Linjära System – Föreläsning 13: Fouriertransformanalys

- Fouriertransformegenskaper
- Signalenergi
- Fouriertransformen av några viktiga signaler

Sammanfattning – Fouriertransformen (från föregående föreläsning)

- Fouriertransformen till $x(t)$:

$$\boxed{\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

$$\left(\text{Jfr. fourierserie: } D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt \right)$$

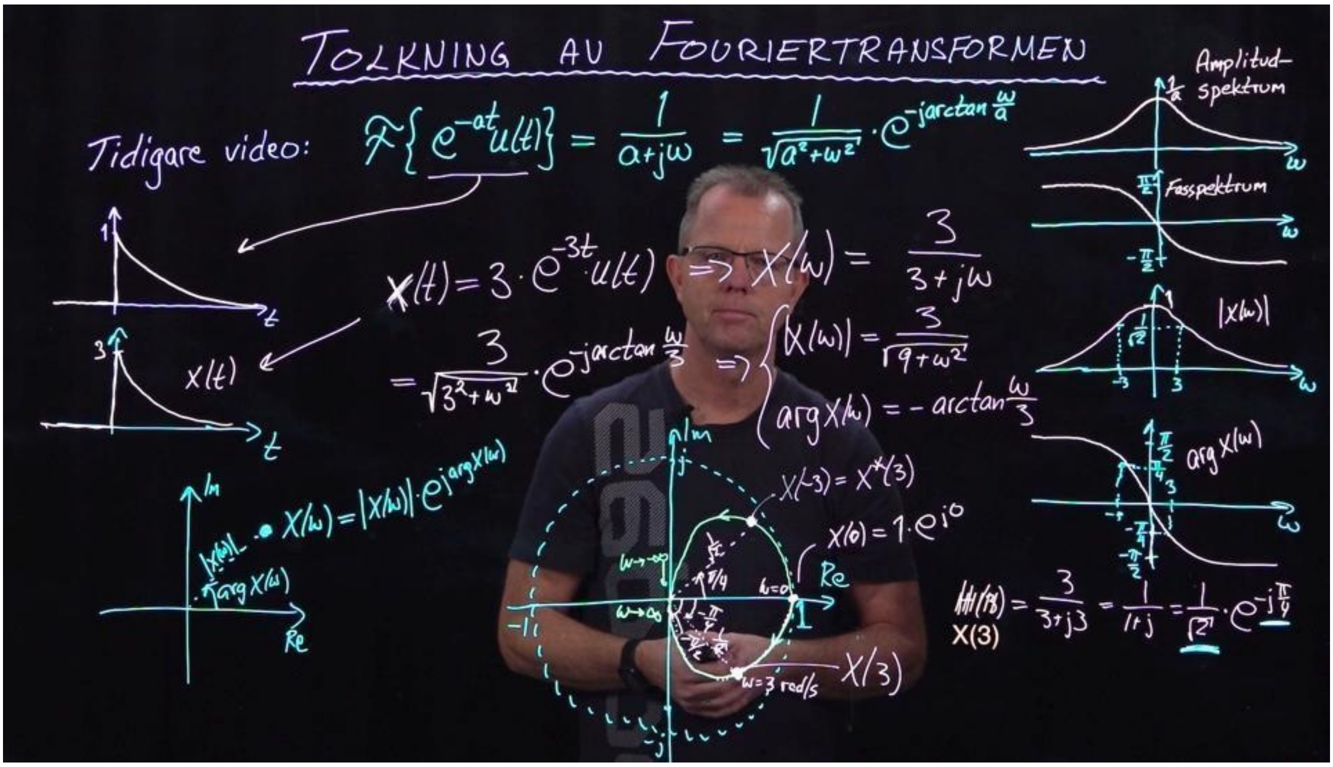
- Inversa fouriertransformen till $X(\omega)$:

$$\boxed{\mathcal{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

$$\left(\text{Jfr. fourierserie: } x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \right)$$

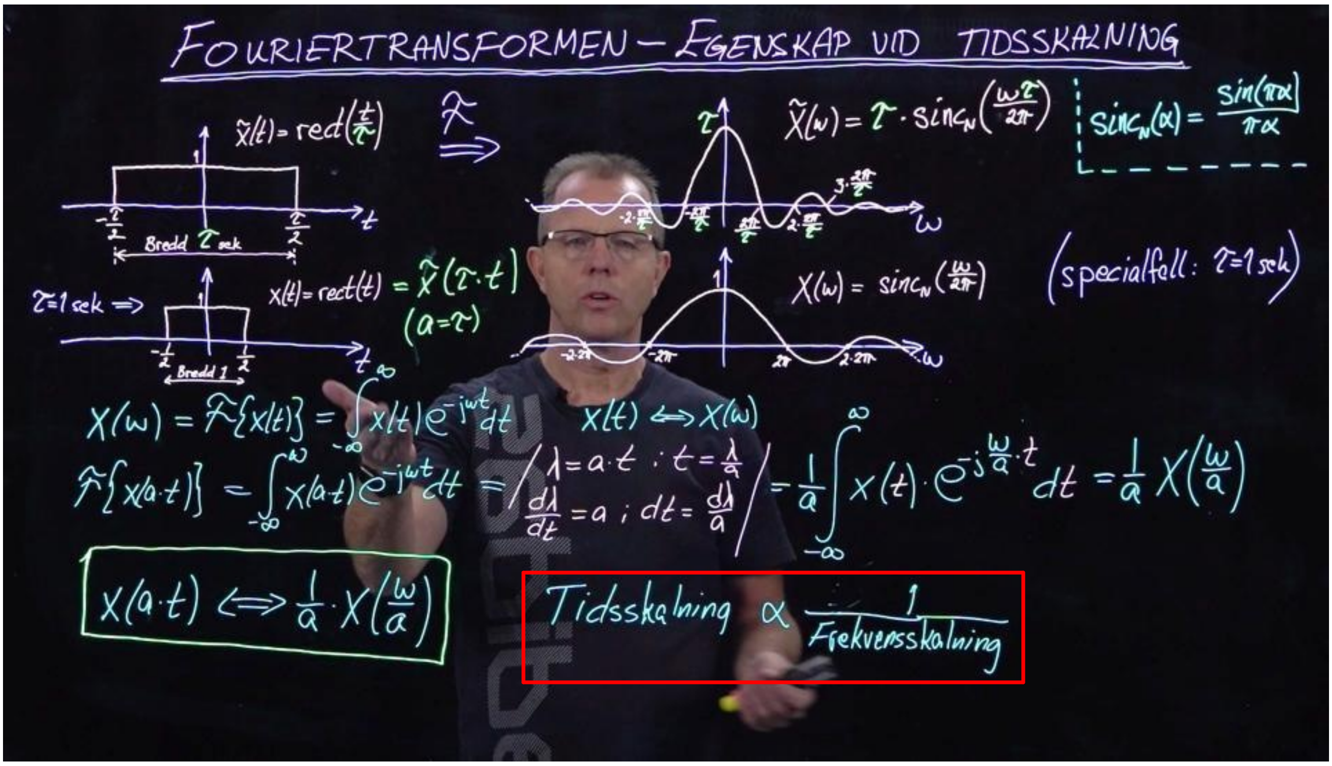
Existensvillkor: $\boxed{\mathcal{F}\{x(t)\} \exists \ \text{ om } \int_{-\infty}^{\infty} |x(t)|\, dt < \infty}$

Transform par: $x(t) \Longleftrightarrow X(\omega)$



VIDEO 1

(Det här pratade jag om redan på förra föreläsningen)



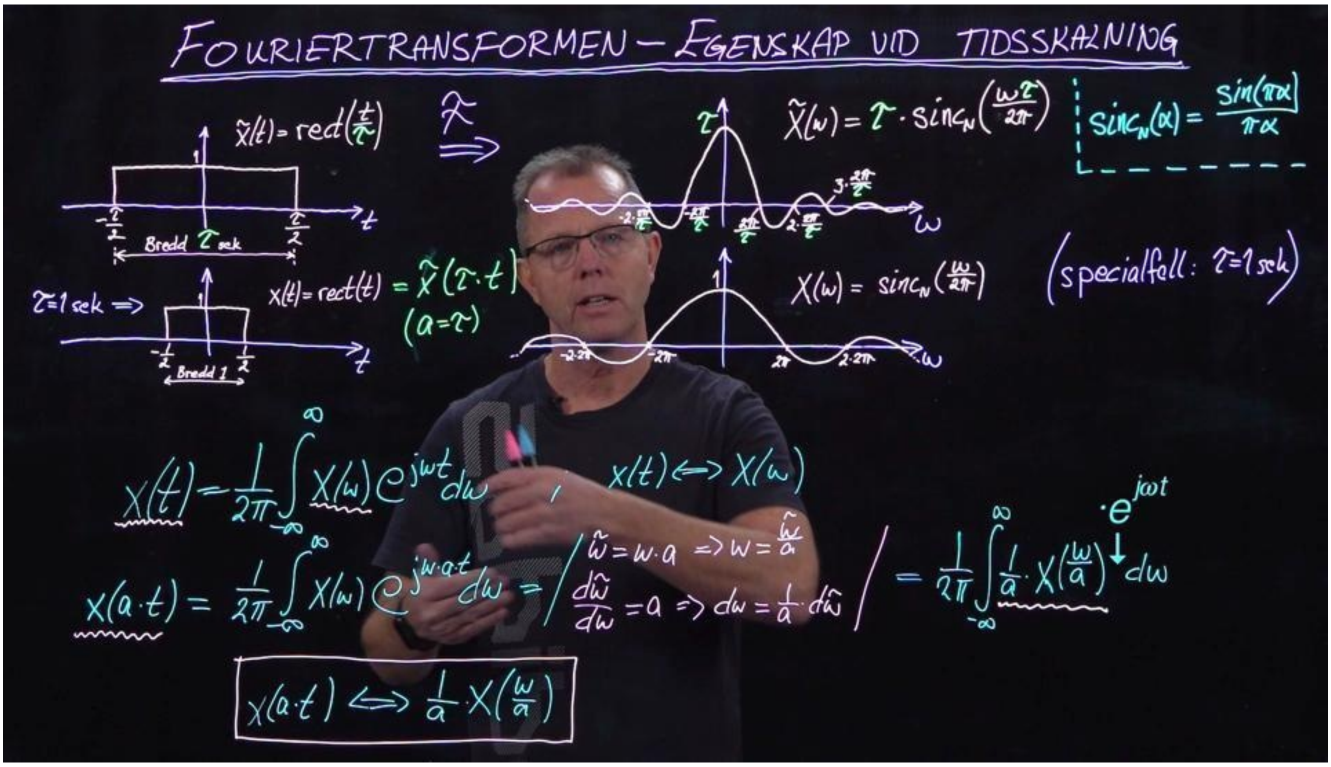
VIDEO 2.1

(Härledning utgående från fouriertransformen)

(På förra föreläsningen demonstrerade jag konsekvensen hos fouriertransformen vid tidsskalning av rect-funktionen. Denna video tar upp konsekvenser av tidsskalning för godtyckliga signaler)

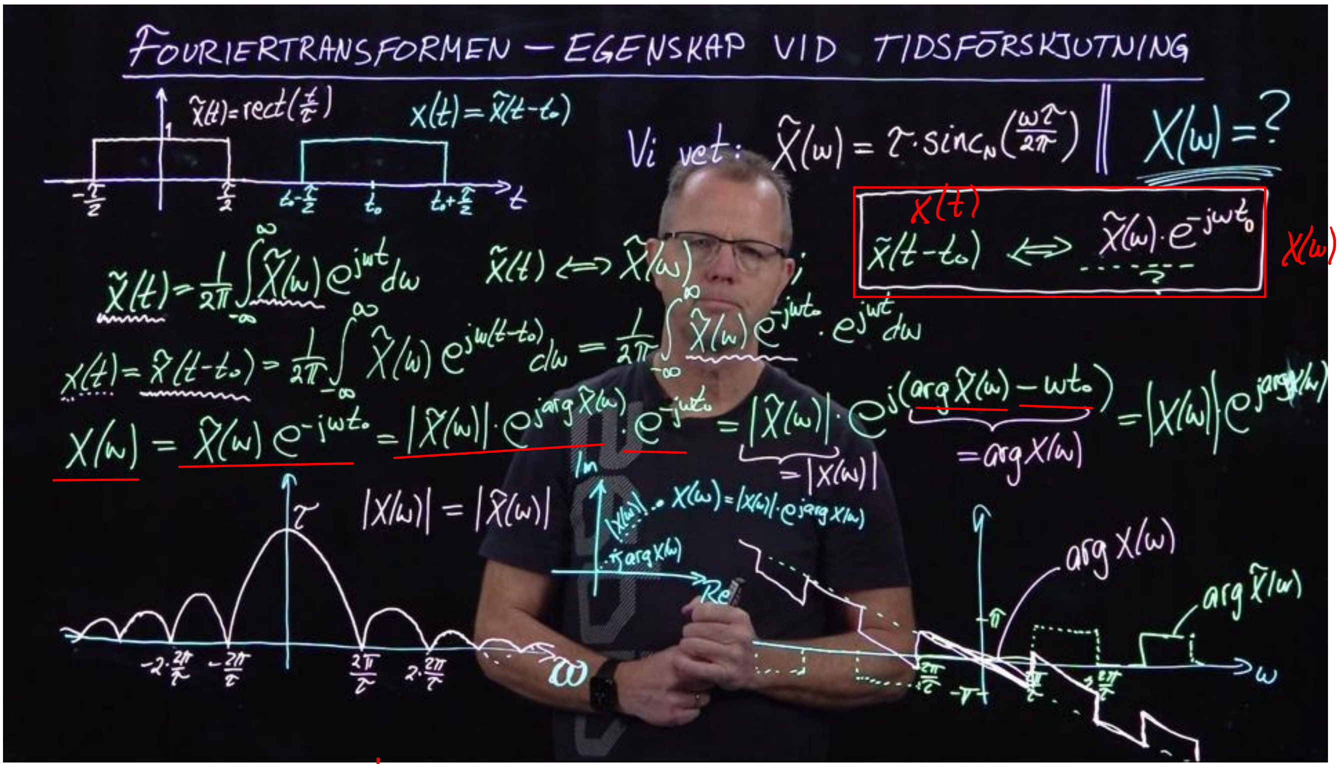
Tidsutbredning		Frekvensutbredning
Liten	$\Rightarrow$	Stor
Stor	$\Rightarrow$	Liten eller stor

En signal kan inte ha en ändlig utbredning i båda domänerna samtidigt!



VIDEO 2.2

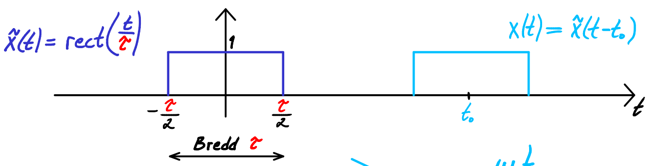
(Härledning utgående från inversa fouriertransformen)

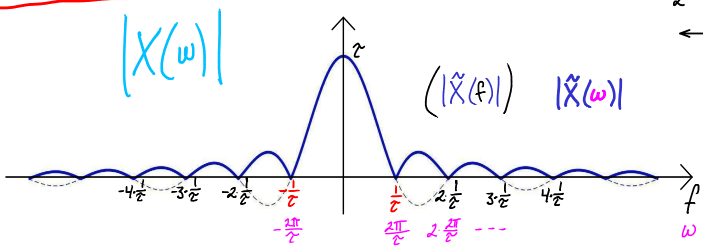


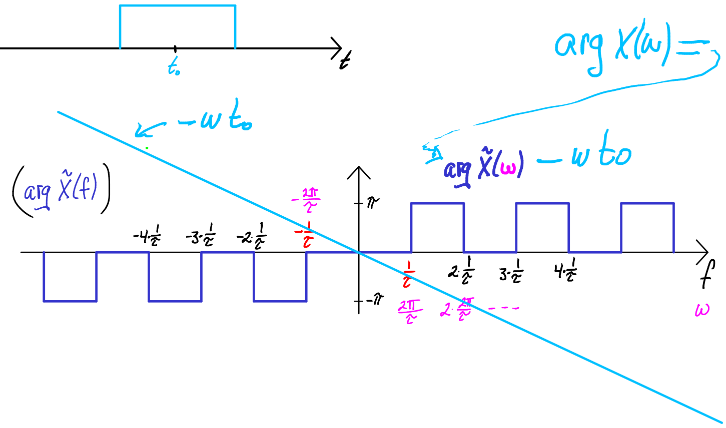
VIDEO 3

(Härledning utgående från inversa fouriertransformen)

Om $x(t) \in \mathbb{R} \ \forall t$
$\Rightarrow X(-\omega) = X^*(\omega)$







Härledning utgående från $\mathcal{F}$.

$x(t) = \tilde{x}(t-t_0) \iff X(\omega) = ?$, $\quad \tilde{x}(t) \iff \tilde{X}(\omega) = \int_{-\infty}^{\infty} \tilde{x}(t) e^{-j\omega t} dt$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \tilde{x}(t-t_0) e^{-j\omega t} dt = \left/ \begin{array}{l} \lambda = t - t_0 \Rightarrow \bullet\ t = \lambda + t_0 \\ \bullet\ \frac{d\lambda}{dt} = 1 \Rightarrow dt = d\lambda \end{array} \right/$$

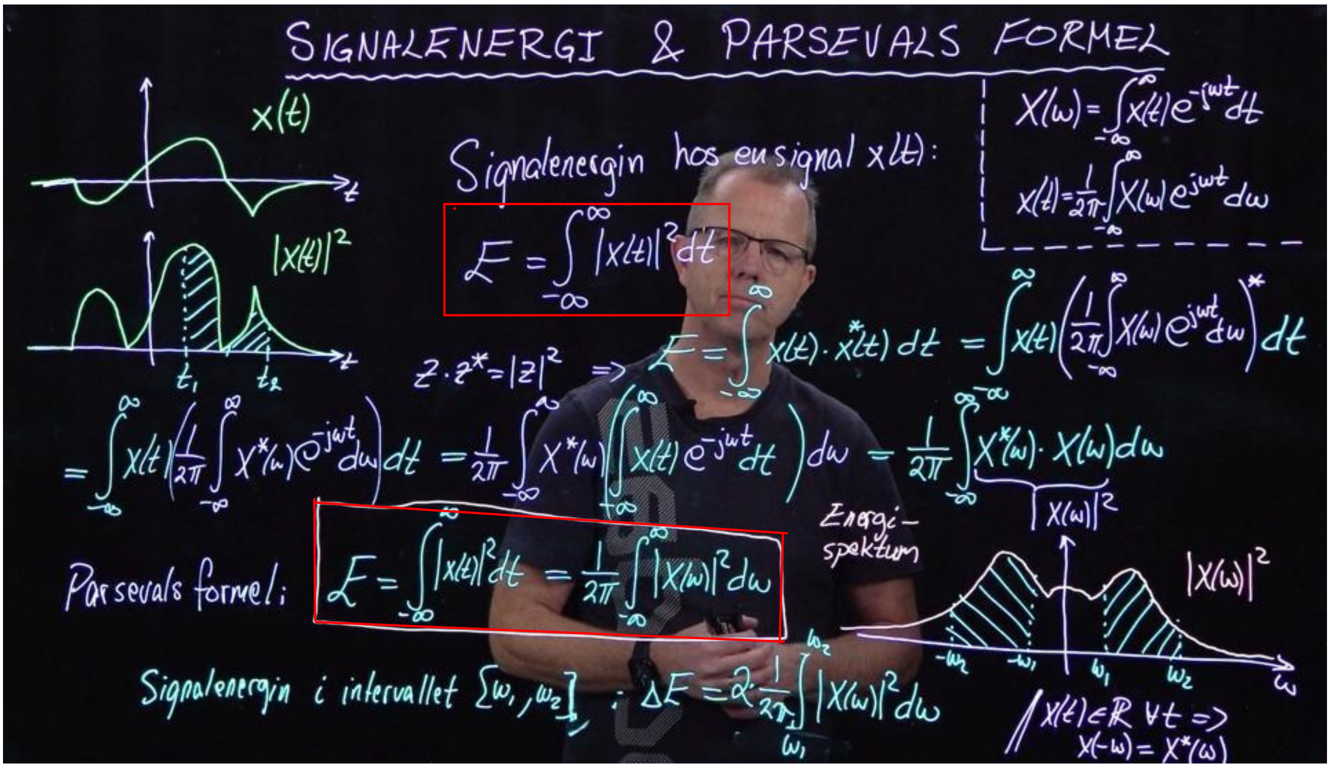
$$= \int_{-\infty}^{\infty} \tilde{x}(\lambda) \underbrace{e^{-j\omega(\lambda+t_0)}}_{e^{-j\omega\lambda}\cdot e^{-j\omega t_0}} d\lambda = \left(\int_{-\infty}^{\infty} \tilde{x}(\lambda) \cdot e^{-j\omega\lambda} d\lambda \right) \cdot e^{-j\omega t_0} = \tilde{X}(\omega) \cdot e^{-j\omega t_0}$$

Se formelsamlingens Tabell 2 för egenskaper hos fouriertransformen:

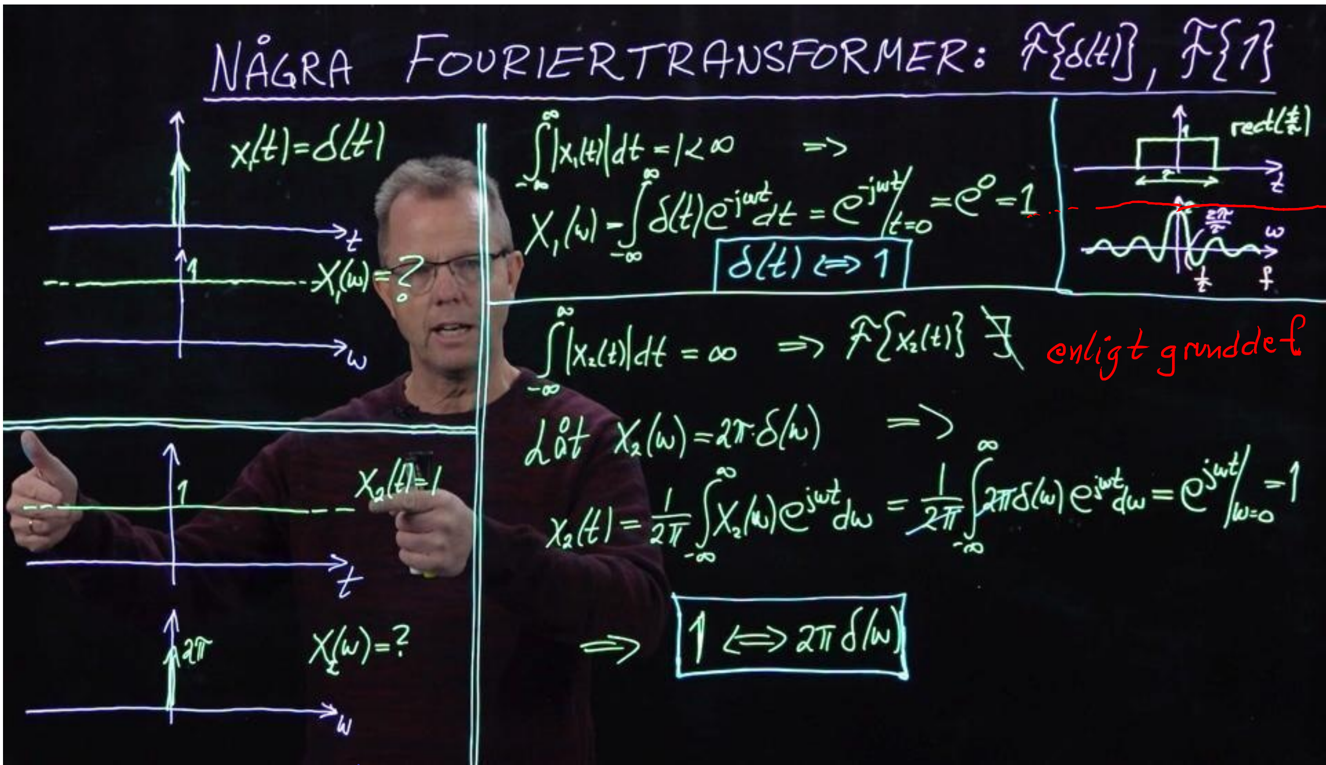
8.2 Egenskaper – fouriertransformen, $x(t) \Leftrightarrow X(\omega)$

Operation	Tidsuttryck, $x(t)$	Transformuttryck, $X(\omega)$
1. Transform	$x(t)$	$X(\omega)$
2. Linjäritet, $a,b \in \mathbb{C}$	$ax_1(t)+bx_2(t)$	$aX_1(\omega)+bX_2(\omega)$
3. Komplexkonjugering	$x^*(t)$	$X^*(-\omega)$
4. Symmetri	$x(t)$ reell	$X(-\omega)=X^*(\omega)$
5. Dualitet	$X(t)$	$2\pi x(-\omega)$
6. Tidsskalning	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
7. Spegling (*tidsskalning med* $a=-1$)	$x(-t)$	$X(-\omega)$
8. Tidsförskjutning	$x(t-t_0)$	$X(\omega)e^{-j\omega t_0}$
9. Frekvensskift, $\omega_0 \in \mathbb{R}$	$x(t)e^{j\omega_0 t}$	$X(\omega-\omega_0)$
10. Derivering	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(\omega)$
11. Multiplikation med t	$t^n x(t)$	$j^n \frac{d^n X(\omega)}{d\omega^n}$

(Plus ytterligare några egenskaper...)



VIDEO 4



VIDEO 5

∞

$\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$

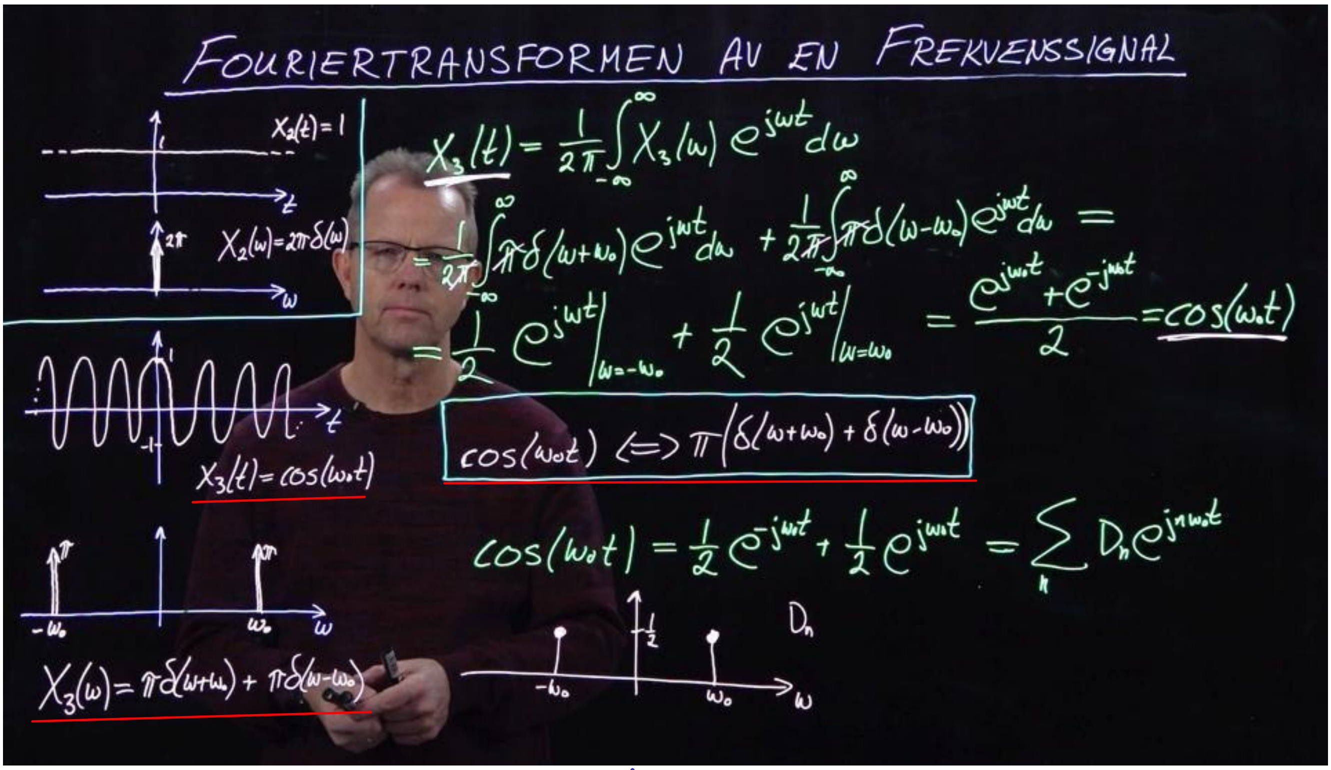
$x(t)$ $y_{zs}(t)=(x*h)(t)$ $H(\omega)=\mathcal{F}\{h(t)\}$

$\mathcal{F}\{x(t)\}=X(\omega)\ \exists \Rightarrow |X(\omega)|<\infty\ \ \forall\omega$

Om man tillåter dirac:er i frekvensdomänen (att fouriertransformen innehåller dirac:er => en utvidgad definition av fouriertransformen!

δ är en distribution $\Rightarrow \mathcal{F}\{1\}\ \exists$ "i distributionsmening"

VIDEO 6



Formelsaml. Tab. 2:9 $\quad x(t)e^{j\omega_0 t} \iff X(\omega-\omega_0)$

Låt $x(t)=1 \iff X(\omega)=2\pi\cdot\delta(\omega)$

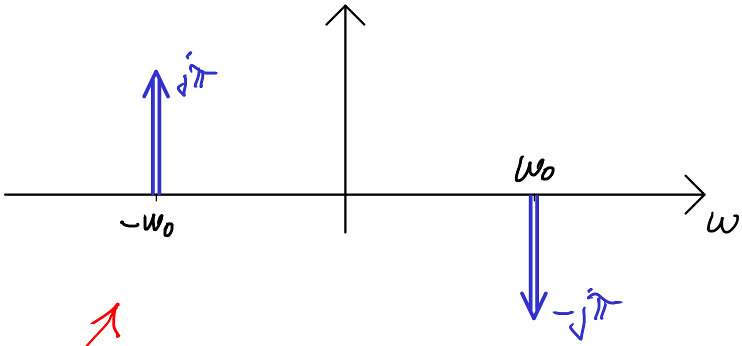
$\Rightarrow e^{j\omega_0 t} \iff 2\pi\delta(\omega-\omega_0)$

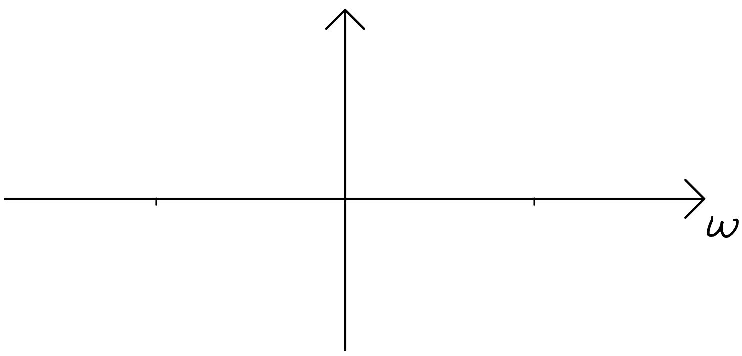
$\sin(\omega_0 t) = \frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\cdot\frac{j}{j}$

$= \frac{-j}{2}e^{j\omega_0 t}+\frac{j}{2}e^{-j\omega_0 t}$

$\iff \frac{-j}{\cancel{2}}\cancel{2}\pi\delta(\omega-\omega_0)+\frac{j}{\cancel{2}}\cdot\cancel{2}\pi\delta(\omega+\omega_0)$

$= j\pi\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)$





$X(f)=X(\omega)\big|_{\omega=2\pi f}$

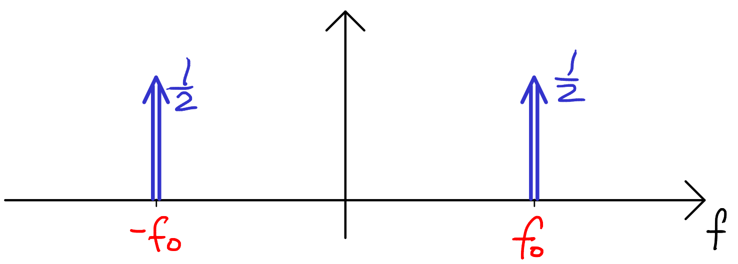
$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right)\Big|_{\omega=2\pi f}$

$=\pi\left(\delta(\underbrace{2\pi}_{a}(f+f_0))+\delta(\underbrace{2\pi}_{a}(f-f_0))\right)$

$=$ / Formels. sid 2: $\delta(a\cdot t)=\frac{1}{|a|}\cdot\delta(t)$ /

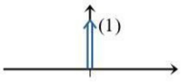
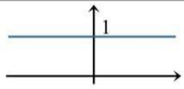
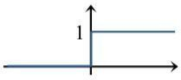
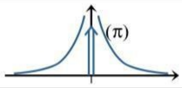
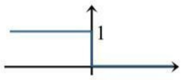
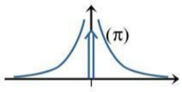
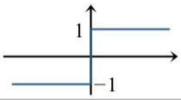
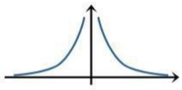
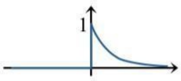
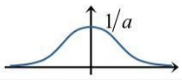
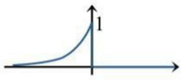
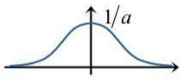
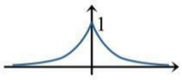
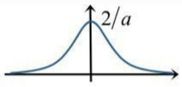
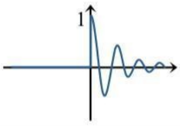
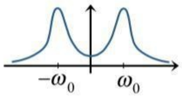
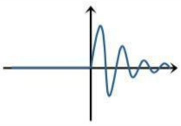
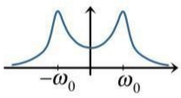
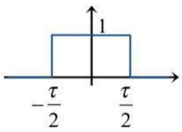
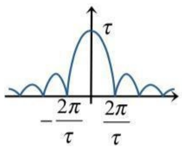
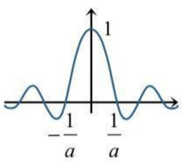
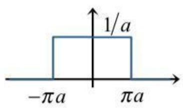
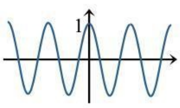
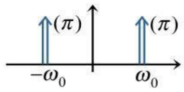
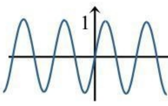
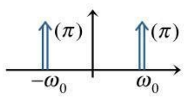
$=\pi\left(\frac{1}{2\pi}\delta(f+f_0)+\frac{1}{2\pi}\delta(f-f_0)\right)$

$=\frac{1}{2}\left(\delta(f+f_0)+\delta(f-f_0)\right)$



Se formelsamlingens Tabell 3 för olika fouriertransformpar x(t) <=> X(w) :

8.3 Fouriertransformer, $x(t) \Leftrightarrow X(\omega)$

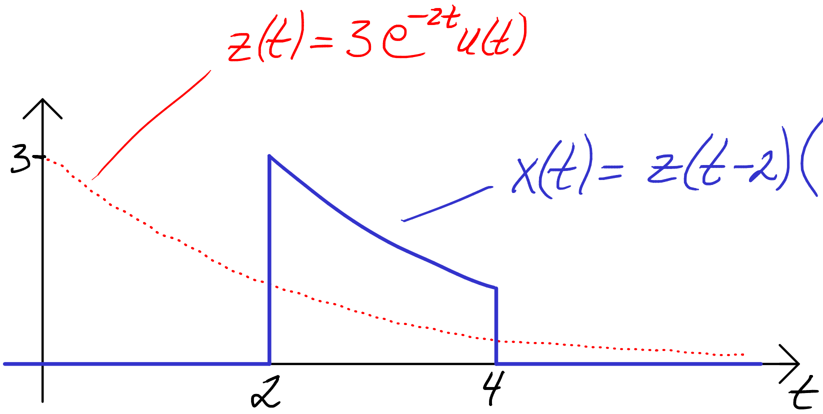
Tidsuttryck, $x(t)$		Transformuttryck, $X(\omega)$	$\lvert X(\omega)\rvert$
1. $\delta(t)$		1	
2. $u(t),\ u_0(t)$		$\mathrm{vp}\left\{\frac{1}{j\omega}\right\}+\pi\delta(\omega)$	
3. $u(-t),\ u_0(-t)$		$\mathrm{vp}\left\{\frac{-1}{j\omega}\right\}+\pi\delta(\omega)$	
4. $\mathrm{sgn}(t)=\frac{t}{\lvert t\rvert}$		$\mathrm{vp}\left\{\frac{2}{j\omega}\right\}$	
5. $e^{-at}u(t)$		$\frac{1}{a+j\omega}$	
6. $e^{at}u_0(-t)$		$\frac{1}{a-j\omega}$	
7. $e^{-a\lvert t\rvert}$		$\frac{2a}{a^2+\omega^2}$	
10. $e^{-at}\cos(\omega_0 t)u(t)$		$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	
11. $e^{-at}\sin(\omega_0 t)u(t)$		$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	
12. $\Pi\left(\frac{t}{\tau}\right)=\mathrm{rect}\left(\frac{t}{\tau}\right)=\begin{cases}1; & \lvert t\rvert\le\frac{\tau}{2}\\ 0; & \lvert t\rvert>\frac{\tau}{2}\end{cases}$		$\tau\,\mathrm{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)=$ $\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$	
13. $\mathrm{sinc}_N(at)=$ $\mathrm{sinc}(a\pi t)$		$\frac{1}{a}\Pi\left(\frac{\omega}{2\pi a}\right)=$ $\frac{1}{a}\mathrm{rect}\left(\frac{\omega}{2\pi a}\right)=$ $\begin{cases}\frac{1}{a}; & \lvert\omega\rvert\le\pi a\\ 0; & \lvert\omega\rvert>\pi a\end{cases}$	
22. $\cos(\omega_0 t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$		$\pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right)$	
23. $\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$		$j\pi\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)$	

(Plus ytterligare ett antal transformpar)

vp = principalvärde

$$= \begin{cases} \frac{1}{j\omega}\ ; & \omega \neq 0 \\ \pi\delta(\omega)\ ; & \omega = 0 \end{cases}$$

Exempel på beräkning av fouriertransformen – hanns inte med på föreläsningen:



$$x(t) = z(t-2)\big(u(t-2) - u(t-4)\big) = \begin{cases} z(t-2); & 2 \leq t < 4 \\ 0; & \text{f.ö.} \end{cases}$$

Beräkna $X(\omega)$

1) Genom direkt användning av fouriertransformintegralen
2) Genom användning av formelsamligens Tab 3:5 & 2:8 på lämpligt sätt.

$$\Longrightarrow \quad X(\omega) = \frac{3e^{-j2\omega}}{2+j\omega}\left(1 + e^{-4-j2\omega}\right)$$

Gör detta själv – det är en bra övning!