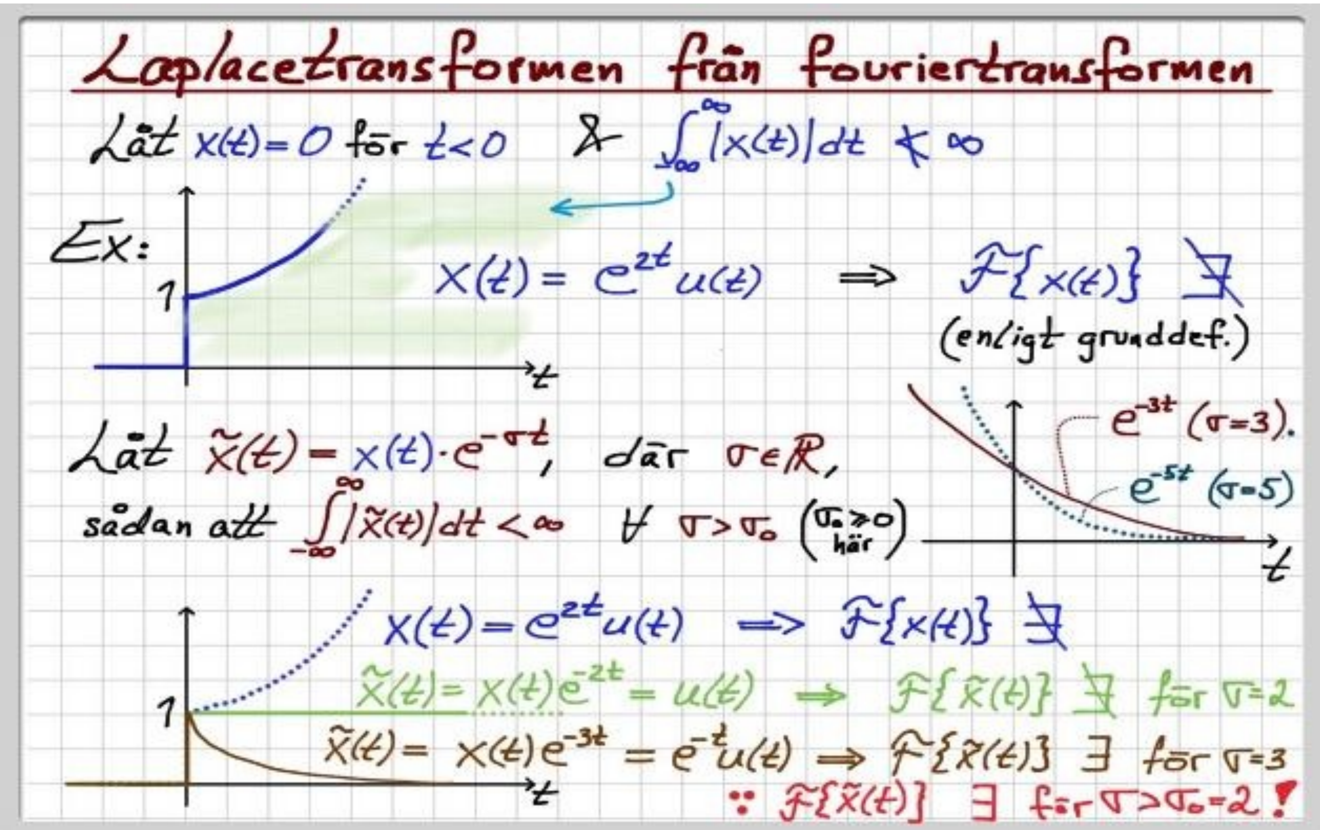
Förberedande videor

- Video 1: Härledning av laplacetransformen från fouriertransformen
- Den enkelsidiga laplacetransformen
- Laplacetransformens konvergensområde
- Video 2: Exempel 1 – laplacetransformen av $x(t) = e^{2t}u(t)$
- Video 3: Exempel 2: Laplacetransformen av $y(t) = e^{-3t}u(t)$
  Exempel 3: Laplacetransformen av $g(t) = \cos(\omega_0 t)u(t)$

Föreläsningsfokus

- Den *enkelsidiga* laplacetransformen, konvergensområde
- Fouriertransformen = laplacetransformen längs $j\omega$-axeln
- Den *dubbelsidiga* laplacetransformen, konvergensområden
  - Räkneexempel: Laplacetransformen av $x(t) = e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$
- Den inversa laplacetransformen



Video 1.1

Video 1.2

Laplacetransformen från fouriertransformen

$$\Rightarrow \mathcal{F}\{\tilde{x}(t)\} = \int_{-\infty}^{\infty} \tilde{x}(t) e^{-j\omega t} dt = \Big/ \begin{matrix} \tilde{x}(t) = x(t)e^{-\sigma t} \\ x(t) = 0 \text{ för } t<0 \end{matrix} \Big/ = \int_{0-}^{\infty} x(t) e^{-\sigma t} \cdot e^{-j\omega t} dt$$

$$= \int_{0-}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \Big/ \boxed{s = \sigma + j\omega} \Big/$$

$$= \underline{\int_{0-}^{\infty} x(t) e^{-st} dt = X(s)}$$

$$\left(\begin{matrix} \text{Jämför med} \\ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \end{matrix} \right)$$

Den enkelsidiga Laplacetransformen till $x(t)$ (där $x(t) = 0$ för $t<0$):

$$\boxed{X(s) = \mathcal{L}_I\{x(t)\} = \int_{0-}^{\infty} x(t) e^{-st} dt}$$

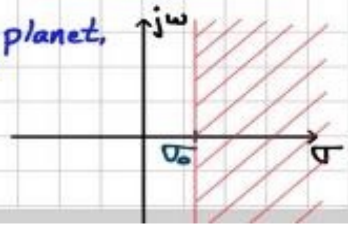
$X_I(s) =$

Existensvillkor, $|X(s)| < \infty$

Konvergensområde:

$\sigma = \mathrm{Re}\{s\} > \sigma_0$

s-planet, $\mathbb{C}$:



Video 2

Laplacetransformexempel $x(t) = e^{2t}u(t)$

$$\underline{X(s)} = \mathcal{L}_I\{x(t)\} = \int_{0-}^{\infty} x(t) e^{-st} dt = \int_{0-}^{\infty} e^{(2-s)t} dt$$

$$= \left[\frac{e^{(2-s)t}}{2-s} \right]_{0-}^{\infty} = \frac{1}{2-s} \left(\lim_{t \to \infty} e^{(2-s)t} - \underbrace{e^{(2-s)\cdot 0}}_{=1} \right)$$

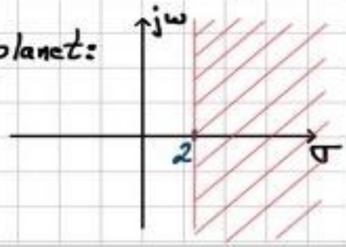
$$= \Big/ \lim_{t \to \infty} e^{(2-s)t} \overset{s = \sigma + j\omega}{=} \lim_{t \to \infty} e^{(2-\sigma)t} \cdot \underbrace{e^{-j\omega t}}_{|e^{j\omega t}| = 1 \ \forall t}$$

$$= 0 \text{ om } 2 - \sigma < 0 \Rightarrow \underline{\sigma > 2} \Big/$$

$$= \frac{0-1}{2-s} = \underline{\underline{\frac{1}{s-2}}}$$

Konvergensområde: $\underline{\mathrm{Re}\{s\} > 2}$

s-planet:



Enkelsidiga laplacetransformen – sammanfattning

Antag $\begin{cases} x(t<0)=0 \\ \int_{-\infty}^{\infty} |x(t)|\,dt = \infty \;\Rightarrow\; \underline{\mathcal{F}\{x(t)\}\;\nexists} \quad \text{(enligt grunddef.)} \end{cases}$

Låt $\tilde{x}(t) = x(t)e^{-\sigma t}$; där $\sigma \in \mathbb{R}$, sådan att $\int_{-\infty}^{\infty} |\tilde{x}(t)|\,dt < \infty \quad \forall \quad \sigma >$ något $\sigma_0 \geq 0$

<u>Följaktligen existerar $\mathcal{F}\{\tilde{x}(t)\}$</u> $\quad$ (dvs. $|\tilde{X}(\omega)| < \infty \;\; \forall \omega$)

$$\Rightarrow \mathcal{F}\{\tilde{x}(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}\,dt = X(\sigma + j\omega) =$$

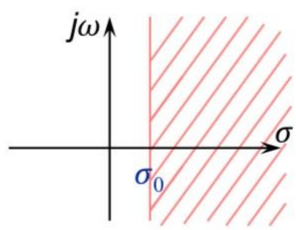
Låt $\boxed{s = \sigma + j\omega}$

$$= \boxed{X_{\mathrm{I}}(s) = \mathcal{L}_{\mathrm{I}}\{x(t)\} = \int_{0-}^{\infty} x(t)e^{-st}\,dt}$$

$X(s) = X_{\mathrm{I}}(s)$: Enkelsidig laplacetransform

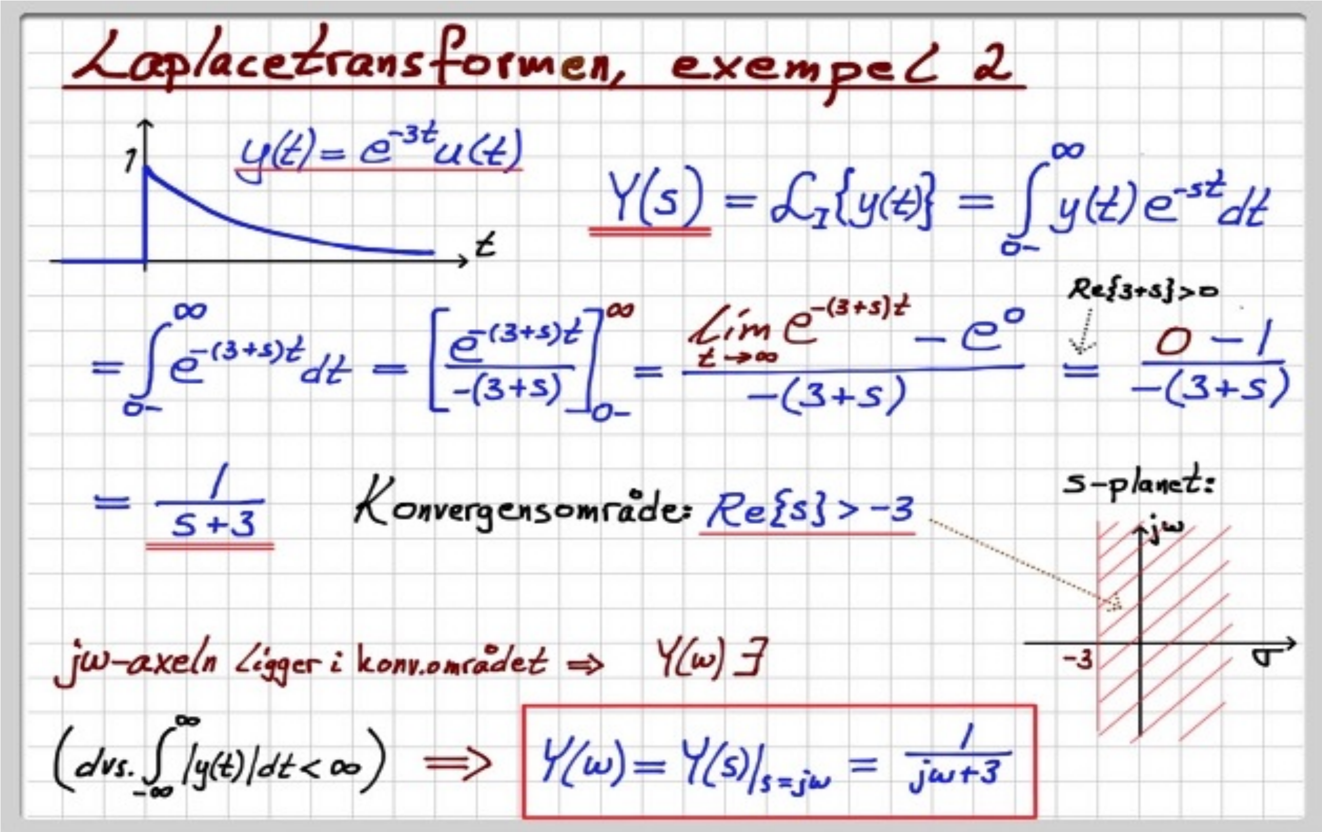
<u>Konvergensområde:</u>

$\sigma = \mathrm{Re}\{s\} > \sigma_0$



OBS! $\begin{cases} \mathcal{F}\{x(t)\}\;\nexists \;\Leftrightarrow\; \sigma_0 > 0 \\ \mathcal{F}\{x(t)\}\;\exists \;\Leftrightarrow\; \sigma_0 < 0 \;\Rightarrow\; \boxed{X(\omega) = X(s)\big|_{s=j\omega}} \end{cases}$

Video 3.1



Laplacetransformen, exempel 2

$y(t) = e^{-3t}u(t)$

$$Y(s) = \mathcal{L}_I\{y(t)\} = \int_{0^-}^{\infty} y(t)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} e^{-(3+s)t}\,dt = \left[\frac{e^{-(3+s)t}}{-(3+s)}\right]_{0^-}^{\infty} = \frac{\lim_{t\to\infty} e^{-(3+s)t} - e^{0}}{-(3+s)} \overset{Re\{3+s\}>0}{=} \frac{0-1}{-(3+s)}$$

$$= \frac{1}{s+3}$$

Konvergensområde: $Re\{s\} > -3$

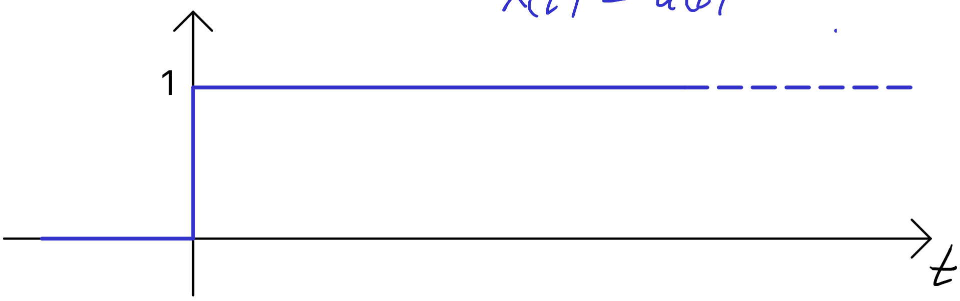
s-planet:

jω-axeln ligger i konv.området $\Rightarrow$ $Y(\omega)\ \exists$

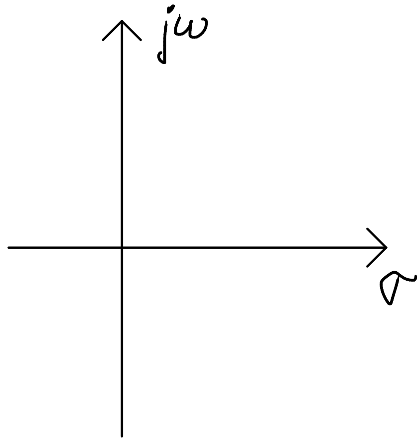
$$\left(\text{dvs. } \int_{-\infty}^{\infty} |y(t)|\,dt < \infty\right) \Rightarrow \boxed{Y(\omega) = Y(s)\big|_{s=j\omega} = \frac{1}{j\omega+3}}$$

$x(t) = u(t)$

$X(s) = ?$



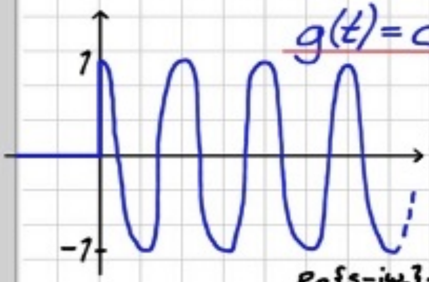
Konvergensområde i s-planet för X(s):



Video 3.2

Laplacetransformen, exempel 3

$g(t)=\cos(\omega_0 t)u(t)$



$$\underline{G(s)} = \int_{0-}^{\infty} g(t)e^{-st}dt =$$

$$\Big/\, g(t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\cdot u(t) \,\Big/ = \frac{1}{2}\int_{0-}^{\infty}\left(e^{-(s-j\omega_0)t}+e^{-(s+j\omega_0)t}\right)dt$$

$Re\{s-j\omega_0\}=Re\{s\}=\sigma>0$ $\quad$ $Re\{s+j\omega_0\}=\sigma>0$

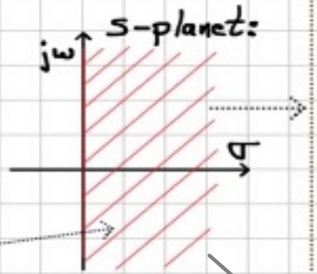
$$=\frac{1}{2}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)}+\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right]_{0-}^{\infty} = \frac{1}{2}\left(\frac{0-e^{0}}{-(s-j\omega_0)}+\frac{0-e^{0}}{-(s+j\omega_0)}\right) = \frac{1}{2}\left(\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right)$$

$$= \underline{\underline{\frac{s}{s^2+\omega_0^2}}}$$

Konvergensområde:

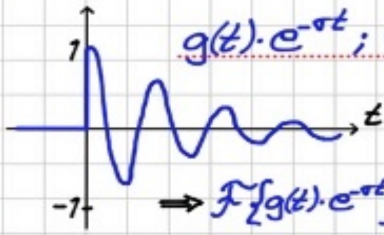
$\underline{Re\{s\}=\sigma>0}$

s-planet:



TOLKNING:

$g(t)\cdot e^{-\sigma t};\ \sigma>0$



$\Rightarrow \mathcal{F}\{g(t)\cdot e^{-\sigma t}\}=\mathcal{L}\{g(t)\}\ \exists$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\Big(\underbrace{\delta(\omega+\omega_0)+\delta(\omega-\omega_0)}\Big)$$

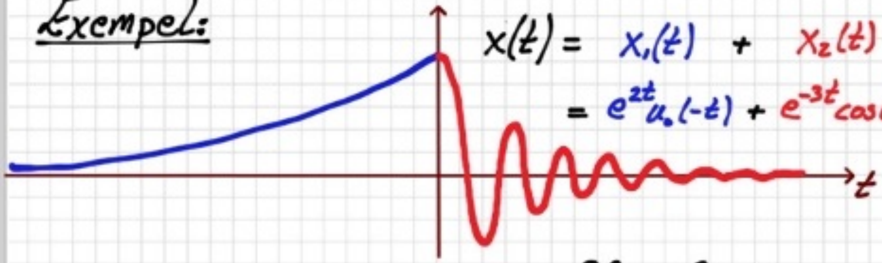
$$\underline{G(\omega)} = vp\left\{G(s)\big|_{s=j\omega}\right\} + \frac{\pi}{2}(\quad)$$

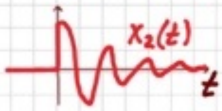
$$\underline{= vp\left\{\frac{j\omega}{\omega_0^2-\omega^2}\right\} + \frac{\pi}{2}\Big(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\Big)}$$

Ett räkneexempel som leder fram till den dubbelsidiga laplacetransformen:

Dubbelsidig Laplacetransform

Exempel:





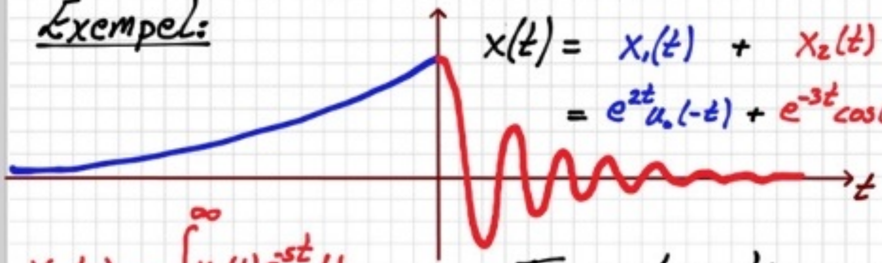
$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$

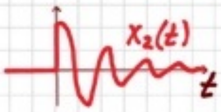
$$\mathcal{L}\{x(t)\} = ?$$

$$\mathcal{L}_{II}\{x(t)\} = X_{II}(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \underbrace{\int_{-\infty}^{0-} x_1(t)e^{-st}\,dt}_{=X_1(s)} + \underbrace{\int_{0-}^{\infty} x_2(t)e^{-st}\,dt}_{=X_2(s)}$$

Dubbelsidig Laplacetransform

Exempel:





$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$

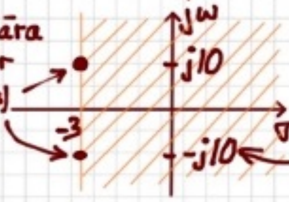
$$X_2(s) = \int_{0-}^{\infty} x_2(t)e^{-st}\,dt$$

Formelsamlingen, Tab. 5:20:

$$e^{-at}\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2} \; ; \; Re\{s\} > -a$$

$$\Rightarrow \boxed{X_2(s) = \frac{s+3}{(s+3)^2+10^2} \; ; \; Re\{s\} > -3} \Longrightarrow$$

Singulära punkter hos $X_2(s)$

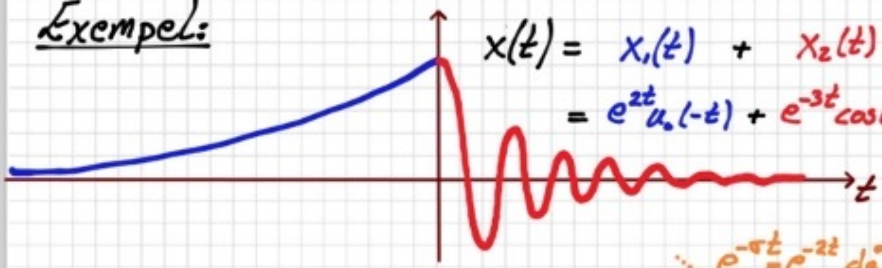


Konvergens-område för $X_2(s)$

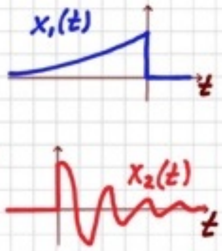
Det komplexa s-planet, $s = \sigma + j\omega$

Dubbelsidig Laplacetransform

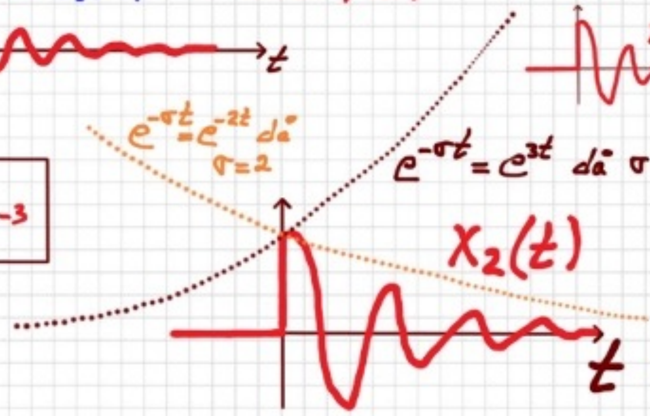
Exempel:

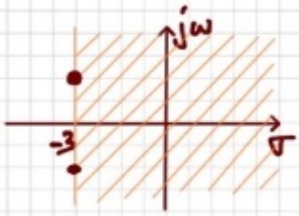


$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



$$\Rightarrow \boxed{X_2(s) = \frac{s+3}{(s+3)^2+10^2} \quad ; \; Re\{s\} > -3}$$

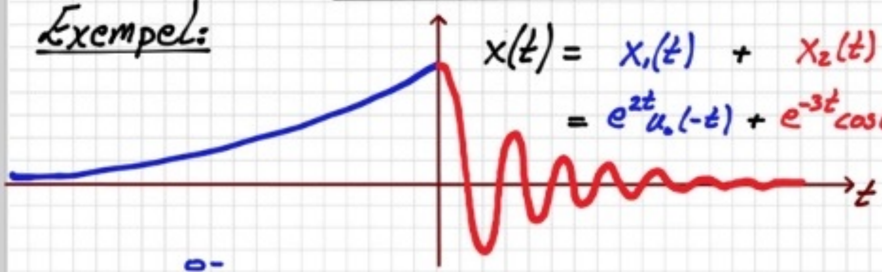




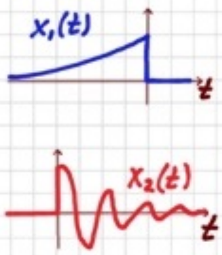
$$\Rightarrow \int_{-\infty}^{\infty} \left| e^{-\sigma t} \cdot x_2(t) \right| dt < \infty \quad \text{för } \sigma > -3$$

Dubbelsidig Laplacetransform

Exempel:



$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



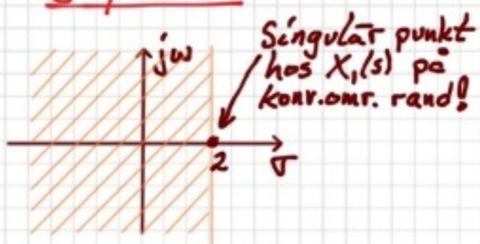
$$X_1(s) = \int_{-\infty}^{0-} x_1(t)e^{-st}dt$$

Formels. Tab. 5:14 : $e^{-at}u_0(-t) \Leftrightarrow \frac{-1}{s+a} \; ; Re\{s\} < -a$

$$\Rightarrow \boxed{\begin{aligned} X_1(s) &= \frac{-1}{s+(-2)} \\ &= \frac{-1}{s-2} \quad ; \; Re\{s\} < 2 \end{aligned}}$$

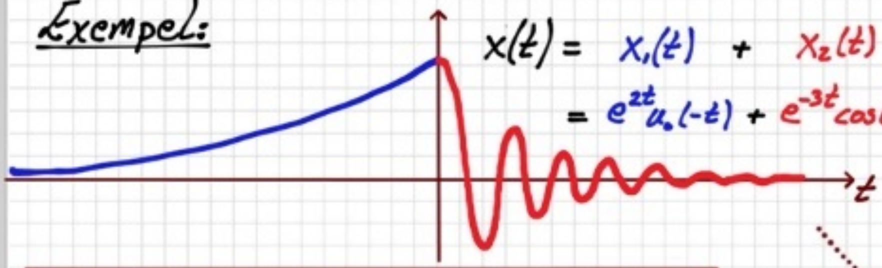
$\Longrightarrow$

s-planet:

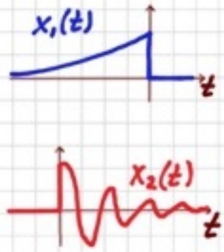


Dubbelsidig Laplacetransform

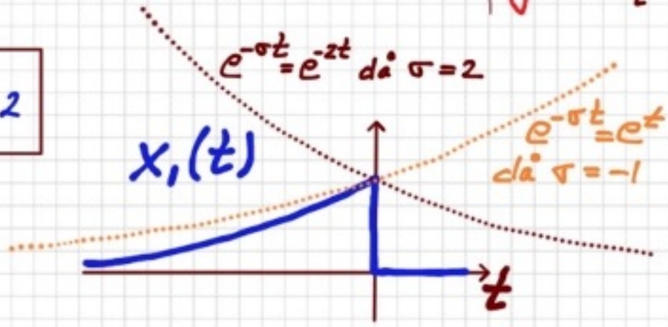
Exempel:

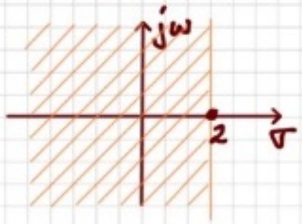


$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



$$\boxed{X_1(s) = \frac{-1}{s+(-2)} = \frac{-1}{s-2} \;;\; \mathrm{Re}\{s\} < 2}$$

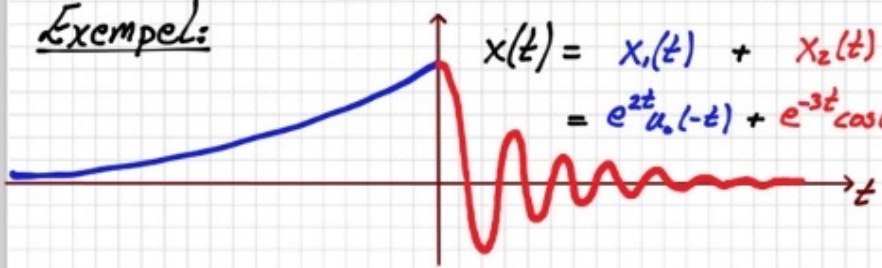




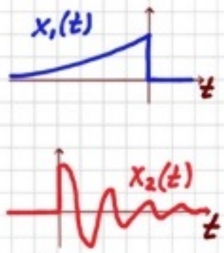
$$\Rightarrow \int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x_1(t)|\,dt < \infty \text{ för } \sigma < 2$$

Dubbelsidig Laplacetransform

Exempel:

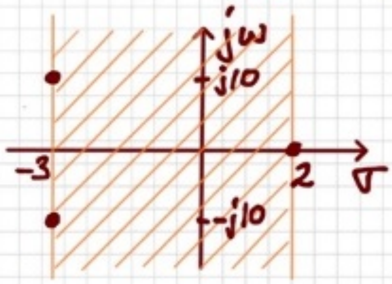


$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



$$\underline{X(s)} = X_1(s) + X_2(s) = \underbrace{\frac{-1}{s-2}}_{\mathrm{Re}\{s\}<2} + \underbrace{\frac{s+3}{(s+3)^2+10^2}}_{\mathrm{Re}\{s\}>-3}$$

$$= \underline{\frac{-5(s+23)}{(s-2)\left((s+3)^2+10^2\right)}} = \underline{\frac{-(5s+115)}{s^3+4s^2+97s-218}}$$



jω-axeln ligger i konv.omr. $\Rightarrow \mathcal{F}\{x(t)\}\ \exists$:

$$X(\omega) = X(s)\big|_{s=j\omega}$$

Konvergensområde för $X(s)$: $\underline{-3 < \mathrm{Re}\{s\} < 2}$

Dubbelsidiga laplacetransformen – sammanfattning

Låt $x(t)$ $\exists\ \forall t$ och låt $\mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$ $\exists$ för något reellt σ i intervallet $\sigma_0 < \sigma < \sigma_1$:

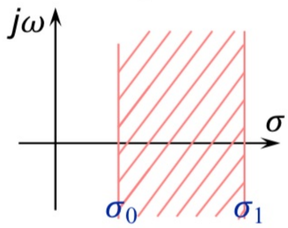
$$\boxed{X_{\mathrm{II}}(s) = \mathcal{L}_{\mathrm{II}}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt}$$

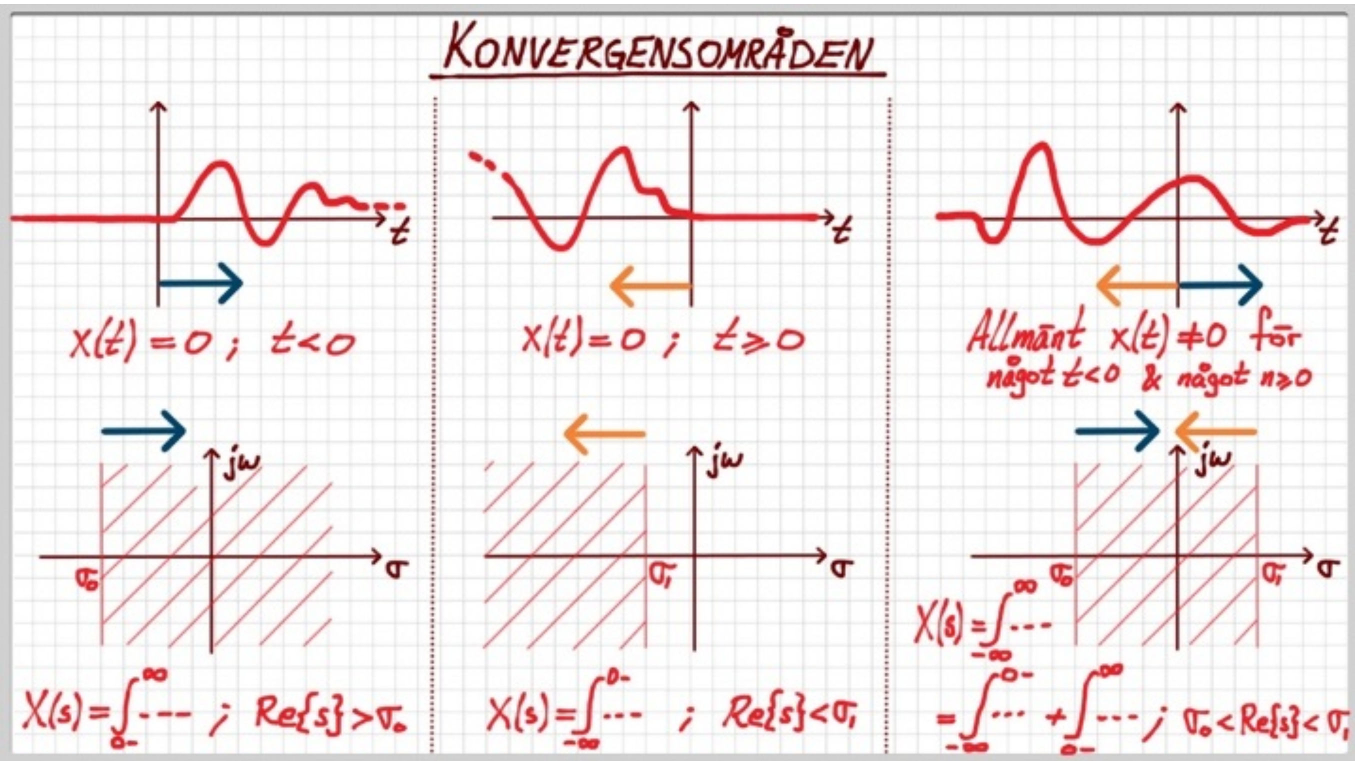
Dubbelsidig laplacetransform

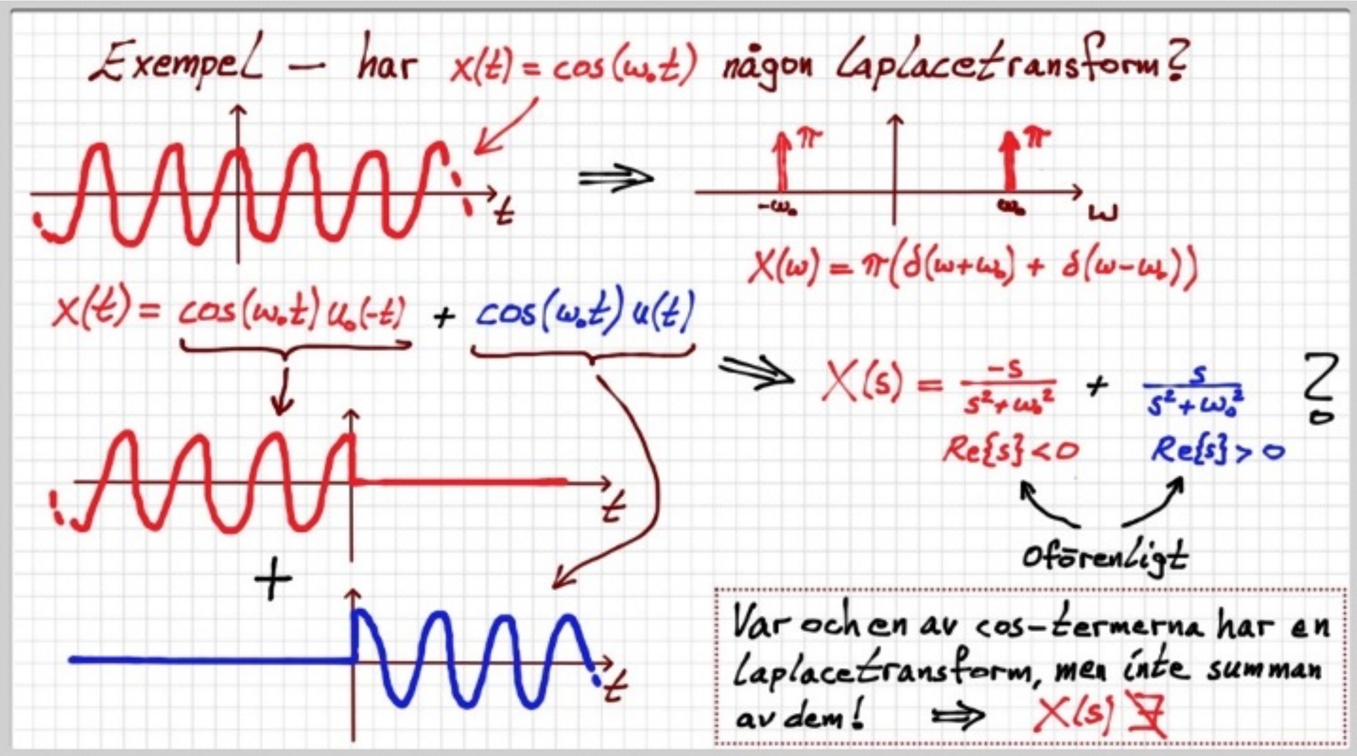
Konvergensområdet för $X_{\mathrm{II}}(s)$: $\sigma_0 < \sigma = \mathrm{Re}\left\{s\right\} < \sigma_1$

(OBS! Om $\mathcal{F}\left\{x(t)\right\}$ $\exists$ $\Leftrightarrow$
$j\omega$-axeln ligger i konvergensområdet för $X(s)$!)

s-planet:

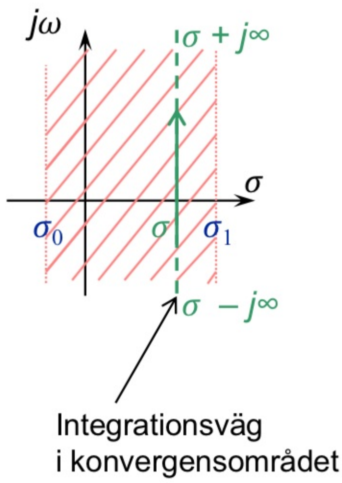






Den inversa laplacetransformen

$$\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds}$$



*I kursen erhåller vi ofta (oftast) transformer och deras inverser från någon **laplacetransformtabell!***

Samma inverstransformsamband för den dubbelsidiga laplacetransformen som för den enkelsidiga!

***OBS:** Laplacetransformanalys finns **i Kap. 6**, men: Definition av laplacetransform, dess existensvillkor samt bevis av olika viktiga transformegenskaper finns i **Appendix D**!!*