi}(t) = \mathcal{H}\left\{ \begin{array}{l} \text{Energilagrard} \\ \text{i systemet} \\ \text{vid } t=0^- \end{array} \right\} x(t)=0 = y(t)|_{x(t)=0}$
 (fria svängningen)

$y_{zs}(t) = \mathcal{H}\{x(t)\}$ Energi-fritt system = $y(t)|_{\text{Inittillst. vid } t=0} = 0$
 (tvingad svängning)

Systemexempel 1 – Mekaniskt svängningssystem, massa i dämpad fjäder

Svängande dämpad fjäder – frilägg och sätt ut krafter:

Insignal: ändrad infästningspunkt



$$\begin{aligned} \text{Fjäderkraften } F_f &= k \cdot y_{\text{tot}}(t) = k \cdot (y_0 + y(t) - x(t)) \\ \text{Dämpkraften } F_d &= c \cdot (y_{\text{tot}}(t))' = c \cdot (y'(t) - x'(t)) \\ \text{Tyngdkraften } F_m &= m \cdot g \quad (g = \text{tyngdaccelerationen}) \end{aligned}$$

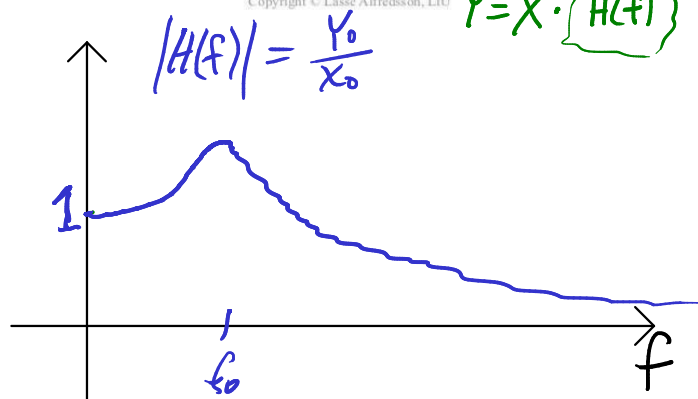
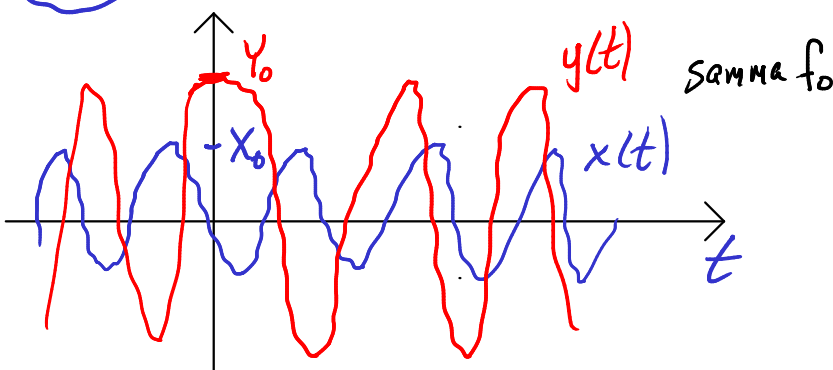
Newtons 2:a lag: $F_m - F_f - F_d = m \cdot y''(t)$

$$\Rightarrow m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = m \cdot g - k \cdot y_0 + c \cdot x'(t) + k \cdot x(t)$$

Vid vila är $x=0, x'=0, y=0, y'=0, y''=0 \Rightarrow m \cdot g = k \cdot y_0$

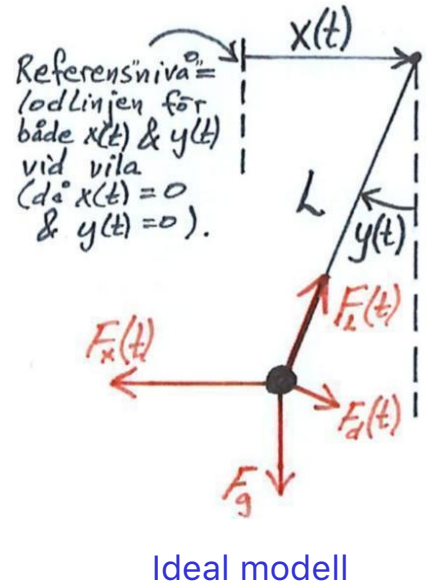
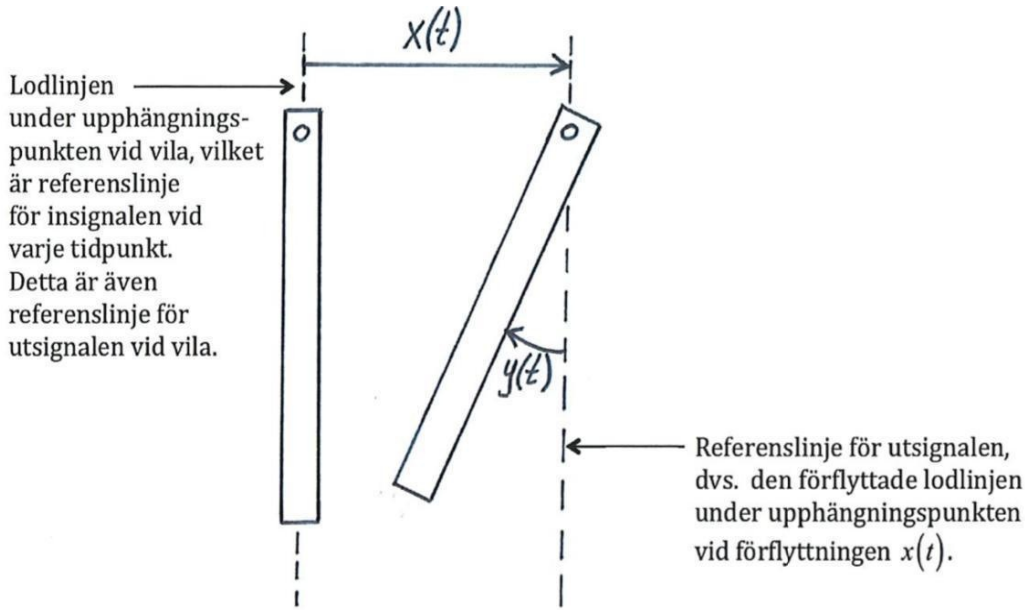
$$m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + k \cdot y(t) = c \frac{dx(t)}{dt} + k \cdot x(t)$$

$y_{zi}(t)$



heror p₀
ω = 2πf
Y = X · H(f)

Systemexempel 2 – Mekaniskt svängningssystem, pendlande linjal



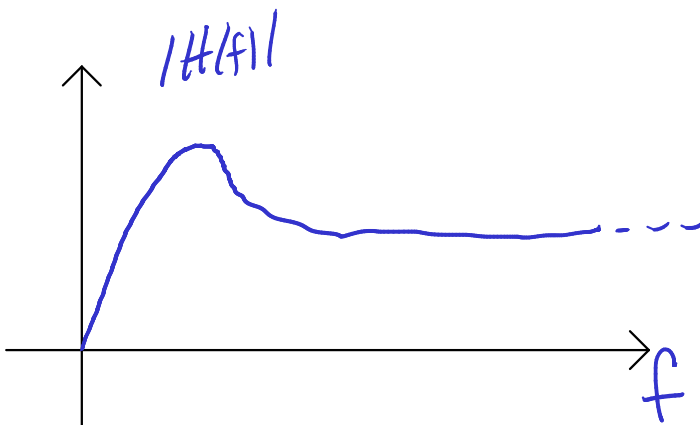
Newtons 2:a lag $\Rightarrow y'' + \frac{c}{m}y' + \frac{g}{l}\sin(y) = \frac{\cos(y)}{l} \cdot x''$

Icke-linjärt system, p.g.a. $\sin(y)$ och $\cos(y)$

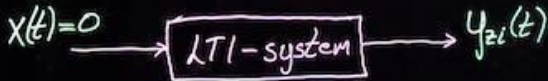
\Rightarrow Linjärisera, dvs. approximera med linjär modell:

Om vinkeln y är liten $\Rightarrow \sin(y) \approx y, \cos(y) \approx 1$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + \frac{c}{m} \frac{dy(t)}{dt} + \frac{g}{l} y(t) = \frac{1}{l} \frac{d^2x(t)}{dt^2}$$



DEN FRIA SVÄNGNINGEN, ZERO-INPUT RESPONSE $y_{zi}(t)$



Differentialekvationsbeskrivning:

$$a_n \cdot \frac{d^n y(t)}{dt^n} + a_{n-1} \cdot \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \cdot \frac{dy(t)}{dt} + a_0 \cdot y(t) = b_m \cdot \frac{d^m x(t)}{dt^m} + b_{m-1} \cdot \frac{d^{m-1} x(t)}{dt^{m-1}} + \dots + b_1 \cdot \frac{dx(t)}{dt} + b_0 \cdot x(t)$$

(Vanligen: $N > M$)

Deriveringsoperatören D : $Dy(t) = \frac{dy(t)}{dt}$, $D^i y(t) = \frac{d^i y(t)}{dt^i}$

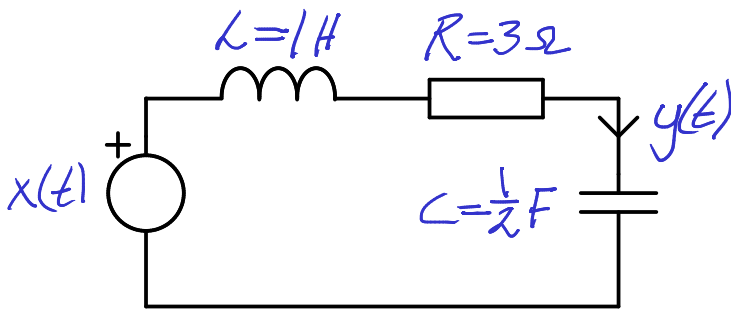
$$\Rightarrow \underline{Q(D)y(t) = P(D)x(t)}$$

där $\begin{cases} Q(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 \\ P(D) = b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0 \end{cases}$

$$\Rightarrow \text{lös } \boxed{Q(D)y_{zi}(t) = 0} \Rightarrow y_{zi}(t) = \sum (\text{karaktäristiska termer})$$

$e^{\lambda t}, t^n e^{\lambda t}, e^{\alpha t} \cdot \cos(\beta t)$

Exempel:



$$(D^2 + 3D + 2)y(t) = Dx(t)$$

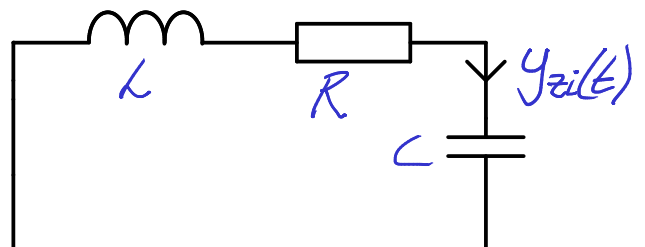
$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$y(0) = 0, y'(0) = -5$$

I VT2 kommer ni, i En- och flervariabelkursen, att lära er hur man löser (dvs. beräknar $y(t)$) sådana här differentialekvationer!

Beräkna $y_{zi}(t)$

Ekvivalent krets då $x(t) = 0$:



$$(D^2 + 3D + 2)y_{zi}(t) = 0$$

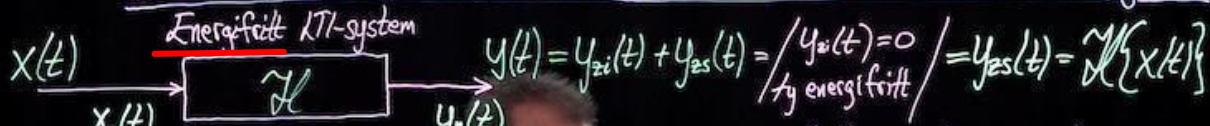
$$\lambda^2 + 3\lambda + 2 = 0 \quad \text{Karaktäristiska ekvationen}$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = -2 \Rightarrow \underline{y_{zi}(t) = K_1 \cdot e^{\lambda_1 t} + K_2 \cdot e^{\lambda_2 t}}$$

Systemets ordning = 2

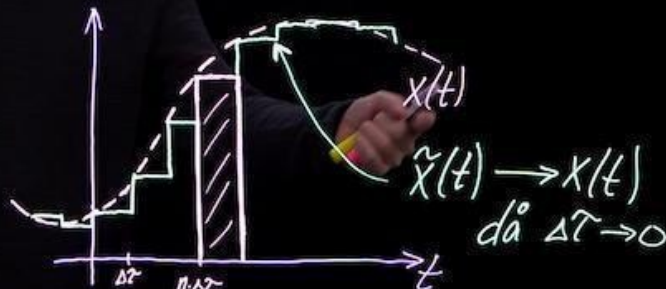
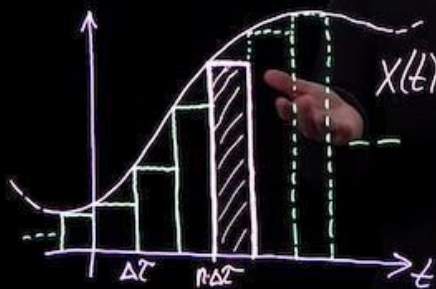
$$h(t) = A \cdot e^{\lambda_1 t} + B e^{\lambda_2 t}$$

DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$

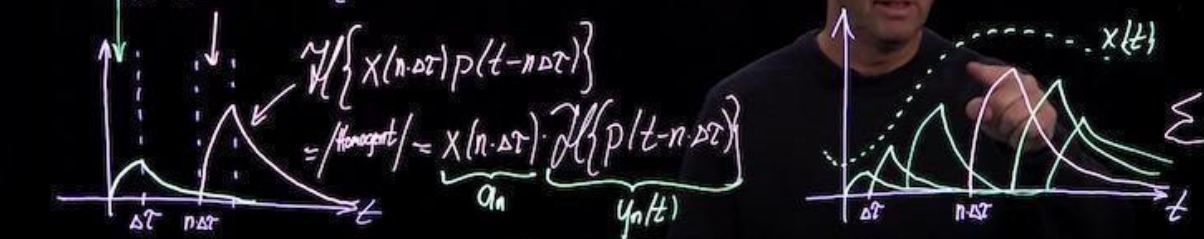
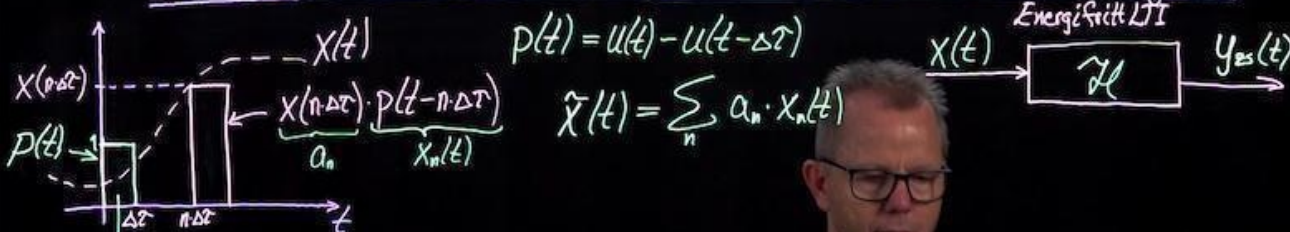


Låt $x(t) = a_1 \cdot x_1(t) + a_2 \cdot x_2(t) + \dots = \sum_n a_n \cdot x_n(t)$
 Linjärt system $\Rightarrow y(t) = a_1 \cdot y_1(t) + a_2 \cdot y_2(t) + \dots = \sum_n a_n \cdot y_n(t)$

Hur väljer vi lämplig uppsättning $\{x_n(t)\}$ för repr. av godtycklig $x(t)$, som ger enkel/enklast beräkning av $\{y_n(t)\}$?



DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$



Linjärt system $\Rightarrow \tilde{y}(t) = \sum_n a_n \cdot y_n(t) = \sum_n x(n\Delta\tau) \cdot \mathcal{H}\{p(t - n\Delta\tau)\} \rightarrow y_{zs}(t)$
 då $\Delta\tau \rightarrow 0$

DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$

$p(t) = u(t) - u(t - \Delta\tau)$
 $\hat{x}(t) = \sum_n a_n \cdot x_n(t)$
 $\hat{y}(t) = \sum_n x(n\Delta\tau) \mathcal{L}\{p(t - n\Delta\tau)\}$

$x(t) \xrightarrow{\text{Energisätt DTI}} \mathcal{L} \rightarrow y_{zs}(t)$

$d(t) \rightarrow \delta(t)$
 $de \Delta\tau \rightarrow 0 \rightarrow \delta(t) \cdot d\tau$

$\hat{x}(t) = \sum_n x(n\Delta\tau) p(t - n\Delta\tau)$
 $\hat{y}(t) = \sum_n x(n\Delta\tau) \mathcal{L}\{p(t - n\Delta\tau)\}$

$\Delta\tau \rightarrow 0$
 $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$
 $y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) \mathcal{L}\{\delta(t - \tau)\} d\tau$

DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$

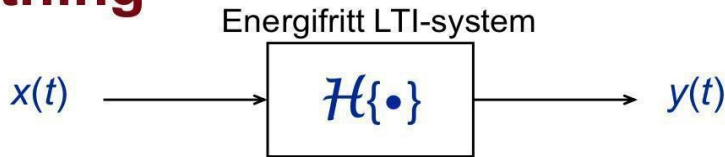
Allmän signalsbeskrivning: $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$
 Linjärt system $\Rightarrow y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) \mathcal{L}\{\delta(t - \tau)\} d\tau$
 Tidsinvariant system $\Rightarrow \mathcal{L}\{\delta(t - \tau)\} = h(t - \tau)$
 Låt $h(t) := \mathcal{L}\{\delta(t)\}$; systemets impulssvar
 Faltungsintegralen: $y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$
 $y_{zs}(t) = (x * h)(t) = x(t) * h(t)$

Viktiga egenskaper: * är kommutativ; $y_{zs}(t) = (x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau$
 $\int_{-\infty}^{\infty} a(\tau) b(t - \tau) d\tau = \begin{cases} \int_{\dots}^{\dots} & ; a(\tau) = 0; \tau > 0 \\ \int_{\dots}^{\dots} & ; b(\tau) = 0; \tau > 0 \end{cases}$

OBS: Kursboken använder λ i stället för τ som integrationsvariabel

Faltning

VIDEO 4.1



\mathcal{H} = systemoperatorn; $y(t) = \mathcal{H}\{x(t)\}$

♦ Från def. av $\delta(t)$ följer: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

\Rightarrow $y(t) = \mathcal{H}\{x(t)\} = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$ =/Linjärt/

$= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(t-\tau)\}d\tau$ =/Tidsinvariant/ $= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Copyright © Lasse Alfredsson, LiTH

Faltningintegralerna konvergerar garanterat om

$\int_{-\infty}^{\infty} |x(t)|dt < \infty$ och $|h(t)| < \infty$ eller $|x(t)| < \infty$ och $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

TS&S06

VIDEO 4.2

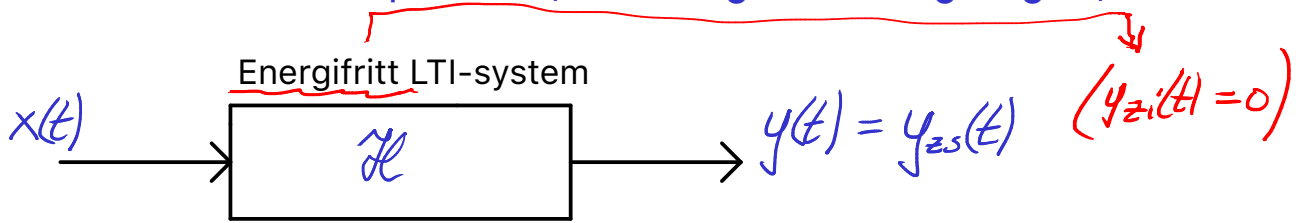
$x(t) \rightarrow$ **LTI** $\rightarrow y(t) = \mathcal{H}\{x(t)\}$
 $x_1(t) \rightarrow y_1(t)$
 $x_2(t) \rightarrow y_2(t)$
 $a_1 x_1(t) + a_2 x_2(t)$ /Linjärt/ $a_1 y_1(t) + a_2 y_2(t)$
 $\delta(t) \rightarrow h(t)$
 $\delta(t-\tau)$ /Tidsinvariant/ $h(t-\tau)$
 $x(\tau) \cdot \delta(t-\tau) \rightarrow x(\tau)h(t-\tau)$
 $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ /LTI/ $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (x * h)(t)$

Copyright © Lasse Alfredsson, LiTH

Slutsats efter video 3 & 4 ovan:

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

Utsignalens zero-statekomponent (den tvingade svängningen):



där

$$y_{zs}(t) = (x * h)(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{t-\tau} \underbrace{h(t-\tau)}_{\tau} d\tau = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

Eller $x(t) * h(t)$

Faltningintegralen/-erna

där $h(t) = \mathcal{L}\{\delta(t)\}$ är LTI-systemets impulssvar

Räkneexempel – faltning

Ett visst energifritt LTI-system har impulssvaret $h(t) = 6e^{-2t}u(t)$.

a) Beräkna systemets utsignal $y(t)$ då dess insignal är

$$x(t) = 4(u(t) - u(t-3))$$

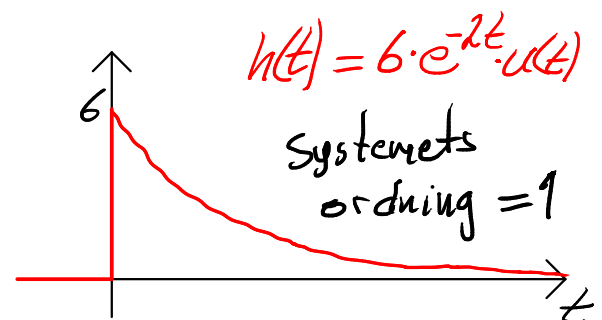
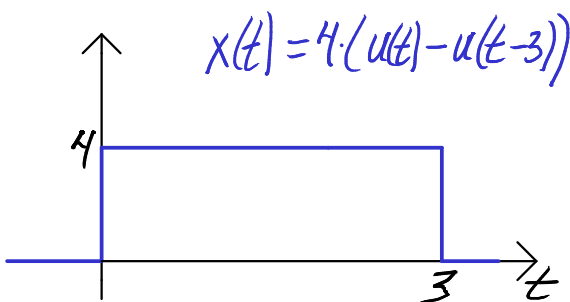
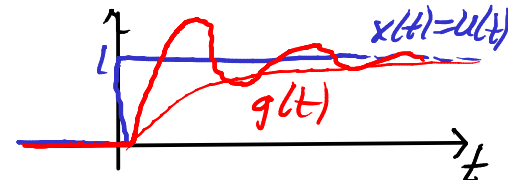
$$y(t) = y_{zi}(t) + y_{zs}(t)$$

b) Beräkna systemets stegsvar $g(t)$

$$= y(t) \text{ då } x(t) = u(t)$$

c) Bestäm systemets kausalitetsegenskap

d) Bestäm systemets stabilitetsegenskap



a) Beräkna systemets utsignal $y(t)$

$$y(t) = y_{zs}(t) \quad \text{ty energifritt system}$$

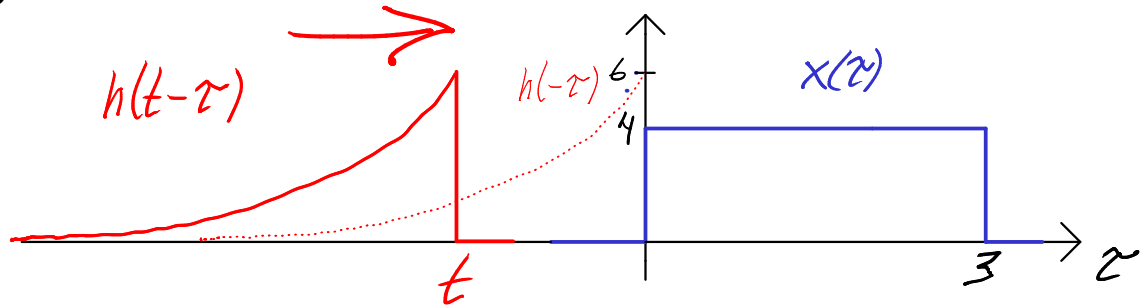
$$h(t-\tau) \stackrel{?}{=} \textcircled{1} \underline{h(-\tau)}$$

$$\textcircled{2} h(t-\tau) = h(-\tau + t)$$

$$y_{zs}(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) \underline{h(t-\tau)} d\tau$$

"Grafisk faltning":

$$\underline{t < 0}$$

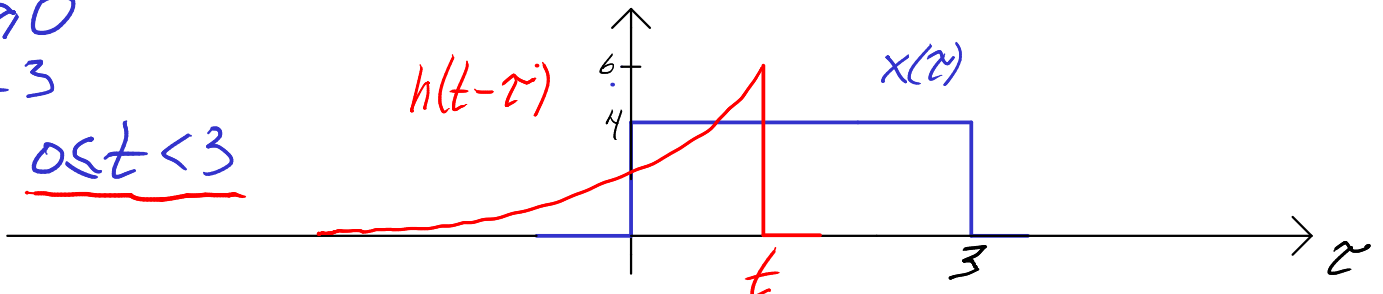


$$\Rightarrow x(\tau)h(t-\tau) = 0 \quad \Rightarrow y_{zs}(t) = \int \dots = 0$$

$$t > 0$$

$$t < 3$$

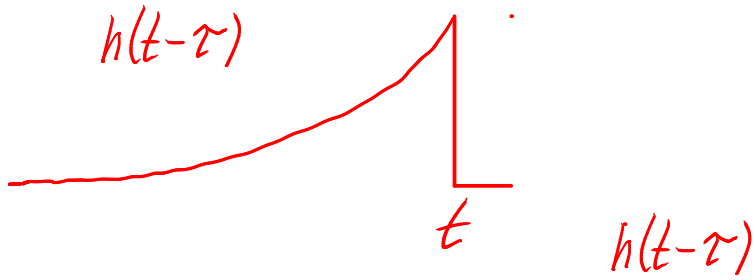
dvs. $0 \leq t < 3$



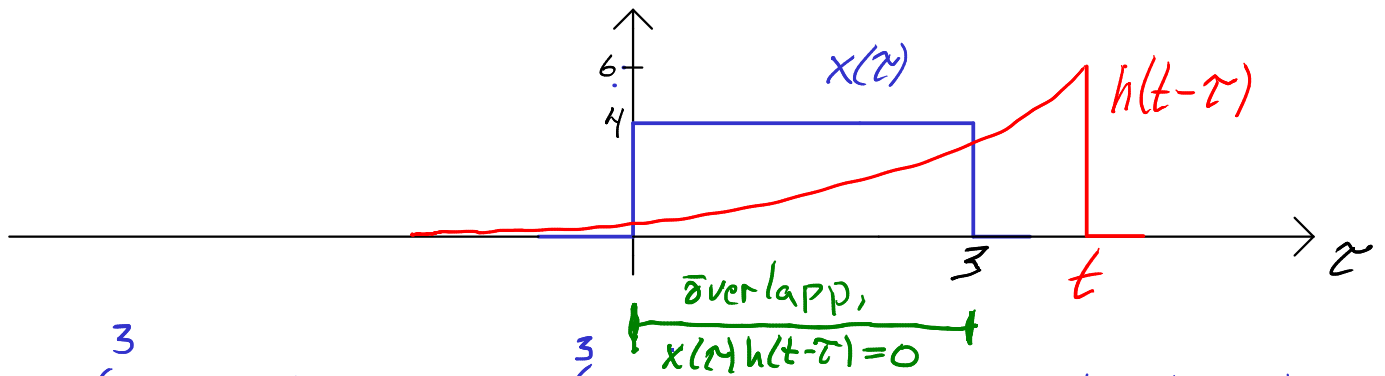
$$\underline{y_{zs}(t)} = \int_{-\infty}^0 0 \cdot h(t-\tau) d\tau + \int_0^t 4 \cdot 6 \cdot e^{-2(t-\tau)} d\tau + \int_t^{\infty} x(\tau) \cdot 0 d\tau$$

overlapp, både x & $h \neq 0$

$$= 24e^{-2t} \int_0^t e^{2\tau} d\tau = \dots = \underline{12(1 - e^{-2t})}$$



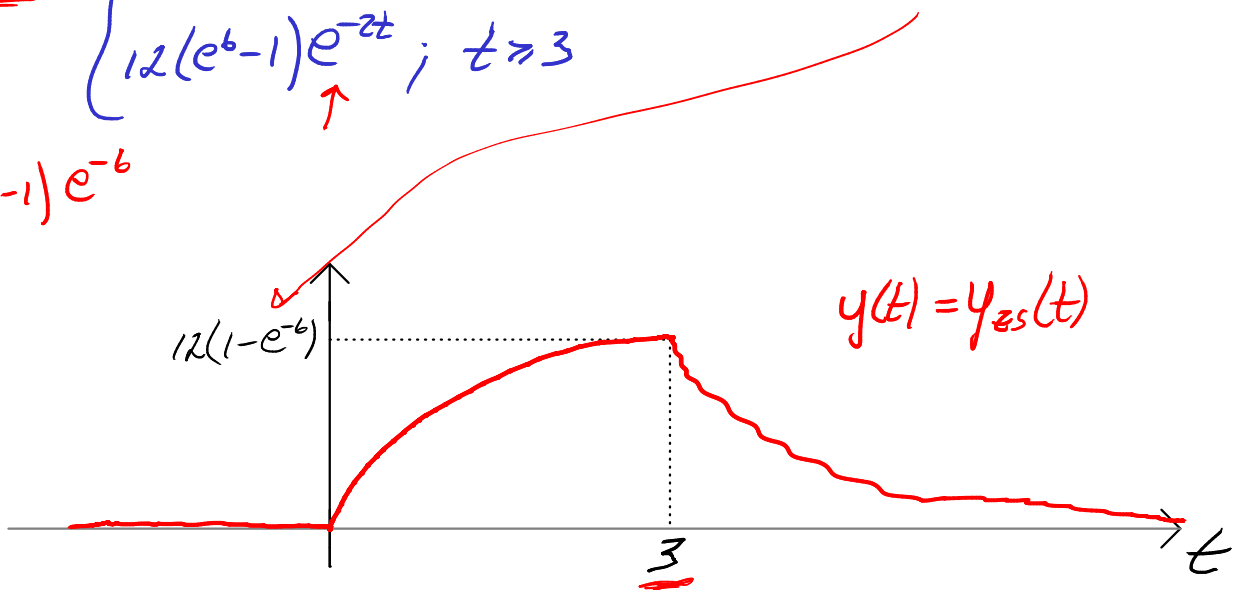
$t \geq 3$



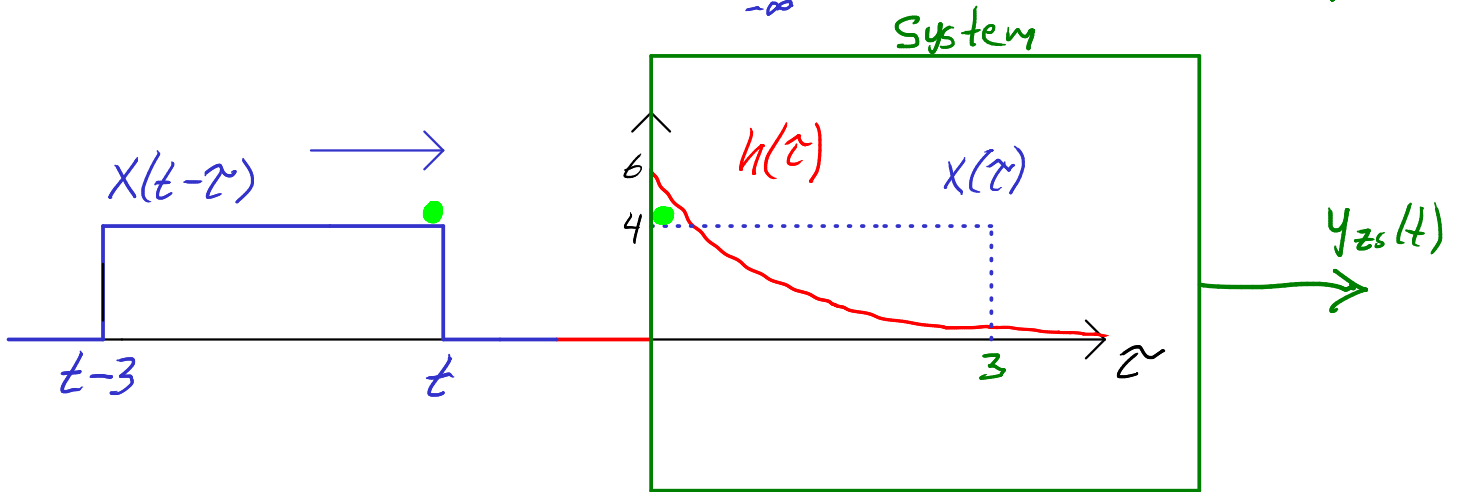
$$\underline{y_{zs}(t)} = \int_0^3 4 \cdot 6 e^{-2(t-\tau)} d\tau = 24 e^{-2t} \int_0^3 e^{2\tau} d\tau = \dots = \underline{12(e^6 - 1)e^{-2t}}$$

$$\therefore \underline{y_{zs}(t)} = \begin{cases} 0; & t < 0 \\ 12(1 - e^{-2t}); & 0 \leq t < 3 \leftarrow t=3 = 12(1 - e^{-2 \cdot 3}) \\ 12(e^6 - 1)e^{-2t}; & t \geq 3 \end{cases}$$

$12(e^6 - 1)e^{-6}$



Testa själv att beräkna $y_{zs}(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$!

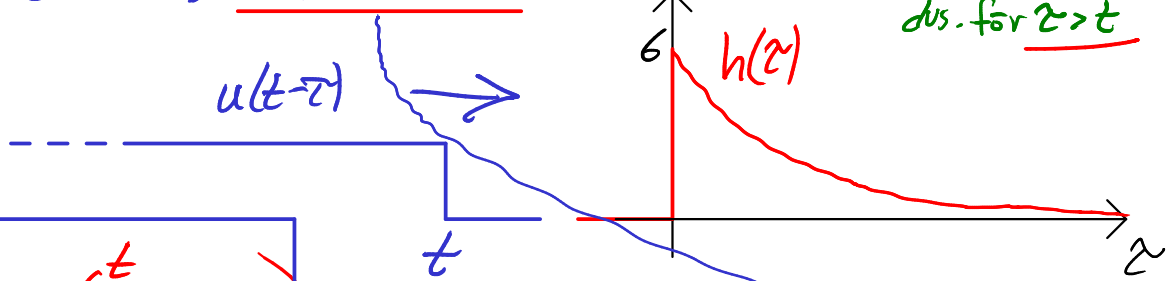


b) Beräkna systemets stegsvar $g(t)$

$$g(t) = \mathcal{L}\{u(t)\} = (u * h)(t) = \int_{-\infty}^{\infty} \underbrace{u(t-\tau)}_{=0 \text{ for } \tau < 0} \underbrace{h(\tau)}_{=0 \text{ for } \tau > t} d\tau$$

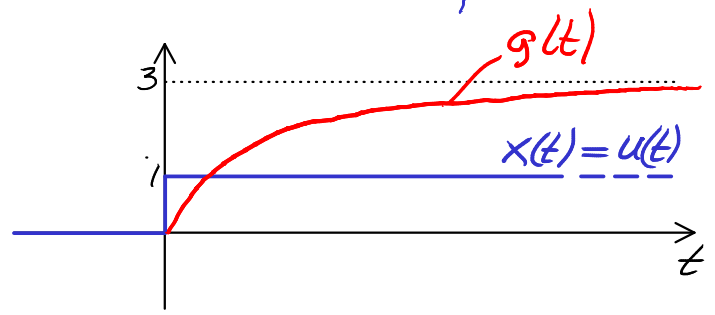
$$= \int_0^t 1 \cdot h(\tau) d\tau = \dots = 3(1 - e^{-2t})u(t)$$

$\bullet: u(t) = 0 \text{ for } t < 0$
 $\Rightarrow u(t-\tau) = 0 \text{ for } t-\tau < 0$
 $\text{dvs. for } \tau > t$

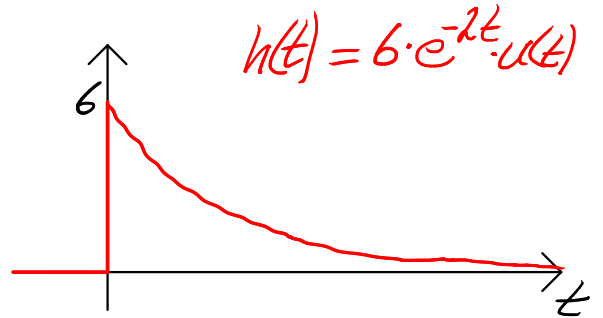
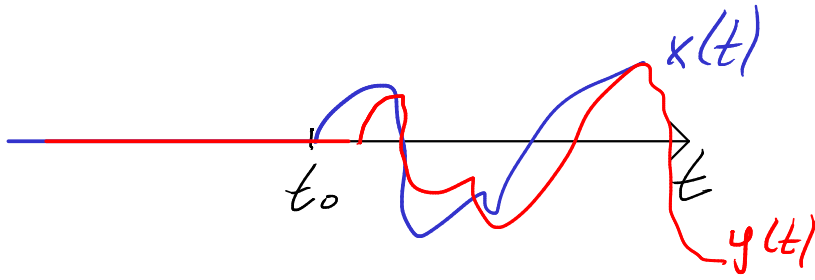


(Allmänt: $g(t) = \int_{-\infty}^t h(\tau) d\tau$)

$h(t) = \frac{dg(t)}{dt}$



c) Bestäm systemets kausalitetsgenskap



Om $x(t) = 0$ för $t < t_0$

$\Rightarrow y(t) = 0$ för $t < t_0$

Kausalt
system

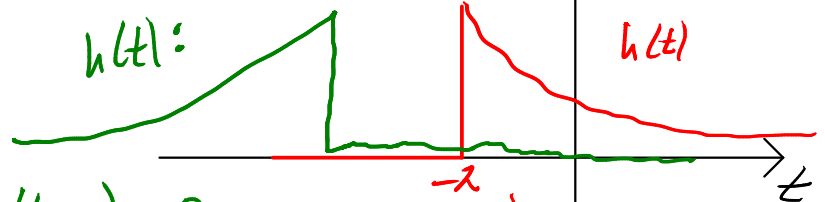
Ex: $x(t) = \delta(t)$; $\delta(t) = 0$ för $t < 0$

Kausalt system $\Rightarrow y(t) = h(t) = 0$ för $t < 0$

Här: $h(t < 0) = 0 \Rightarrow$

Kausalt system

$h(t)$:



$h(t > 0) = 0$

\Rightarrow Anti-kausalt system

Ikäusalt
system

$h(t)$

$\checkmark h(t < 0) \neq 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$y(0) = \int_{-\infty}^0 x(\tau) h(0-\tau) d\tau$$

$$+ \int_{0+}^{\infty} x(\tau) h(0-\tau) d\tau$$

$= 0$ för kausalt system

$h(-\tau) = 0$ för $\tau > 0$

$\Rightarrow h(\tau) = 0$ för $\tau < 0$

d) Bestäm systemets stabilitetsegenskap

Stabilitet – för LTI-system

Stabilt system:

Varje begränsad insignal ger upphov till en begränsad utsignal.

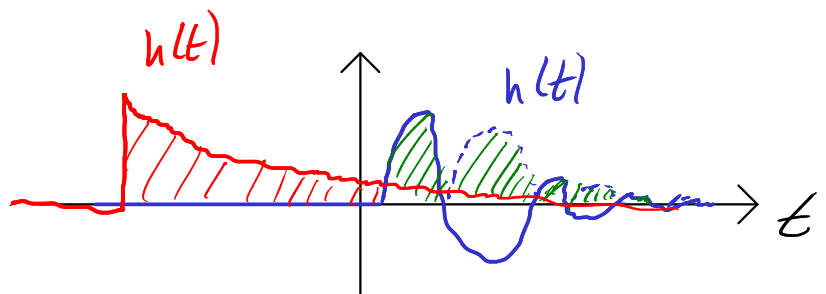
\Leftrightarrow

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

$$|y_{zs}(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau \right| \leq \int_{-\infty}^{\infty} \underbrace{|x(t-\tau)|}_{\leq M < \infty} |h(\tau)| d\tau \leq M \int_{-\infty}^{\infty} |h(\tau)| d\tau \leq N < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Exempel på impulssvar $h(t)$ för stabilt system:

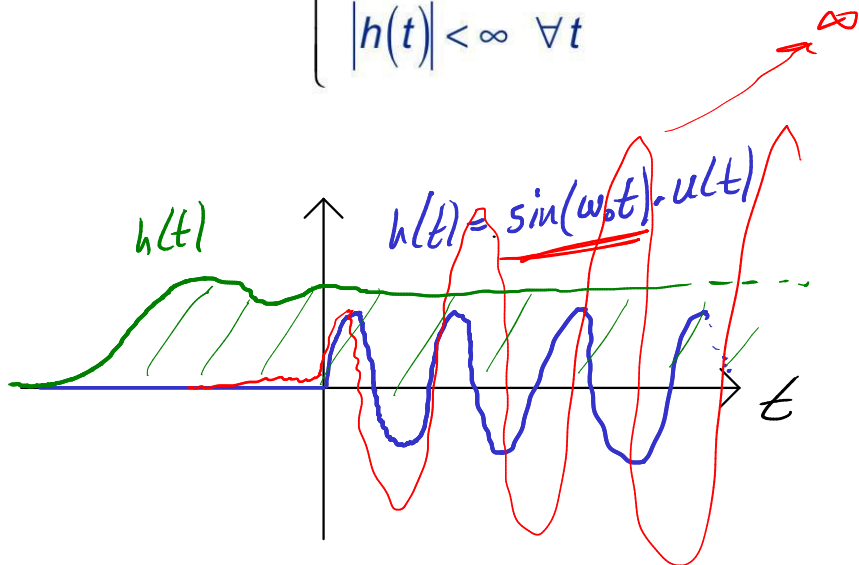


Marginellt stabilt system:

De flesta begränsade insignaler ger upphov till begränsade utsignaler.

$$\Leftrightarrow \begin{cases} \int_{-\infty}^{\infty} |h(t)| dt < \infty \\ |h(t)| < \infty \quad \forall t \end{cases}$$

Exempel på impulssvar $h(t)$ för marginellt stabilt system:

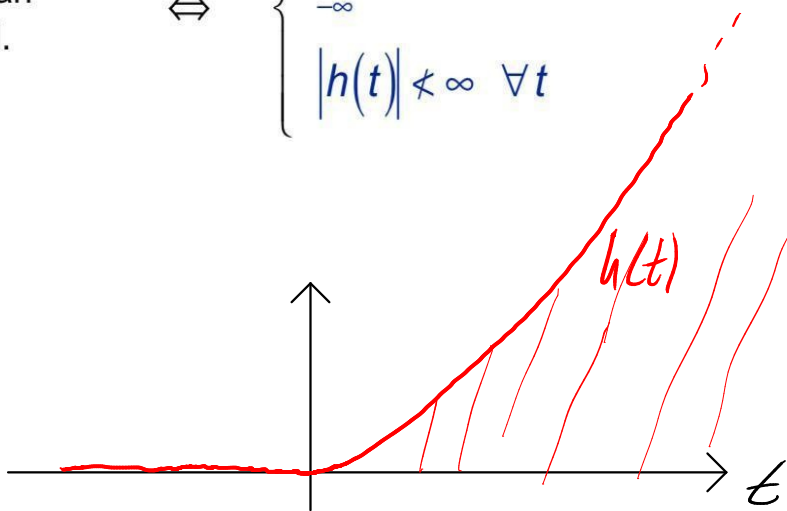


Instabilt system:

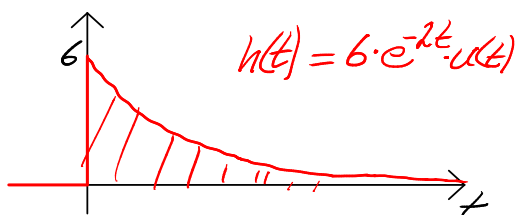
Ingen begränsad nollskild insignal kan ge upphov till en begränsad utsignal.

$$\Leftrightarrow \begin{cases} \int_{-\infty}^{\infty} |h(t)| dt < \infty \\ |h(t)| < \infty \quad \forall t \end{cases}$$

Exempel på impulssvar $h(t)$ för instabilt system:



d) Stabilitetsegenskap för LTI-systemet i räkneuppgiften?



$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_0^{\infty} 6 \cdot e^{-2t} dt \\ &= 6 \left[\frac{e^{-2t}}{-2} \right]_0^{\infty} = -3 \left(\underset{=0}{e^{-\infty}} - \underset{=1}{e^0} \right) = 3 \end{aligned}$$

\Rightarrow Stabilt LTI-system