

- Fouriertransformegenskaper
- Signalenergi
- Fouriertransformen av några viktiga signaler

Sammanfattning – Fouriertransformen (från föregående föreläsning)

- Fouriertransformen till $x(t)$:

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\left(\begin{array}{l} \text{Jfr. fourierserie:} \\ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt \end{array} \right)$$

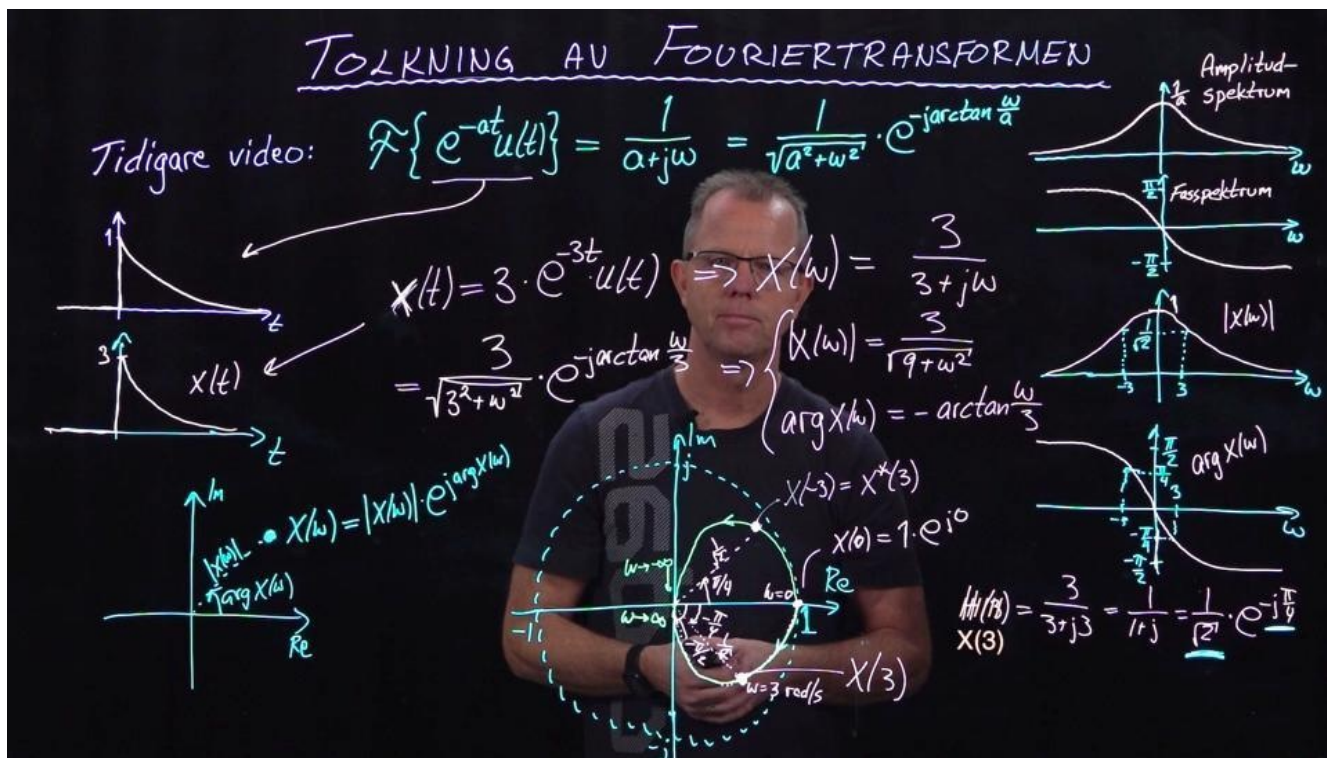
- Inversa fouriertransformen till $X(\omega)$:

$$\mathcal{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\left(\begin{array}{l} \text{Jfr. fourierserie:} \\ x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{array} \right)$$

Existensvillkor:

$$\mathcal{F}\{x(t)\} \exists \text{ om } \int_{-\infty}^{\infty} |x(t)| dt < \infty$$



VIDEO 1

(Det här pratade jag om redan på förra föreläsningen)

VIDEO 2.1

(Härledning utgående från fouriertransformen)

FOURIERTRANSFORMEN – EGENSKAP VID TIDSSKALNING

$\tilde{x}(t) = \text{rect}\left(\frac{t}{\tau}\right) \xrightarrow{\tau} \tilde{X}(\omega) = \tau \cdot \text{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)$ $\text{sinc}_N(x) = \frac{\sin(\pi x)}{\pi x}$

$\tau = 1 \text{ sek} \Rightarrow x(t) = \text{rect}(t) = \tilde{x}(\tau \cdot t) \quad (a = \tau)$

$X(\omega) = \text{sinc}_N\left(\frac{\omega}{2\pi}\right)$ (specialfall: $\tau = 1 \text{ sek}$)

$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ $x(t) \leftrightarrow X(\omega)$

$\mathcal{F}\{x(a \cdot t)\} = \int_{-\infty}^{\infty} x(a \cdot t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j\frac{\omega}{a} t} \frac{dt}{a} = \frac{1}{a} \int_{-\infty}^{\infty} x(t) e^{-j\frac{\omega}{a} t} dt = \frac{1}{a} X\left(\frac{\omega}{a}\right)$

$x(a \cdot t) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$ Tidsskalning $\propto \frac{1}{\text{Frekvensskalning}}$

(På förra föreläsningen demonstrerade jag konsekvensen hos fouriertransformen vid tidsskalning av rect-funktionen. Denna video tar upp konsekvenser av tidsskalning för godtyckliga signaler)

Tidsutbredning	Frekvensutbredning

VIDEO 2.2

(Härledning utgående från inversa fouriertransformen)

FOURIERTRANSFORMEN – EGENSKAP VID TIDSSKALNING

$\tilde{x}(t) = \text{rect}\left(\frac{t}{\tau}\right) \xrightarrow{\tau} \tilde{X}(\omega) = \tau \cdot \text{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)$ $\text{sinc}_N(x) = \frac{\sin(\pi x)}{\pi x}$

$\tau = 1 \text{ sek} \Rightarrow x(t) = \text{rect}(t) = \tilde{x}(\tau \cdot t) \quad (a = \tau)$

$X(\omega) = \text{sinc}_N\left(\frac{\omega}{2\pi}\right)$ (specialfall: $\tau = 1 \text{ sek}$)

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ $x(t) \leftrightarrow X(\omega)$

$x(a \cdot t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega a t} d\omega = \int_{-\infty}^{\infty} \frac{1}{2\pi} X\left(\frac{\omega}{a}\right) e^{j\omega t} \frac{d\omega}{a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a} X\left(\frac{\omega}{a}\right) e^{j\omega t} d\omega$

$x(a \cdot t) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$

VIDEO 3

(Härledning
utgående från
inversa
fouriertransformen)

FOURIERTRANSFORMEN – EGENSKAP VID TIDSFÖRSKJUTNING

$\tilde{x}(t) = \text{rect}\left(\frac{t}{\tau}\right)$ $x(t) = \tilde{x}(t - t_0)$

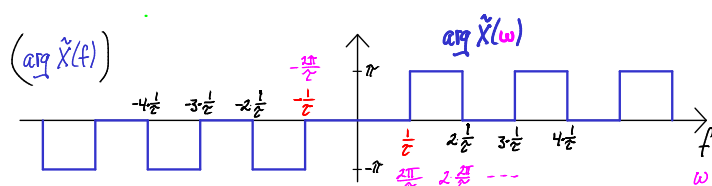
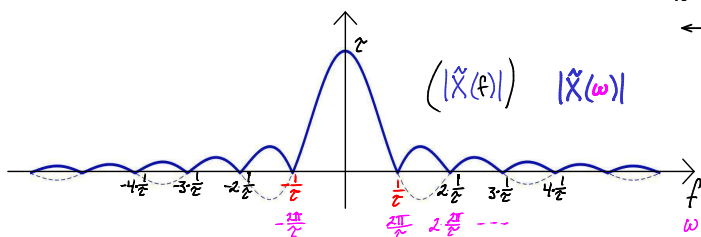
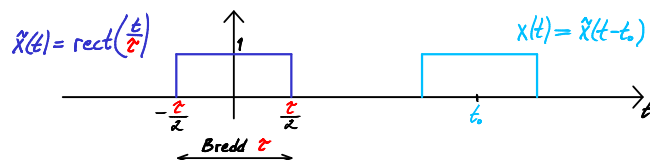
Vi vet: $\tilde{X}(\omega) = \tau \cdot \text{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)$ || $X(\omega) = ?$

$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\omega) e^{j\omega t} d\omega$ $\tilde{x}(t) \leftrightarrow \tilde{X}(\omega)$

$x(t) = \tilde{x}(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\omega) e^{j\omega(t-t_0)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\omega) e^{-j\omega t_0} e^{j\omega t} d\omega$

$X(\omega) = \tilde{X}(\omega) e^{-j\omega t_0} = |\tilde{X}(\omega)| \cdot e^{j(\arg \tilde{X}(\omega) - \omega t_0)} = |X(\omega)| e^{j\arg X(\omega)}$

$|X(\omega)| = |\tilde{X}(\omega)|$ $\arg X(\omega) = \arg \tilde{X}(\omega) - \omega t_0$



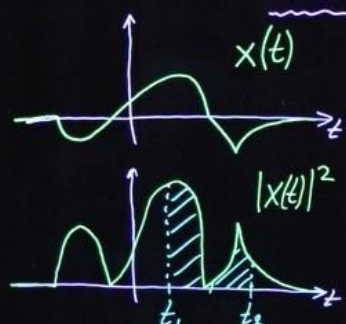
Se formelsamlingens Tabell 2 för egenskaper hos fouriertransformen:

8.2 Egenskaper – fouriertransformen, $x(t) \Leftrightarrow X(\omega)$

Operation	Tidsuttryck, $x(t)$	Transformuttryck, $X(\omega)$
1. Transform	$x(t)$	$X(\omega)$
2. Linjäritet, $a, b \in \mathbb{C}$	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
3. Komplexkonjugering	$x^*(t)$	$X^*(-\omega)$
4. Symmetri	$x(t)$ reell	$X(-\omega) = X^*(\omega)$
5. Dualitet	$X(t)$	$2\pi x(-\omega)$
6. Tidsskalning	$x(at)$	$\frac{1}{ a } X\left(\frac{\omega}{a}\right)$
7. Spegling (tidsskalning med $a = -1$)	$x(-t)$	$X(-\omega)$
8. Tidsförskjutning	$x(t - t_0)$	$X(\omega) e^{-j\omega t_0}$
9. Frekvensskift, $\omega_0 \in \mathbb{R}$	$x(t) e^{j\omega_0 t}$	$X(\omega - \omega_0)$
10. Derivering	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(\omega)$
11. Multiplikation med t	$t^n x(t)$	$j^n \frac{d^n X(\omega)}{d\omega^n}$

(Plus ytterligare några egenskaper...)

SIGNALENERGI & PARSEVALS FORMEL



Signalenergi hos en signal $x(t)$:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

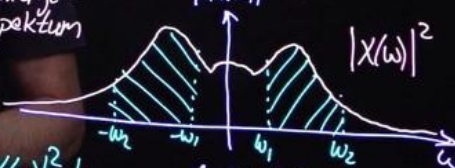
$$E = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt = \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right)^* dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega) d\omega$$

Parsevals formel:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

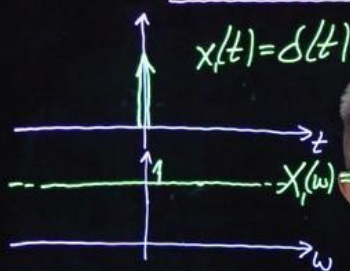
Energi-
spektrum



Signalenergi i intervallet $[\omega_1, \omega_2]$: $\Delta E = 2 \cdot \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |X(\omega)|^2 d\omega$

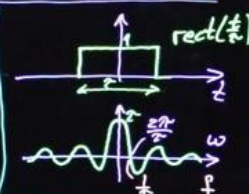
$$x(t) \in \mathbb{R} \forall t \Rightarrow x(-\omega) = X^*(\omega)$$

NÅGRA FOURIERTRANSFORMER: $\mathcal{F}\{\delta(t)\}$, $\mathcal{F}\{1\}$



$$\int_{-\infty}^{\infty} |x_1(t)| dt = 1 < \infty \Rightarrow X_1(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t} \Big|_{t=0} = e^0 = 1$$

$$\boxed{\delta(t) \Leftrightarrow 1}$$



$$\int_{-\infty}^{\infty} |x_2(t)| dt = \infty \Rightarrow \mathcal{F}\{x_2(t)\} \nexists$$

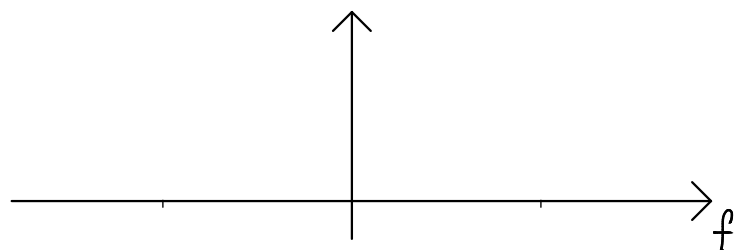
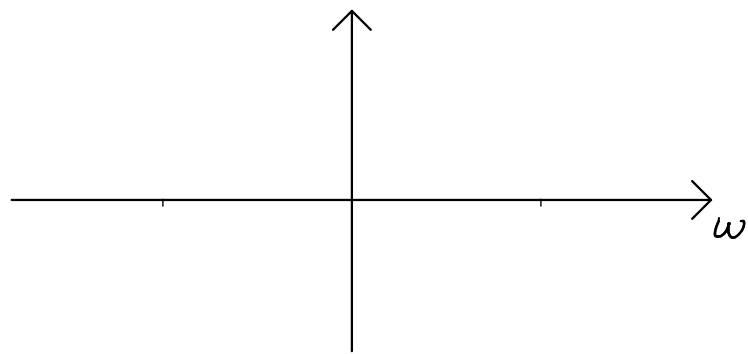
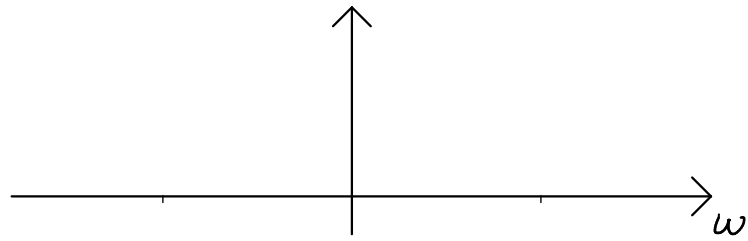
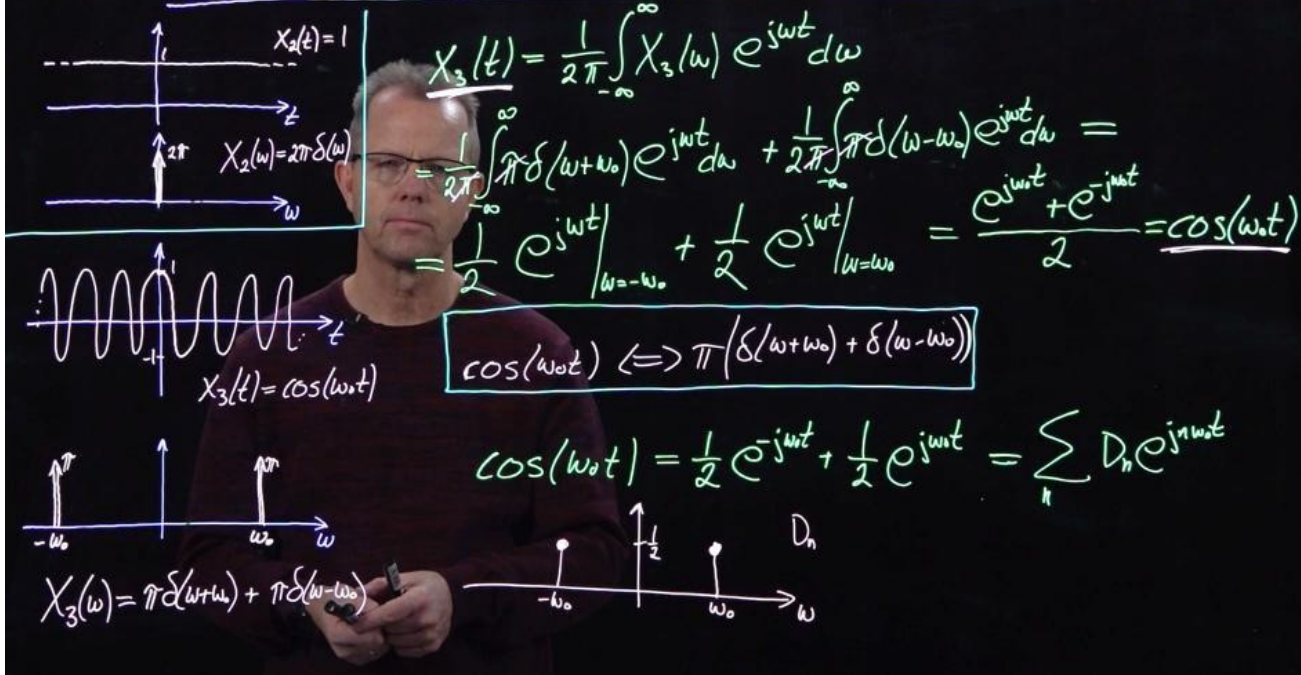
$$\frac{d}{dt} x_2(\omega) = 2\pi \delta(\omega) \Rightarrow$$

$$x_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega) e^{j\omega t} d\omega = e^{j\omega t} \Big|_{\omega=0} = 1$$

$$\Rightarrow \boxed{1 \Leftrightarrow 2\pi \delta(\omega)}$$

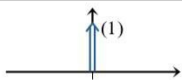
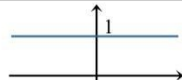
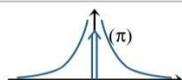
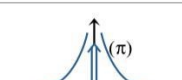
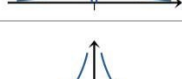
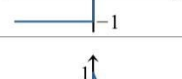
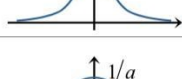
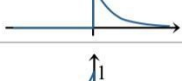
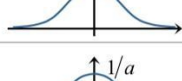
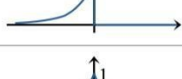
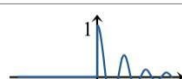
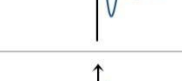
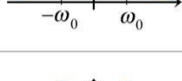
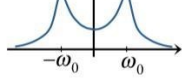
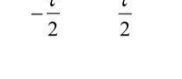
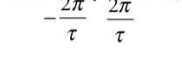


FOURIERTRANSFORMEN AV EN FREKVENSSIGNAL



Se formelsamlingens Tabell 3 för olika fouriertransformpar $x(t) \Leftrightarrow X(\omega)$:

8.3 Fouriertransformer, $x(t) \Leftrightarrow X(\omega)$

Tidsuttryck, $x(t)$	Transformuttryck, $X(\omega)$	$ X(\omega) $
1. $\delta(t)$ 	1	
2. $u(t), u_0(t)$ 	$\text{vp} \left\{ \frac{1}{j\omega} \right\} + \pi\delta(\omega)$	
3. $u(-t), u_0(-t)$ 	$\text{vp} \left\{ \frac{-1}{j\omega} \right\} + \pi\delta(\omega)$	
4. $\text{sgn}(t) = \frac{t}{ t }$ 	$\text{vp} \left\{ \frac{2}{j\omega} \right\}$	
5. $e^{-at}u(t)$ 	$\frac{1}{a + j\omega}$	
6. $e^{at}u_0(-t)$ 	$\frac{1}{a - j\omega}$	
7. $e^{-a t }$ 	$\frac{2a}{a^2 + \omega^2}$	
10. $e^{-at} \cos(\omega_0 t) u(t)$ 	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	
11. $e^{-at} \sin(\omega_0 t) u(t)$ 	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	
12. $\Pi\left(\frac{t}{\tau}\right) = \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1; & t \leq \frac{\tau}{2} \\ 0; & t > \frac{\tau}{2} \end{cases}$ 	$\tau \text{sinc}_N\left(\frac{\omega\tau}{2\pi}\right) = \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$	
13. $\text{sinc}_N(at) = \text{sinc}(a\pi t)$ 	$\frac{1}{a} \Pi\left(\frac{\omega}{2\pi a}\right) = \frac{1}{a} \text{rect}\left(\frac{\omega}{2\pi a}\right) = \begin{cases} \frac{1}{a}; & \omega \leq \pi a \\ 0; & \omega > \pi a \end{cases}$	
22. $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$ 	$\pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$	
23. $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$ 	$j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$	

(Plus ytterligare ett antal transformpar)

