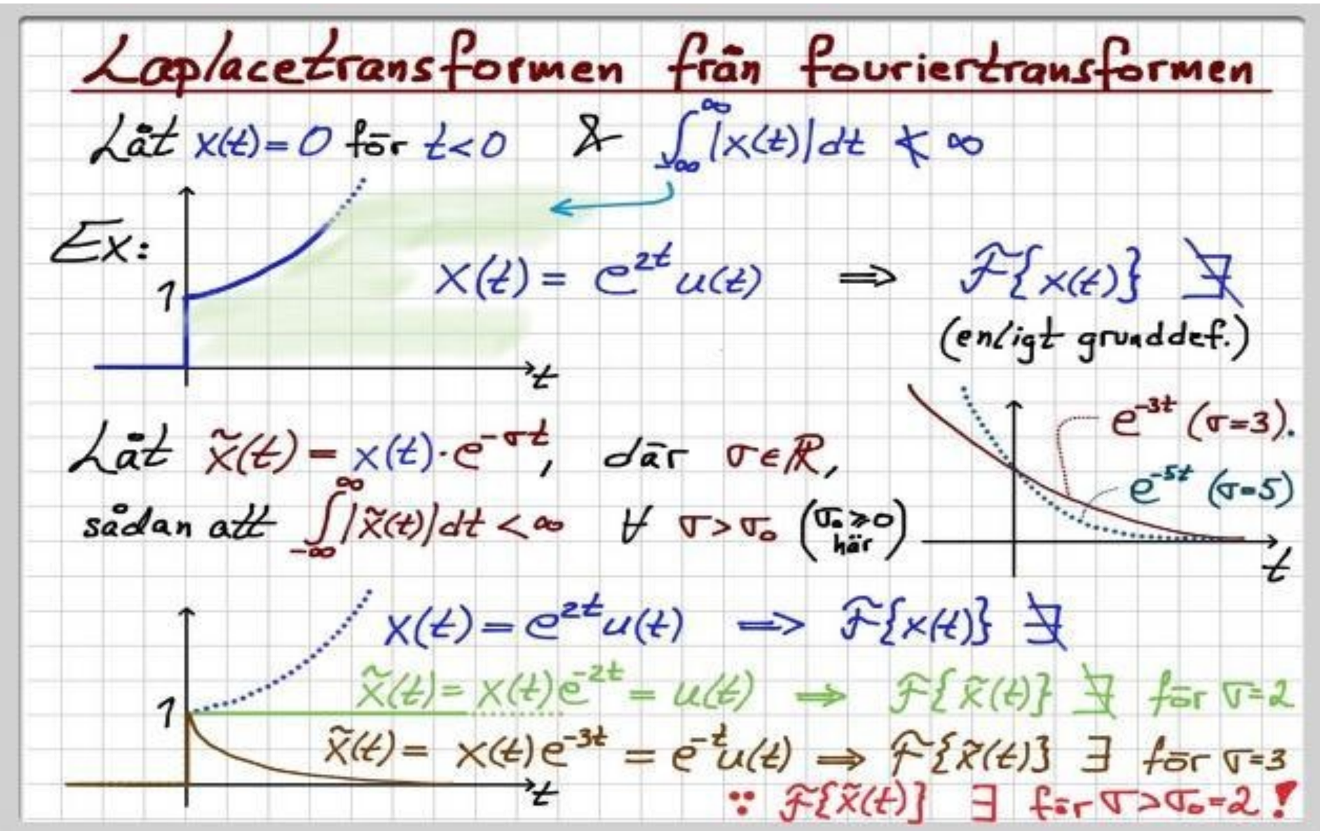
Förberedande videor

- Video 1: Härledning av laplacetransformen från fouriertransformen
- Den enkelsidiga laplacetransformen
- Laplacetransformens konvergensområde
- Video 2: Exempel 1 – laplacetransformen av $x(t) = e^{2t}u(t)$
- Video 3: Exempel 2: Laplacetransformen av $y(t) = e^{-3t}u(t)$
  Exempel 3: Laplacetransformen av $g(t) = \cos(\omega_0 t)u(t)$

Föreläsningsfokus

- Den *enkelsidiga* laplacetransformen, konvergensområde
- Fouriertransformen = laplacetransformen längs $j\omega$-axeln
- Den *dubbelsidiga* laplacetransformen, konvergensområden
  - Räkneexempel: Laplacetransformen av $x(t) = e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$
- Den inversa laplacetransformen



Video 1.1

dvs $\mathcal{L}\{x(t)\} \exists$ för $\sigma > 2$

$\uparrow$

$Re\{s\}$

$s = \sigma + j\omega$

Laplacetransformen från fouriertransformen

Video 1.2

$$\Rightarrow \mathcal{F}\{\tilde{x}(t)\} = \int_{-\infty}^{\infty} \tilde{x}(t)\, e^{-j\omega t}\, dt = \Big/ \begin{matrix}\tilde{x}(t)=x(t)e^{-\sigma t} \\ x(t)=0 \text{ för } t<0\end{matrix} \Big/ = \int_{0-}^{\infty} x(t)\, e^{-\sigma t} e^{-j\omega t}\, dt$$

$$= \int_{0-}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt = \Big/ \boxed{s=\sigma+j\omega} \Big/$$

$$= \int_{0-}^{\infty} x(t)\, e^{-st}\, dt = X(s)$$

(Jämför med $X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$)

Den enkelsidiga Laplacetransformen till $x(t)$ (där $x(t)=0$ för $t<0$):

$$\boxed{X(s) = \mathcal{L}_I\{x(t)\} = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt}$$

$X_I(s) =$

Existensvillkor, $|X(s)|<\infty$

Konvergensområde:

$\sigma = Re\{s\} > \sigma_0$

s-planet, $\mathbb{C}$:

$X(s_0) = \dots$ $\qquad s_0 = -12+j2 \qquad s=\sigma+j\omega$

Laplacetransformen är bara fullständigt definierad tillsammans med sitt konvergensområde

Laplacetransformexempel $\quad x(t)=e^{2t}u(t)$

Video 2

$$X(s) = \mathcal{L}_I\{x(t)\} = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt = \int_{0-}^{\infty} e^{(2-s)t}\, dt$$

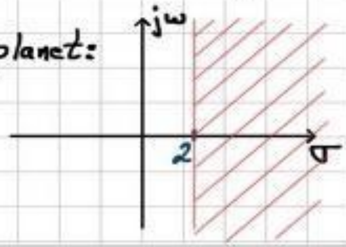
$$= \left[\frac{e^{(2-s)t}}{2-s}\right]_{0-}^{\infty} = \frac{1}{2-s}\left(\lim_{t\to\infty} e^{(2-s)t} - \underbrace{e^{(2-s)\cdot 0}}_{=1}\right)$$

$$= \Big/ \lim_{t\to\infty} e^{(2-s)t} \overset{s=\sigma+j\omega}{=} \lim_{t\to\infty} e^{(2-\sigma)t}\cdot \underbrace{e^{-j\omega t}}_{|e^{j\omega t}|=1\ \forall t} = 0 \text{ om } 2-\sigma<0 \Rightarrow \sigma>2 \Big/$$

$$= \frac{0-1}{2-s} = \frac{1}{s-2}$$

Konvergensområde: $Re\{s\}>2$

s-planet:

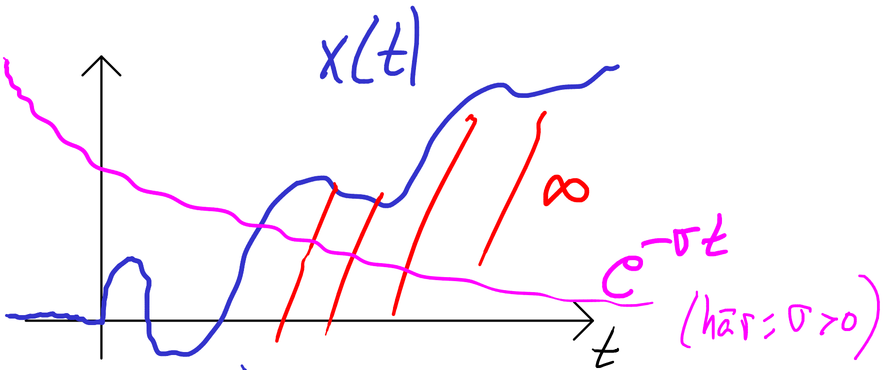


Enkelsidiga laplacetransformen – sammanfattning

Antag $\begin{cases} x(t<0)=0 \\ \int_{-\infty}^{\infty} |x(t)|\,dt = \infty \;\Rightarrow\; \underline{\mathcal{F}\{x(t)\} \nexists} \end{cases}$ (enligt grunddef.)

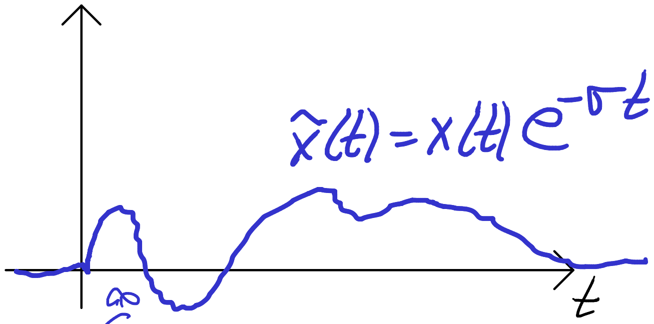
Låt $\tilde{x}(t) = x(t)e^{-\sigma t}$; där $\sigma \in \mathbb{R}$, sådan att $\int_{-\infty}^{\infty} |\tilde{x}(t)|\,dt < \infty \quad \forall \quad \sigma >$ något $\sigma_0 \geq 0$

<u>Följaktligen existerar $\mathcal{F}\{\tilde{x}(t)\}$</u> $\left(\text{dvs. } |\tilde{X}(\omega)| < \infty \;\; \forall\omega\right)$



$X(\omega) \nexists$

fy $\int_{-\infty}^{\infty} |x(t)|\,dt = \infty$



$\int_{-\infty}^{\infty} |\tilde{x}(t)|\,dt < \infty$ för $\sigma > \sigma_0$

$\Rightarrow \mathcal{F}\{\tilde{x}(t)\} = \mathcal{L}\{x(t)\} = X(s)$

$$\Rightarrow \mathcal{F}\{\tilde{x}(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = X(\sigma + j\omega) =$$

Låt $\boxed{s = \sigma + j\omega}$

$$= \boxed{X_{\mathrm{I}}(s) = \mathcal{L}_{\mathrm{I}}\{x(t)\} = \int_{0-}^{\infty} x(t)e^{-st}\,dt}$$

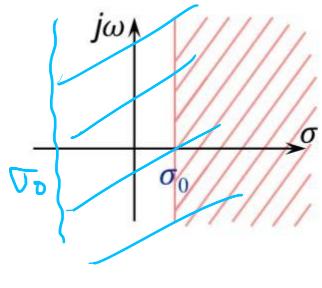
$e^{-j\omega t}$

$\sigma = 0$

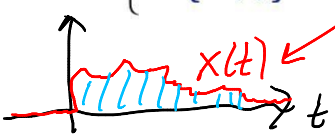
$X(s) = X_{\mathrm{I}}(s)$: Enkelsidig laplacetransform

<u>Konvergensområde:</u>

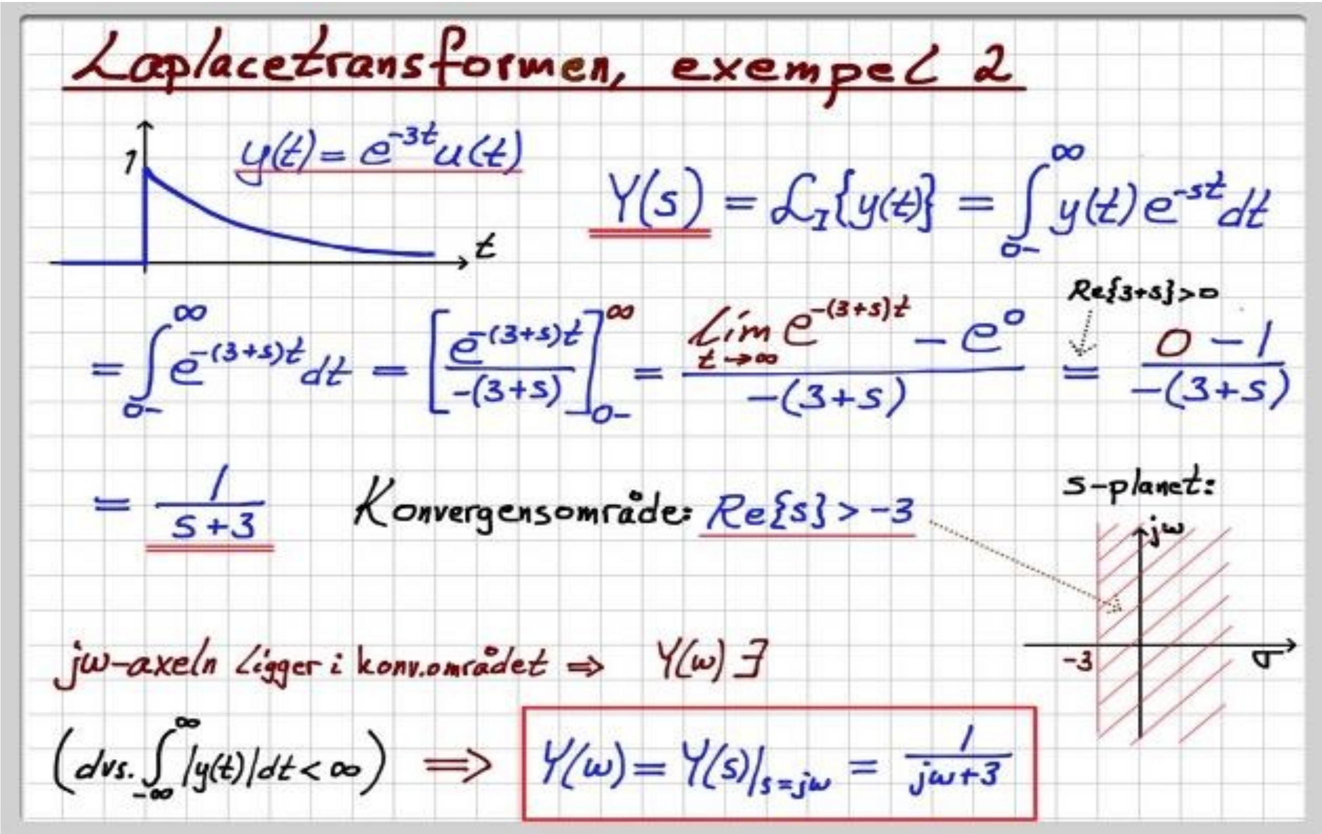
$\sigma = \mathrm{Re}\{s\} > \sigma_0$



OBS! $\begin{cases} \mathcal{F}\{x(t)\} \nexists \;\Leftrightarrow\; \sigma_0 > 0 \\ \mathcal{F}\{x(t)\} \exists \;\Leftrightarrow\; \sigma_0 < 0 \;\Rightarrow\; \boxed{X(\omega) = X(s)\big|_{s=j\omega}} \end{cases}$



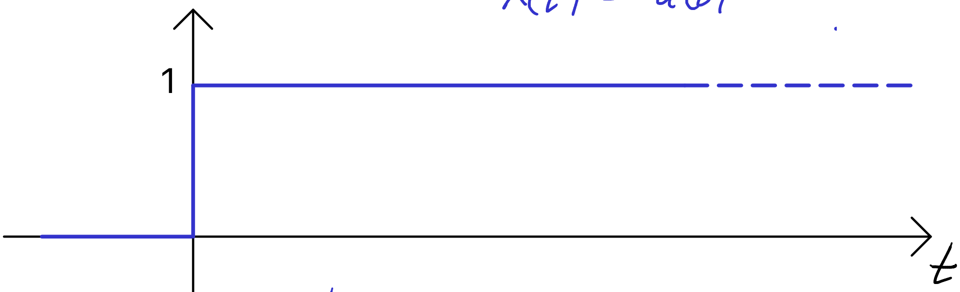
Video 3.1



$$e^{-at}u(t) \Longleftrightarrow \frac{1}{s+a}\ ;\ Re\{s\} > -a$$

t.ex. $e^{-3t}u(t) \Longleftrightarrow \frac{1}{s+3}\ ;\ Re\{s\}>-3$

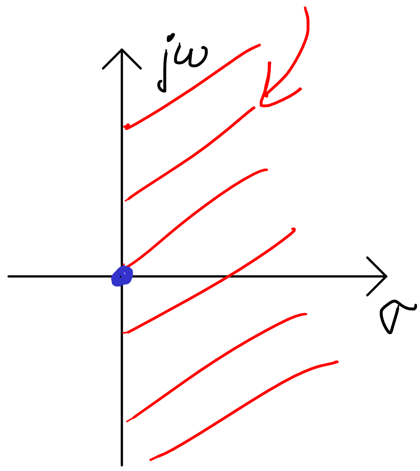
$$x(t) = u(t) \qquad X(s) = ?$$



$$x(t) = u(t) = e^{-at}\cdot u(t) \text{ för } a=0$$

$$\Rightarrow X(s) = \frac{1}{s+a} = \frac{1}{s}\ ;\ Re\{s\} > 0$$

Konvergensområde i s-planet för X(s):



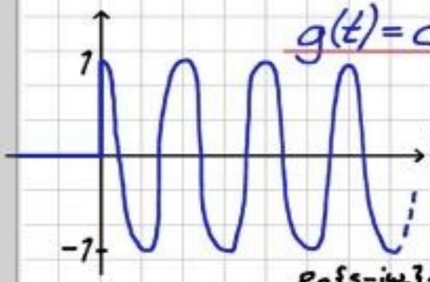
$$X(\omega) \neq X(s)\Big|_{s=j\omega}$$

$$X(\omega) = vp\left\{\frac{1}{j\omega}\right\} + \pi\delta(\omega)$$

formels.

Video 3.2

Laplacetransformen, exempel 3



$g(t) = \cos(\omega_0 t)\, u(t)$

$$\underline{G(s)} = \int_{0-}^{\infty} g(t)\, e^{-st}\, dt = \Big/ \, g(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot u(t) \, \Big/ = \frac{1}{2}\int_{0-}^{\infty} \left(e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t} \right) dt$$

$Re\{s - j\omega_0\} = Re\{s\} = \sigma > 0$ $\qquad$ $Re\{s + j\omega_0\} = \sigma > 0$

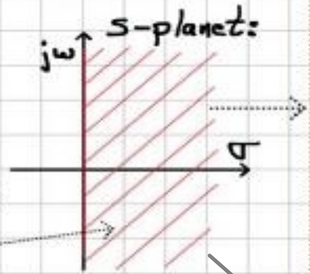
$$= \frac{1}{2}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_{0-}^{\infty} = \frac{1}{2}\left(\frac{0 - e^{0}}{-(s-j\omega_0)} + \frac{0 - e^{0}}{-(s+j\omega_0)} \right) = \frac{1}{2}\left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right)$$

$$= \underline{\underline{\frac{s}{s^2 + \omega_0^2}}}$$

Konvergensområde:

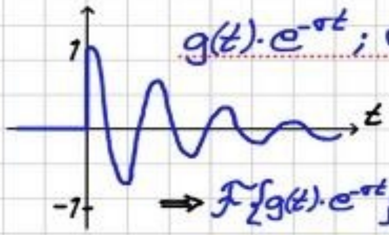
$\underline{Re\{s\} = \sigma > 0}$

s-planet:



TOLKNING:

$g(t) \cdot e^{-\sigma t}\, ; \ \sigma > 0$



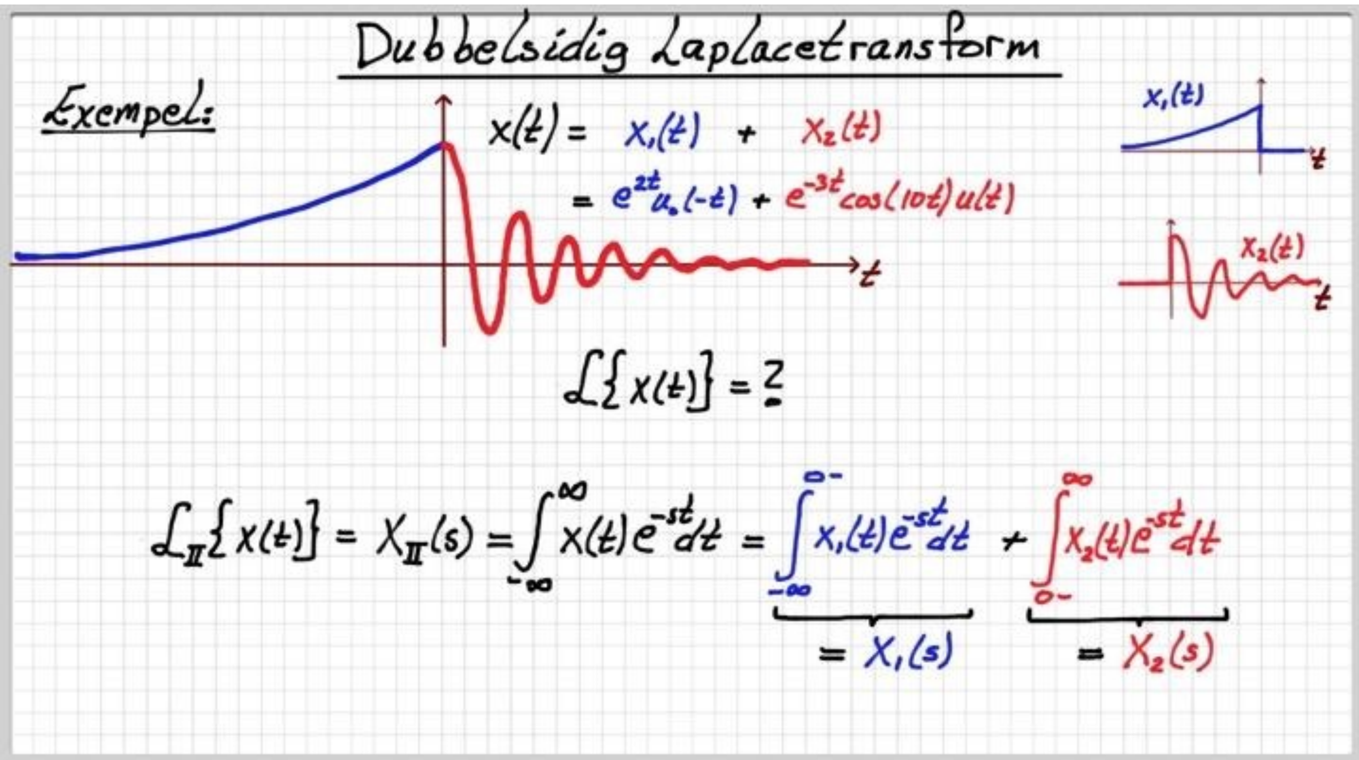
$\Rightarrow \mathcal{F}\{g(t) \cdot e^{-\sigma t}\} = \mathcal{L}\{g(t)\} \ \exists$

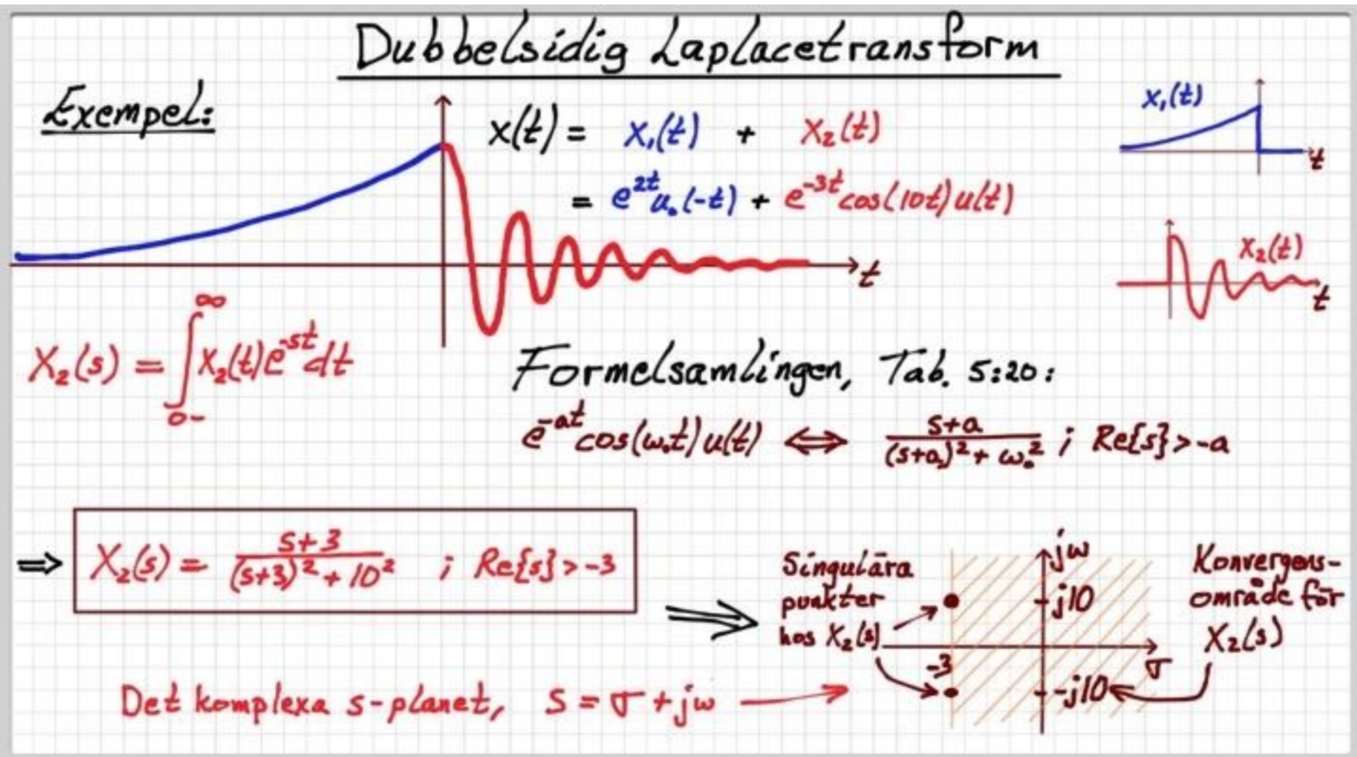
$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi \underbrace{\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right)}$$

$$\underline{G(\omega)} = vp\left\{ G(s)\big|_{s = j\omega} \right\} + \frac{\pi}{2}(\quad)$$

$$= \underline{vp\left\{ \frac{j\omega}{\omega_0^2 - \omega^2} \right\} + \frac{\pi}{2}\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right)}$$

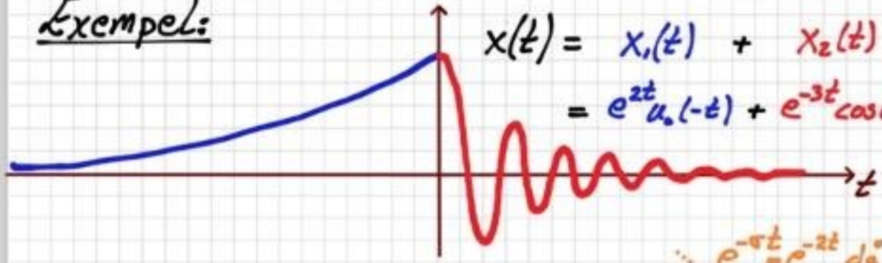
Ett räkneexempel som leder fram till den dubbelsidiga laplacetransformen:



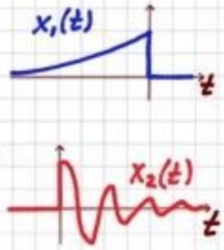


Dubbelsidig Laplacetransform

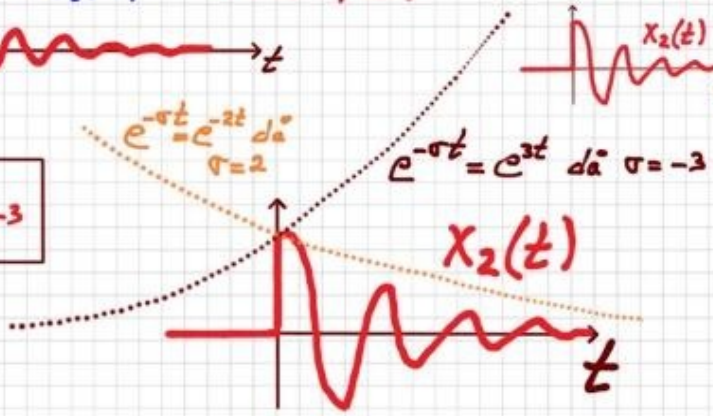
Exempel:

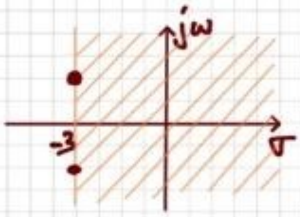


$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



$$\Rightarrow \boxed{X_2(s) = \frac{s+3}{(s+3)^2+10^2} \quad ; \ Re\{s\} > -3}$$

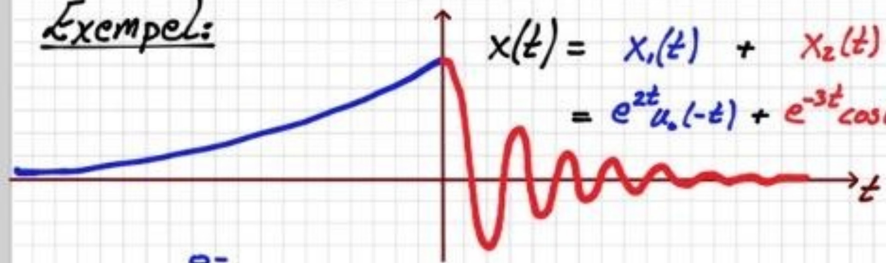




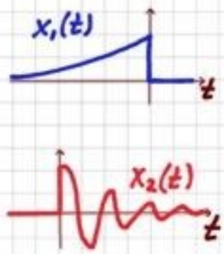
$$\Rightarrow \int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x_2(t)|\, dt < \infty \quad \text{för } \sigma > -3$$

Dubbelsidig Laplacetransform

Exempel:



$$x(t) = x_1(t) + x_2(t)$$
$$= e^{2t}u_0(-t) + e^{-3t}\cos(10t)u(t)$$



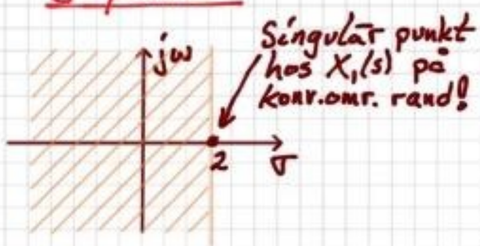
$$X_1(s) = \int_{-\infty}^{0^-} x_1(t)e^{-st}\, dt$$

Formels. Tab. 5:14: $e^{-at}u_0(-t) \Leftrightarrow \frac{-1}{s+a}$; $Re\{s\} < -a$

$$\Rightarrow \boxed{X_1(s) = \frac{-1}{s+(-2)} = \frac{-1}{s-2} \quad ; \ Re\{s\} < 2}$$

$\Longrightarrow$

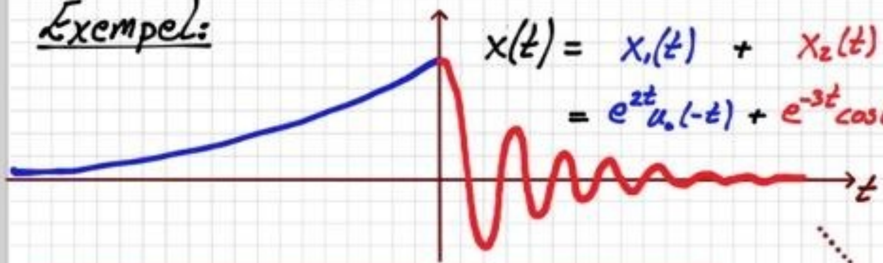
s-planet:

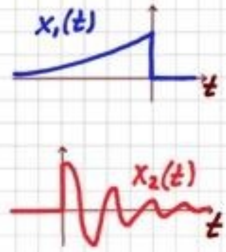


Dubbelsidig Laplacetransform

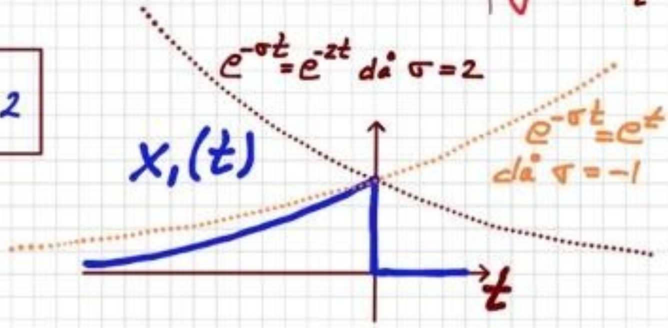
Exempel:

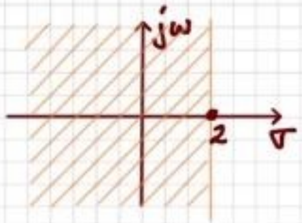
$$x(t) = x_1(t) + x_2(t) = e^{2t}u_0(-t) + e^{-3t}\cos(10t)\,u(t)$$





$$\boxed{X_1(s) = \frac{-1}{s+(-2)} = \frac{-1}{s-2} \; ; \; Re\{s\} < 2}$$



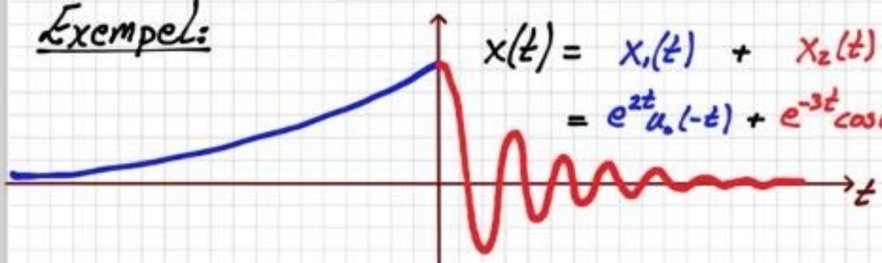


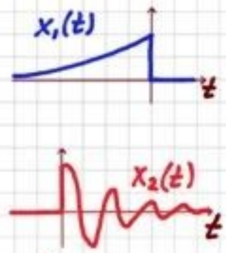
$$\Rightarrow \int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x_1(t)|\,dt < \infty \text{ för } \sigma < 2$$

Dubbelsidig Laplacetransform

Exempel:

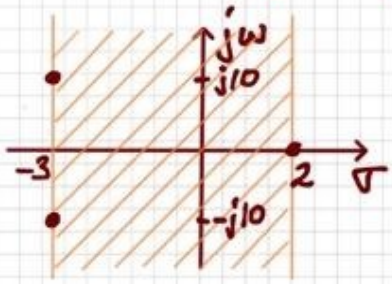
$$x(t) = x_1(t) + x_2(t) = e^{2t}u_0(-t) + e^{-3t}\cos(10t)\,u(t)$$





$$\underline{X(s)} = X_1(s) + X_2(s) = \underbrace{\frac{-1}{s-2}}_{Re\{s\}<2} + \underbrace{\frac{s+3}{(s+3)^2+10^2}}_{Re\{s\}>-3}$$

$$= \underline{\frac{-5(s+23)}{(s-2)((s+3)^2+10^2)}} = \underline{\frac{-(5s+115)}{s^3+4s^2+97s-218}}$$



jω-axeln ligger i konv.omr. $\Rightarrow \mathcal{F}\{x(t)\}\ \exists$:

$$X(\omega) = X(s)\big|_{s=j\omega}$$

Konvergensområde för $X(s)$: $\underline{-3 < Re\{s\} < 2}$

Dubbelsidiga laplacetransformen – sammanfattning

Låt $x(t)$ $\exists\ \forall t$ och låt $\mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$ $\exists$ för något reellt σ i intervallet $\sigma_0 < \sigma < \sigma_1$:

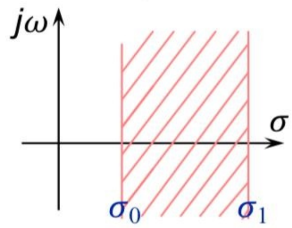
$$\boxed{X_{\mathrm{II}}(s) = \mathcal{L}_{\mathrm{II}}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt}$$

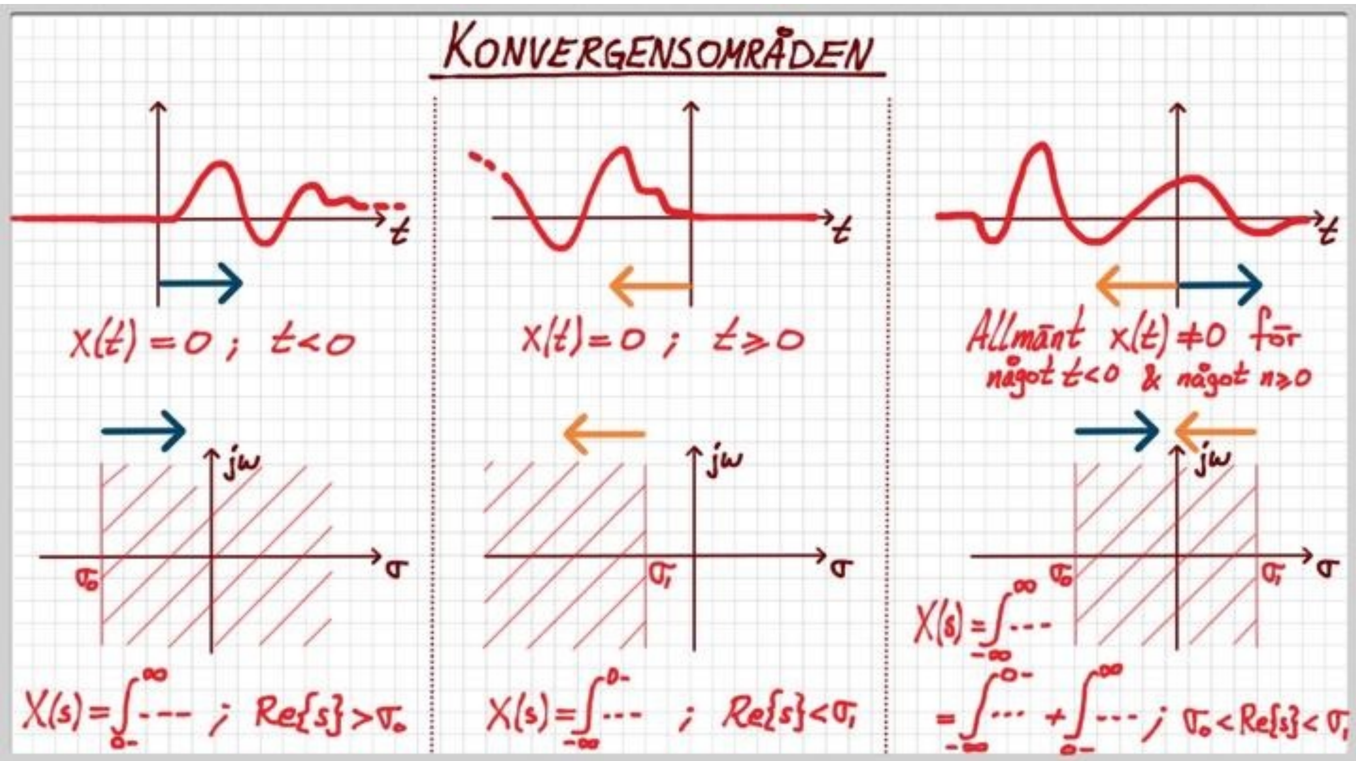
Dubbelsidig laplacetransform

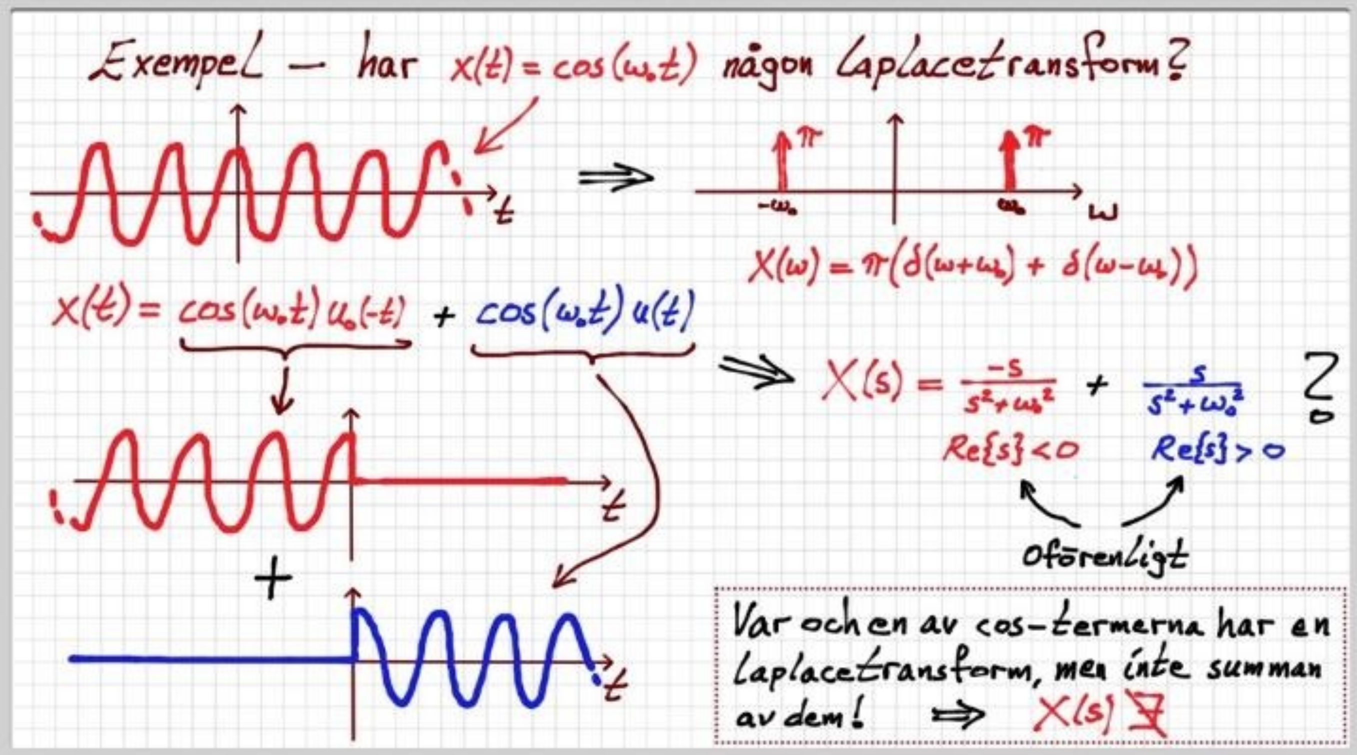
Konvergensområdet för $X_{\mathrm{II}}(s)$: $\sigma_0 < \sigma = \mathrm{Re}\left\{s\right\} < \sigma_1$

(OBS! Om $\mathcal{F}\left\{x(t)\right\}$ $\exists$ $\Leftrightarrow$
$j\omega$-axeln ligger i konvergensområdet för $X(s)$!)

s-planet:

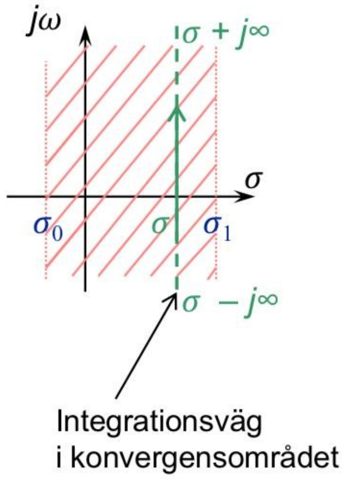






Den inversa laplacetransformen

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}\,ds$$



I kursen erhåller vi ofta (oftast) transformer och deras inverser från någon ***laplacetransformtabell!***

Samma inverstransformsamband för den dubbelsidiga laplacetransformen som för den enkelsidiga!

OBS: *Laplacetransformanalys finns* ***i Kap. 6****, men:*
Definition av laplacetransform, dess existensvillkor samt bevis av olika viktiga transformegenskaper finns i ***Appendix D****!!*