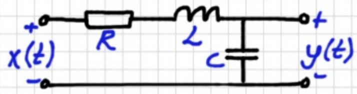
(Större) Räkneexempel – Systemanalys med fouriertransformen



$R = 1\,\Omega$
$L = 1\,H$
$C = 1\,F$

a) Beräkna/bestäm LTI-systemets
- frekvensfunktion $H(\omega)$
- impulssvar $h(t)$, samt skissera detta
- kausalitetsegenskap & stabilitetsegenskap
- systembeskrivande differentialekvation
- ordning
- amplitudkaraktäristik $|H(\omega)|$

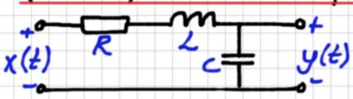
b) Skissera amplitudkaraktäristiken och ange/motivera vilken typ av frekvens-selektivt filter (LP, BP, HP osv.) det elektriska systemet utgör.

c) Beräkna utsignalen $y_1(t)$ då insignalen är $x_1(t) = 3 \cdot e^{-2t} u(t)$ [V] och systemet är energifritt.

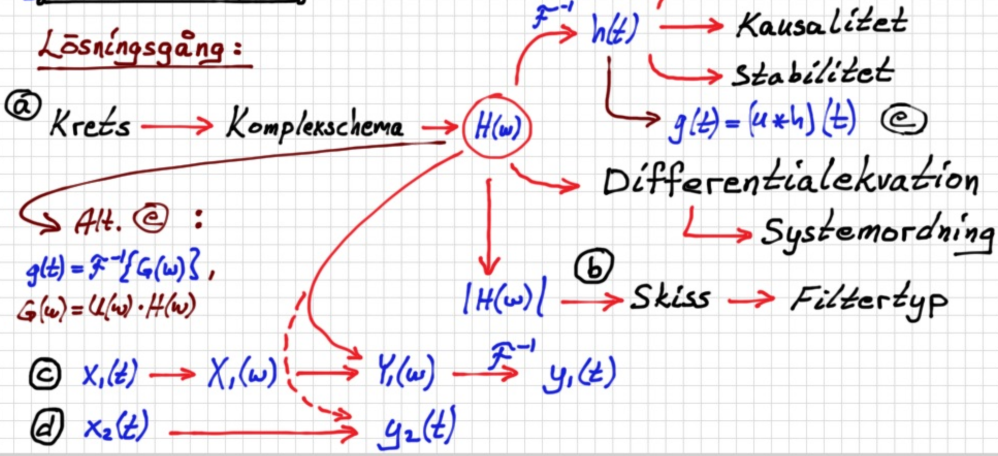
d) Beräkna utsignalen $y_2(t)$ då insignalen är $x_2(t) = 4 + 3\cos(2t + \frac{3\pi}{4})$ [V]

e) Beräkna stegsvaret $g(t)$

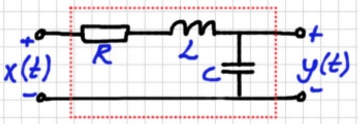
(Större) Räkneexempel – Systemanalys med fouriertransformen



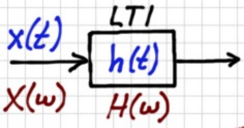
$R = 1\,\Omega$
$L = 1\,H$
$C = 1\,F$



(Större) Räkneexempel – Systemanalys med fouriertransformen



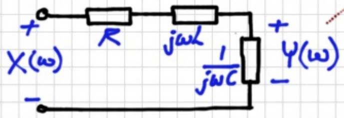
$\Longrightarrow$



$y_{zs}(t) = (x * h)(t)$

$Y_{zs}(\omega) = X(\omega)\cdot H(\omega)$

a) Komplexschema:

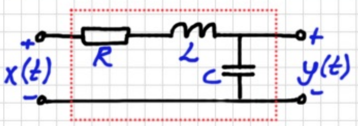


Spänningsdelning $\Rightarrow$

$$Y(\omega) = X(\omega)\cdot \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + (R + j\omega L)}$$

$$\Longrightarrow \underline{H(\omega)} = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC + j^2\omega^2 LC} \underset{R=1\Omega,\ L=1H,\ C=1F}{=} \underline{\frac{1}{1-\omega^2+j\omega}}$$

(Större) Räkneexempel – Systemanalys med fouriertransformen



a) $H(\omega) = \frac{1}{1-\omega^2+j\omega}$

$h(t) = ?$ $\quad h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega$ Svår att beräkna här!

Sök istället transformpar i formelsamlingen!

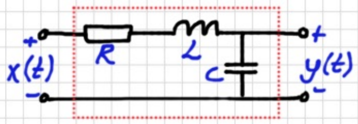
Nämnaren i $H(\omega)$: $(j\omega)^2 + j\omega + 1 = (j\omega + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2$

Formels. Tab. 3:11 $\Rightarrow$ $e^{-at}\sin(\omega_0 t)u(t) \Longleftrightarrow \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$

Här: $\begin{cases} a = \frac{1}{2} \\ \omega_0 = \frac{\sqrt{3}}{2} \text{ rad/s} \end{cases}$

Dvs. $H(\omega) = \frac{2}{\sqrt{3}}\cdot\frac{1\cdot\frac{\sqrt{3}}{2}}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \Longrightarrow \underline{h(t) = \frac{2}{\sqrt{3}}\cdot e^{-\frac{t}{2}}\sin(\frac{\sqrt{3}}{2}t)u(t)}$

(Större) Räkneexempel – Systemanalys med fouriertransformen



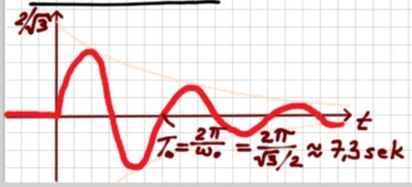
a) $H(\omega) = \frac{1}{1-\omega^2+j\omega}$

Formels. Tab. 3:11 $\Rightarrow$ $e^{-at}\sin(\omega_0 t)u(t) \Longleftrightarrow \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$

Här: $\begin{cases} a = \frac{1}{2} \\ \omega_0 = \frac{\sqrt{3}}{2} \\ \text{rad/s} \end{cases}$

Dvs. $H(\omega) = \frac{2}{\sqrt{3}} \cdot \frac{1 \cdot \frac{\sqrt{3}}{2}}{(j\omega+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow \underline{h(t) = \frac{2}{\sqrt{3}} \cdot e^{-\frac{t}{2}} \sin(\frac{\sqrt{3}}{2}t)u(t)}$

Skissera $h(t)$:

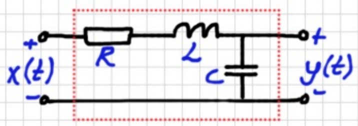


$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{3}/2} \approx 7{,}3$ sek

- $h(t) = 0$ för $t<0$ $\Rightarrow$ Systemet är kausalt
- $\int_{-\infty}^{\infty} |h(t)|\,dt \;\left(= \frac{2}{\sqrt{3}} \cdot \frac{1}{1/2} = \frac{4}{\sqrt{3}}\right) < \infty$

$\Rightarrow$ Systemet är stabilt

(Större) Räkneexempel – Systemanalys med fouriertransformen



a) $H(\omega) = \frac{1}{1-\omega^2+j\omega}$

$h(t) = \frac{2}{\sqrt{3}} \cdot e^{-\frac{t}{2}} \sin(\frac{\sqrt{3}}{2}t)u(t)$

Kausalt & stabilt LTI-system

Differentialekvationsbeskrivning?

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + j\omega + 1} \Rightarrow (j\omega)^2 \cdot Y(\omega) + j\omega \cdot Y(\omega) + Y(\omega) = X(\omega)$

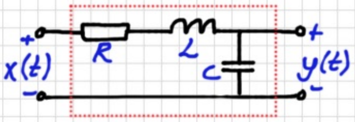
$\mathcal{F}^{-1}\{VL\} = \mathcal{F}^{-1}\{HL\}$, med $\frac{d^n y(t)}{dt^n} \Longleftrightarrow (j\omega)^n \cdot Y(\omega) \Rightarrow$

$$\boxed{\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)}$$

Systemets ordning = differentialekvationens ordning = $\underline{2}$

(= antalet reaktiva nätelement, L & C)

(Större) Räkneexempel – Systemanalys med fouriertransformen



a) $H(\omega) = \dfrac{1}{1-\omega^2+j\omega}$

$h(t) = \frac{2}{\sqrt{3}} \cdot e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) u(t)$

$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$

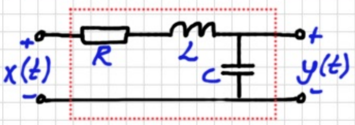
Kausalt & stabilt LTI-system

Ordning = 2

Beräkna systemets amplitudkaraktäristik $|H(\omega)|$

$$|H(\omega)| = \left|\frac{1}{1-\omega^2+j\omega}\right| = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}$$

(Större) Räkneexempel – Systemanalys med fouriertransformen



a) $H(\omega) = \dfrac{1}{1-\omega^2+j\omega}$

$h(t) = \frac{2}{\sqrt{3}} \cdot e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) u(t)$

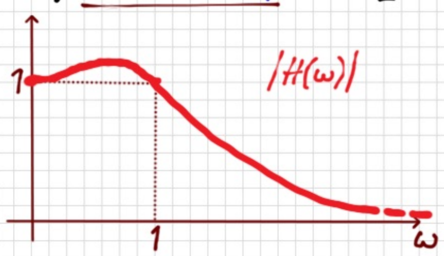
$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$

Kausalt & stabilt LTI-system

Ordning = 2

$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}$

b) Skissera $|H(\omega)|$ och ange filtertyp.



1) $H(0) = \frac{1}{1-0^2+j0} = 1$

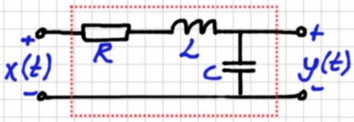
2) $\lim_{\omega\to\infty} |H(\omega)| = \lim_{\omega\to\infty} \left|\frac{\frac{1}{\omega^2}}{\frac{1}{\omega^2}-1+j\frac{1}{\omega}}\right| = \frac{0}{1} = 0$

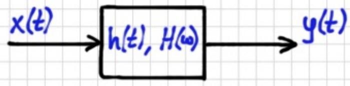
3) $|H(1)| = \left|\frac{1}{0+j}\right| = 1$

LP-filter!

(Examinatorn motiverar muntligen)

(Större) Räkneexempel – Systemanalys med fouriertransformen





$$H(\omega)=\frac{1}{1-\omega^2+j\omega}$$

c) $x_1(t)=3\cdot e^{-2t}u(t)$ [V]. Beräkna $y_1(t)$. // $y_1(t)=(x*h)(t) \Rightarrow$

$$Y_1(\omega)=X_1(\omega)H(\omega)=\Big/\text{Formels. Tab. 3:5, } e^{-at}u(t)\Leftrightarrow\frac{1}{a+j\omega}\Big/=\frac{3}{2+j\omega}\cdot\frac{1}{1-\omega^2+j\omega}=\big/\text{P.B.U.}\big/$$

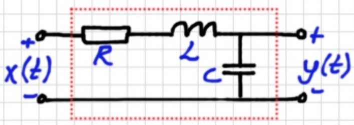
$$=\frac{A}{2+j\omega}+\frac{B(j\omega)+C}{(j\omega)^2+j\omega+1}=\left/\left.\begin{array}{l} A+B=0\\ A+2B+C=0\\ A+2C=3\end{array}\right\}\Rightarrow A=1,\ B=-1,\ C=1\right/$$

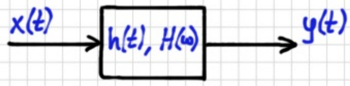
$$=\frac{1}{2+j\omega}+\frac{-j\omega+1}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$\mathcal{F}^{-1}\left\{\frac{1}{2+j\omega}\right\}=e^{-2t}u(t)$ enligt Tab. 3:5

men vad är $\mathcal{F}^{-1}$ av den andra termen?

(Större) Räkneexempel – Systemanalys med fouriertransformen





$$H(\omega)=\frac{1}{1-\omega^2+j\omega}$$

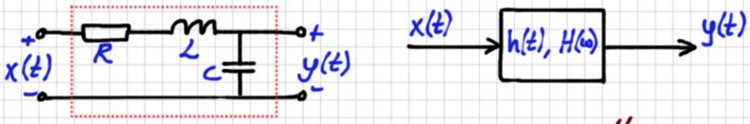
c) $x_1(t)=3\cdot e^{-2t}u(t)$ [V]. Beräkna $y_1(t)$. // $y_1(t)=(x*h)(t) \Rightarrow$

$$Y_1(\omega)=X_1(\omega)H(\omega)=\frac{1}{2+j\omega}+\frac{-j\omega+1}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

Formelsaml. Tab. 3:10 & 3:11 $\Rightarrow$

$$\mathcal{F}\{e^{-at}\cos(\omega_0 t)u(t)\}=\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}\qquad \mathcal{F}\{e^{-at}\sin(\omega_0 t)u(t)\}=\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$$

(Större) Räkneexempel – Systemanalys med fouriertransformen



$$H(\omega) = \frac{1}{1-\omega^2+j\omega}$$

c) $x_1(t) = 3\cdot e^{-2t}u(t)$ [V]. Beräkna $y_1(t)$. // $y_1(t) = (x*h)(t) \Rightarrow$

$$Y_1(\omega) = X_1(\omega)H(\omega) = \frac{1}{2+j\omega} + \frac{-j\omega+1}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \qquad -j\omega+1 = -(j\omega+\tfrac{1}{2}) + \tfrac{3}{2}$$

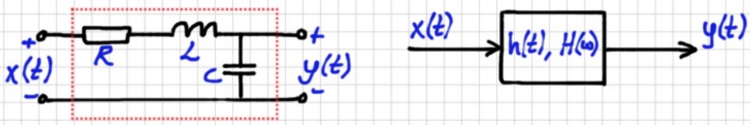
$$= \frac{1}{2+j\omega} - \frac{j\omega+\frac{1}{2}}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{2}{\sqrt{3}}\cdot\frac{3}{2}\,\frac{\frac{\sqrt{3}}{2}}{(j\omega+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

↓Tab 3:5 ↓Tab 3:10 ↓Tab 3:11

$$\Rightarrow y_1(t) = e^{-2t}\cdot u(t) - e^{-\frac{t}{2}}\cos(\tfrac{\sqrt{3}}{2}t)\cdot u(t) + \sqrt{3}\,e^{-\frac{t}{2}}\sin(\tfrac{\sqrt{3}}{2}t)\cdot u(t)$$

$$= \left(e^{-2t} - e^{-\frac{t}{2}}\left(\cos(\tfrac{\sqrt{3}}{2}t) - \sqrt{3}\sin(\tfrac{\sqrt{3}}{2}t)\right)\right)u(t)$$

(Större) Räkneexempel – Systemanalys med fouriertransformen



$$H(\omega) = \frac{1}{1-\omega^2+j\omega}$$

d) $x_2(t) = 4 + 3\cos(2t + \frac{3\pi}{4})$ [V]. Beräkna $y_2(t)$.

Vi vet: För stabilt LTI-system (som här!) med stationär insignal

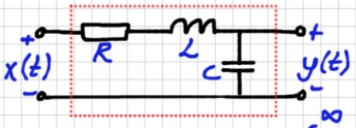
$$x(t) = C_0 + C\cdot \cos/\sin(\omega_0 t) \Rightarrow y(t) = C_0\cdot H(0) + C\cdot|H(\omega_0)|\cdot\cos/\sin(\omega_0 t + \arg H(\omega_0))$$

Här: $$\boxed{y_2(t) = 4\cdot H(0) + 3\cdot|H(2)|\cdot\cos(2t + \tfrac{3\pi}{4} + \arg H(2))}$$

där $H(0) = \frac{1}{1-0^2+j\cdot 0} = 1$ & $H(2) = \frac{1}{1-2^2+j2} = \frac{1}{-3+j2} = \frac{1}{\sqrt{(-3)^2+2^2}\cdot e^{j(\arctan\frac{2}{-3}+\pi)}}$ ↑ Ty neg. realdel

$$= \underbrace{\frac{1}{\sqrt{13}}}_{|H(2)|}\cdot e^{j\underbrace{(\arctan\frac{2}{3}+\pi)}_{\arg H(2)}} \Rightarrow y(t) = \overset{4\cdot 1}{4} + \frac{3}{\sqrt{13}}\cos(2t + \tfrac{3\pi}{4} + \arctan\tfrac{2}{3} + \pi) \text{ [V]}$$

(Större) Räkneexempel – Systemanalys med fouriertransformen



$$h(t)=\frac{2}{\sqrt{3}}\cdot e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t) \qquad H(\omega)=\frac{1}{1-\omega^2+j\omega}$$

e) $g(t)=(u*h)(t)=\int_{-\infty}^{\infty}u(t-\tau)h(\tau)\,d\tau=\int_{-\infty}^{t}h(\tau)\,d\tau=A\int_0^t e^{a\tau}\sin(\omega_0\tau)\,d\tau$ $\quad \begin{cases}A=\frac{2}{\sqrt{3}}\\ a=-\frac{1}{2}\\ \omega_0=\frac{1}{A}\end{cases}$

$$=\Big/\begin{array}{l}\int u\cdot v=u\cdot V-\int u'\cdot V\\ \int\sin=-\cos,\ \int\cos=\sin\end{array}\Big/=A\left(\frac{1}{-\omega_0}\left[e^{a\tau}\cos(\omega_0\tau)\right]_0^t-\frac{a}{-\omega_0}\int_0^t e^{a\tau}\cos(\omega_0\tau)\,d\tau\right)$$

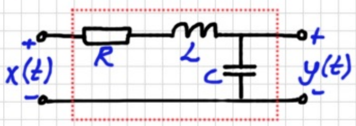
$$=A\left(\frac{-1}{\omega_0}\left[e^{a\tau}\cos(\omega_0\tau)\right]_0^t+\frac{a}{\omega_0}\left(\frac{1}{\omega_0}\left[e^{a\tau}\sin(\omega_0\tau)\right]_0^t-\frac{a}{\omega_0}\int_0^t e^{a\tau}\sin(\omega_0\tau)\,d\tau\right)\right)$$

$$\Rightarrow A\left(1+\frac{a^2}{\omega_0^2}\right)\int_0^t e^{a\tau}\sin(\omega_0\tau)\,d\tau=\frac{A}{\omega_0}\left[e^{a\tau}\left(\frac{a}{\omega_0}\sin(\omega_0 t)-\cos(\omega_0 t)\right)\right]_0^t=\frac{A}{\omega_0}\Big(e^{at}\big(\sin(\omega_0 t)-\cos(\omega_0 t)\big)\underbrace{-e^{0}\big(\sin(0)-\cos(0)\big)}_{=-1(0-1)=1}\Big)$$

$$\Rightarrow g(t)=\underbrace{\frac{A\cdot\omega_0^2}{\omega_0(\omega_0^2+a^2)}}_{=1}\left(e^{-\frac{t}{2}}\left(\frac{-1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)-\cos\left(\frac{\sqrt{3}}{2}t\right)\right)+1\right)u(t)$$

Phu, det var jobbigt!

(Större) Räkneexempel – Systemanalys med fouriertransformen



$$h(t)=\frac{2}{\sqrt{3}}\cdot e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t) \qquad H(\omega)=\frac{1}{1-\omega^2+j\omega}$$

e) Enklare: Använd fouriertransformen! (Laplacetransformen ännu bättre...!)

$$g(t)=(u*h)(t) \Rightarrow G(\omega)=U(\omega)\cdot H(\omega)=\left(vp\left\{\frac{1}{j\omega}\right\}+\pi\delta(\omega)\right)\frac{1}{1-\omega^2+j\omega}$$

$$=vp\left\{\frac{1}{j\omega((j\omega)^2+j\omega+1)}\right\}+\underbrace{\frac{\pi}{1-\omega^2+j\omega}\delta(\omega)}_{=\frac{\pi}{1-0+j0}\delta(\omega)}=\Big/\begin{array}{c}\text{P.B.U.}\\ \text{som i c)}\end{array}\Big/=vp\left\{\frac{1}{j\omega}-\underbrace{\frac{j\omega+1}{(j\omega)^2+j\omega+1}}\right\}+\pi\delta(\omega)$$

$$=\frac{j\omega+\frac{1}{2}+\frac{1}{2}}{(j\omega+\frac{1}{2})^2+\frac{3}{4}}$$

Omforma, som i c)

$$=vp\left\{\frac{1}{j\omega}\right\}+\pi\delta(\omega)-\frac{\frac{1}{2}+j\omega}{(\frac{1}{2}+j\omega)^2+(\frac{\sqrt{3}}{2})^2}-\frac{1}{2}\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}/2}{(\frac{1}{2}+j\omega)^2+(\frac{\sqrt{3}}{2})^2}$$

Formels. Tab. 3:2, 3:10, 3:11

$$\Rightarrow g(t)=\left(1-e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)u(t)$$