

Signaler & System – Föreläsning 2: Tidsdomänanalys av tidskontinuerliga LTI-system

VIDEO 1

UTSIGNALEN FRÅN KAUSALA LTI-SYSTEM

Kausalt LTI-system

$x(t) \rightarrow \mathcal{H} \rightarrow y(t) = \mathcal{H}\{\text{lagrad energi \& } x(t)\} = y_{zi}(t) + y_{zs}(t)$

Låt $x(t < 0) = 0$

$x(t) = u(t)$

$y(t) = y_{zi}(t) + y_{zs}(t)$

$y_{zi}(t) = \mathcal{H}\left\{ \begin{array}{l} \text{Energilagrard} \\ \text{i systemet} \\ \text{vid } t=0^- \end{array} \right\} x(t) = 0 = y(t)|_{x(t)=0}$
 zero-input response
 fria svängningen

$y_{zs}(t) = \mathcal{H}\{x(t)\} = y(t)|_{\text{init-tillst. vid } t=0} = 0$
 zero-state response
 zvingad svängning

Energifritt system

Med bevarad energi

$y_{zi}(t)$

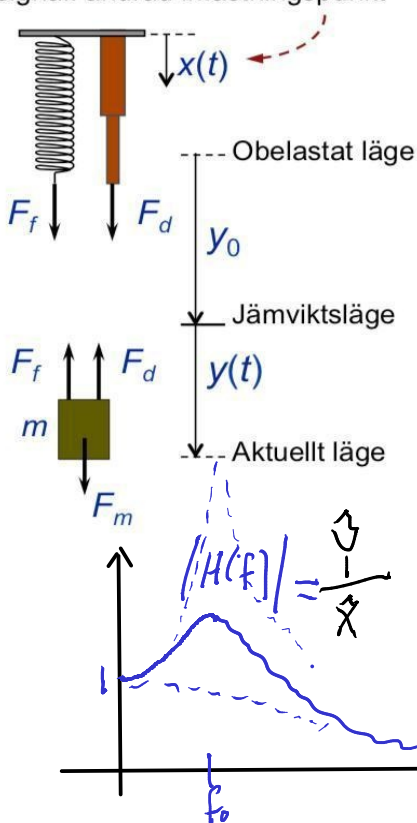
$y_{zs}(t)$

$y(t) = y_{zi}(t) + y_{zs}(t)$

Systemexempel 1 – Mekaniskt svängningssystem, massa i dämpad fjäder

Svängande dämpad fjäder – frilägg och sätt ut krafter:

Insignal: ändrad infästningspunkt



Fjäderkraften $F_f = k \cdot y_{\text{tot}}(t) = k \cdot (y_0 + y(t) - x(t))$

Dämpkraften $F_d = c \cdot (y_{\text{tot}}(t))' = c \cdot (y'(t) - x'(t))$

Tyngdkraften $F_m = m \cdot g$ ($g =$ tyngdaccelerationen)

Newtons 2:a lag: $F_m - F_f - F_d = m \cdot y''(t)$

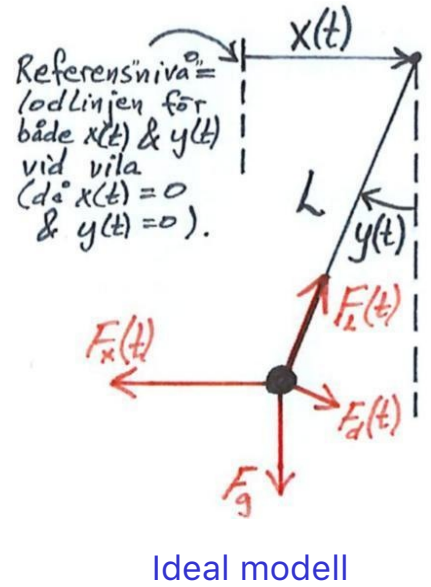
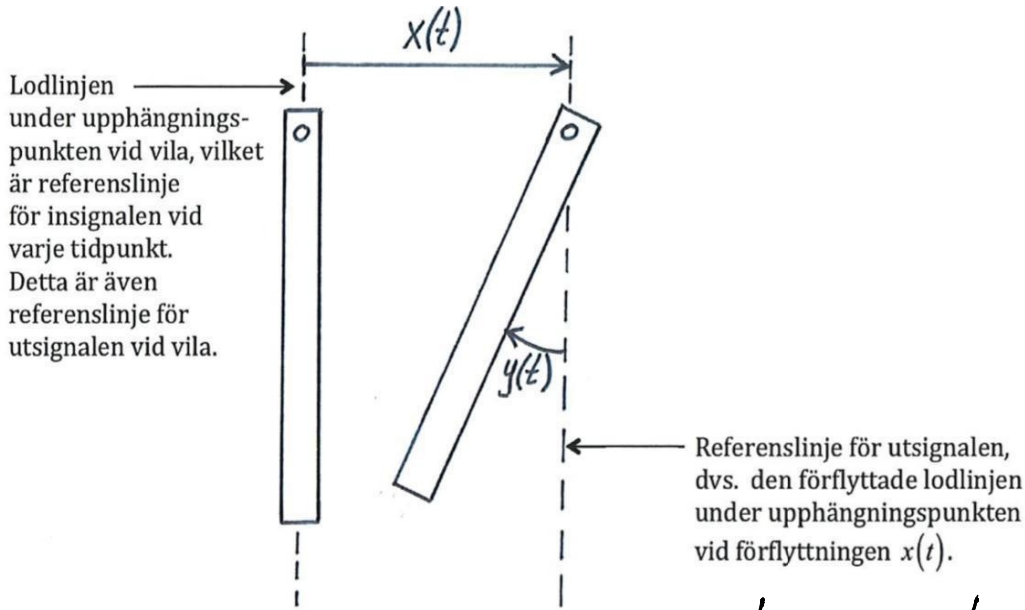
$\Rightarrow m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = m \cdot g - k \cdot y_0 + c \cdot x'(t) + k \cdot x(t)$

Vid vila är $x=0, x'=0, y=0, y'=0, y''=0 \Rightarrow m \cdot g = k \cdot y_0$

\Rightarrow

$$m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + k \cdot y(t) = c \frac{dx(t)}{dt} + k \cdot x(t)$$

Systemexempel 2 – Mekaniskt svängningssystem, pendlande linjal



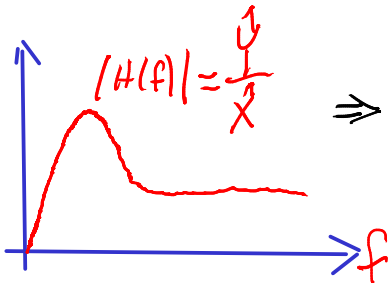
Newtons 2:a lag \Rightarrow

$$y'' + \frac{c}{m}y' + \frac{g}{L}\sin(y) = \frac{\cos(y)}{L} \cdot x''$$

Icke-linjärt system, p.g.a. $\sin(y)$ och $\cos(y)$

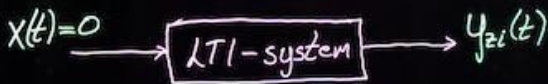
\Rightarrow Linjärisera, dvs. approximera med linjär modell:

Om vinkeln y är liten $\Rightarrow \sin(y) \approx y, \cos(y) \approx 1$



$$\frac{d^2y(t)}{dt^2} + \frac{c}{m} \frac{dy(t)}{dt} + \frac{g}{L} y(t) = \frac{1}{L} \frac{d^2x(t)}{dt^2}$$

Den fria svängningen, zero-input response $y_{zi}(t)$



Differentialsystembeskrivning:

$$a_n \frac{d^N y(t)}{dt^N} + a_{n-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^M x(t)}{dt^M} + b_{m-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

(Vanligen: $N > M$)

Deriveringsoperatören D : $Dy(t) = \frac{dy(t)}{dt}$, $D^i y(t) = \frac{d^i y(t)}{dt^i}$

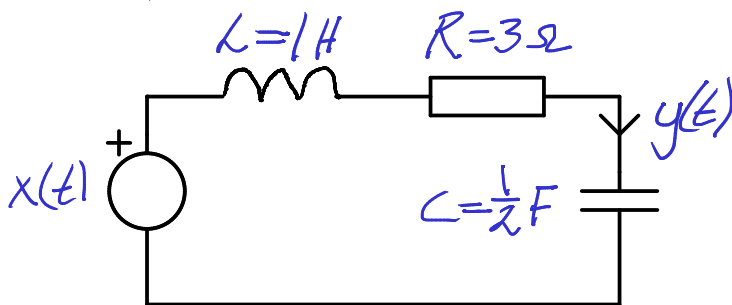
$$\Rightarrow Q(D)y(t) = P(D)x(t) \quad \text{där} \quad \begin{cases} Q(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 \\ P(D) = b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0 \end{cases}$$

$$\Rightarrow \text{lös } \boxed{Q(D)y_{zi}(t) = 0} \Rightarrow y_{zi}(t) = \sum (\text{karaktäristiska termer})$$

$e^{\lambda t}$, $t^n e^{\lambda t}$, $e^{\alpha t} \cdot \cos(\beta t)$

OBS: $y_{zi}(t)$ betecknas $y_0(t)$ i kursboken!

Exempel:

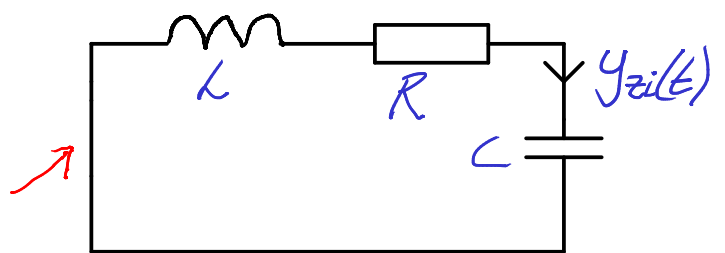


$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$y(0) = 0, \quad y'(0) = -5$$

Beräkna $y_{zi}(t)$

Ekvivalent krets:



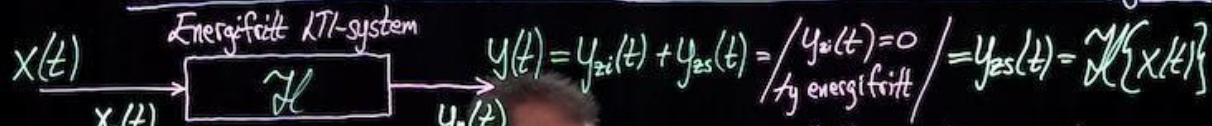
$$(D^2 + 3D + 2)y_{zi}(t) = 0$$

$$\Rightarrow \text{Kar. ekv. } \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases} \Rightarrow y_{zi}(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} = \dots$$

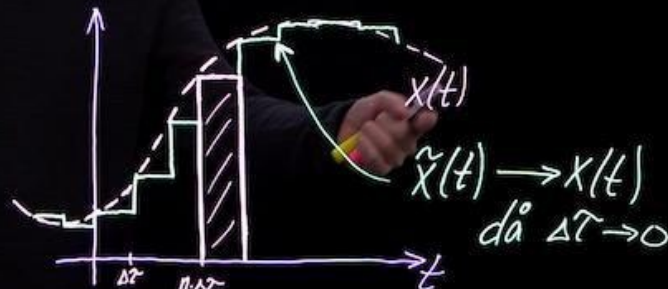
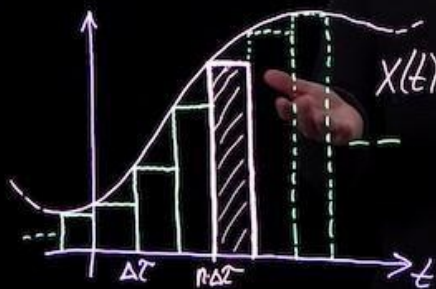
$$= -5e^{-t} + 5e^{-2t}; \quad t \geq 0$$

DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$

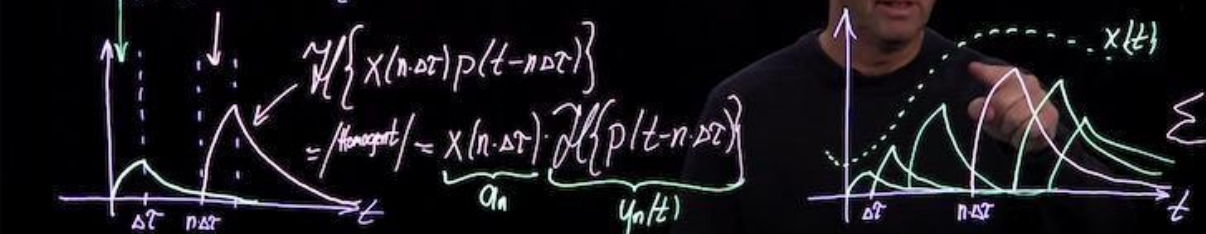
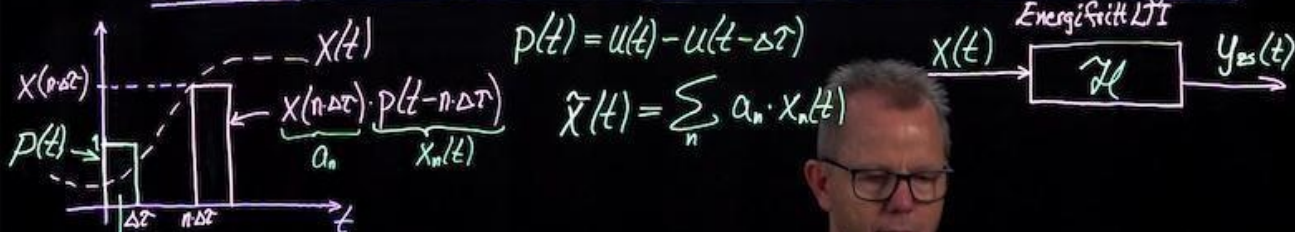


Låt $x(t) = a_1 \cdot x_1(t) + a_2 \cdot x_2(t) + \dots = \sum_n a_n \cdot x_n(t)$
 Linjärt system $\Rightarrow y(t) = a_1 \cdot y_1(t) + a_2 \cdot y_2(t) + \dots = \sum_n a_n \cdot y_n(t)$

Hur väljer vi lämplig uppsättning $\{x_n(t)\}$ för repr. av godtycklig $x(t)$, som ger enkel/enklast beräkning av $\{y_n(t)\}$?



DEN TVINGADE SVÄNGNINGEN, ZERO-STATE RESPONSE $y_{zs}(t)$



Linjärt system $\Rightarrow \tilde{y}(t) = \sum_n a_n \cdot y_n(t) = \sum_n x(n\Delta\tau) \cdot \mathcal{H}\{p(t - n\Delta\tau)\} \rightarrow y_{zs}(t)$
 då $\Delta\tau \rightarrow 0$

DEN Tvingade Svängningen, ZERO-STATE RESPONSE $y_{zs}(t)$

$p(t) = u(t) - u(t - \Delta\tau)$
 $\hat{x}(t) = \sum_n a_n \cdot x_n(t)$
 $\mathcal{L}\{p(t - n \cdot \Delta\tau)\} = ?$
 $d(t) \rightarrow \delta(t)$ $p(t) = d(t) \cdot \Delta\tau$
 $de \Delta\tau \rightarrow 0 \rightarrow \delta(t) \cdot d\tau$
 Dvs. $\hat{x}(t) = \sum_n x(n\Delta\tau) p(t - n\Delta\tau)$
 $\hat{y}(t) = \sum_n x(n\Delta\tau) \mathcal{L}\{p(t - n\Delta\tau)\}$
 $\Delta\tau \rightarrow 0 \rightarrow x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$
 $\Delta\tau \rightarrow 0 \rightarrow y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) \mathcal{L}\{\delta(t - \tau)\} d\tau$

DEN Tvingade Svängningen, ZERO-STATE RESPONSE $y_{zs}(t)$

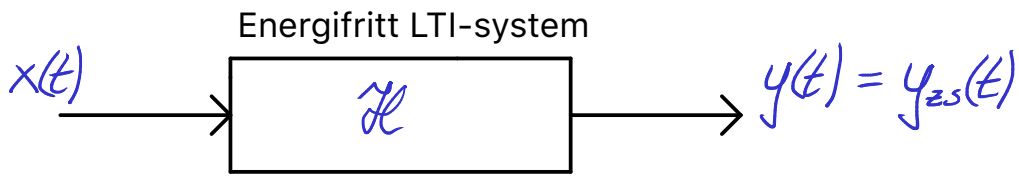
Allmän signalsbeskrivning: $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$
 Linjärt system $\Rightarrow y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) \mathcal{L}\{\delta(t - \tau)\} d\tau$
 Låt $h(t) := \mathcal{L}\{\delta(t)\}$; systemets impulssvar
 Tidsinvariant system $\Rightarrow \mathcal{L}\{\delta(t - \tau)\} = h(t - \tau)$
 Faltningintegralen: $y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$
 $y_{zs}(t) = (x * h)(t) = x(t) * h(t)$
 Viktiga egenskaper: * är kommutativ; $y_{zs}(t) = (x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau$
 $\int_{-\infty}^{\infty} a(\tau) b(t - \tau) d\tau = \begin{cases} \int_{\dots}^{\infty} \dots & ; a(\tau) = 0; \tau > 0 \\ \int_{-\infty}^t \dots & ; b(\tau) = 0; \tau > 0 \end{cases}$

Notera: Om vi utgår från $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \Rightarrow$

$y_{zs}(t) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau\right\} = \text{Linjärt system}$
 $= \int_{-\infty}^{\infty} x(\tau) \mathcal{L}\{\delta(t - \tau)\} d\tau = \text{Tidsinvariant system} = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$
 $\Rightarrow \mathcal{L}\{\delta(t - \tau)\} = h(t - \tau)$

Slutsats efter videon ovan:

Utsignalens zero-statekomponent (den tvingade svängningen):



där

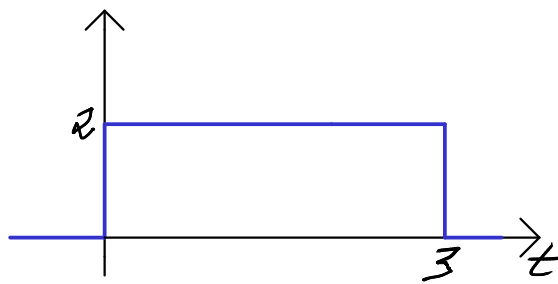
$$y_{zs}(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

Eller $x(t) * h(t)$

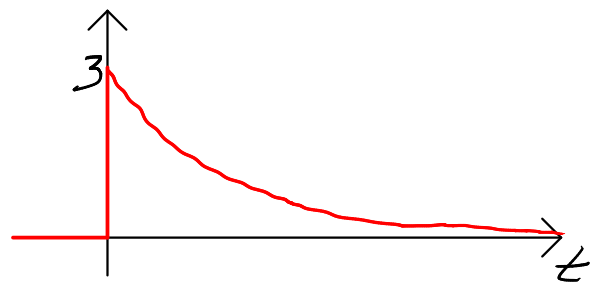
Faltningsintegralen/-erna

där $h(t) = \mathcal{L}\{\delta(t)\}$ är LTI-systemets impulssvar

Exempel:



$$x(t) = 2(u(t) - u(t-3))$$

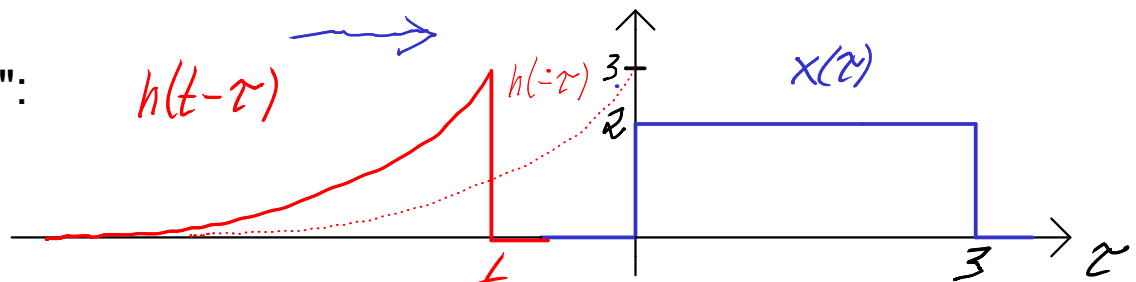


$$h(t) = 3e^{-\frac{1}{2}t} \cdot u(t)$$

ordn. 1

$$y_{zs}(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

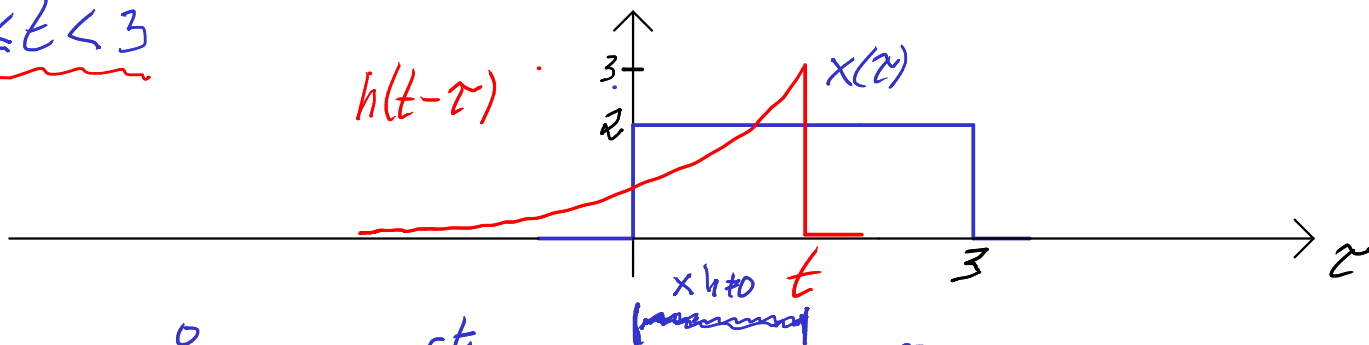
"Grafisk faltning":



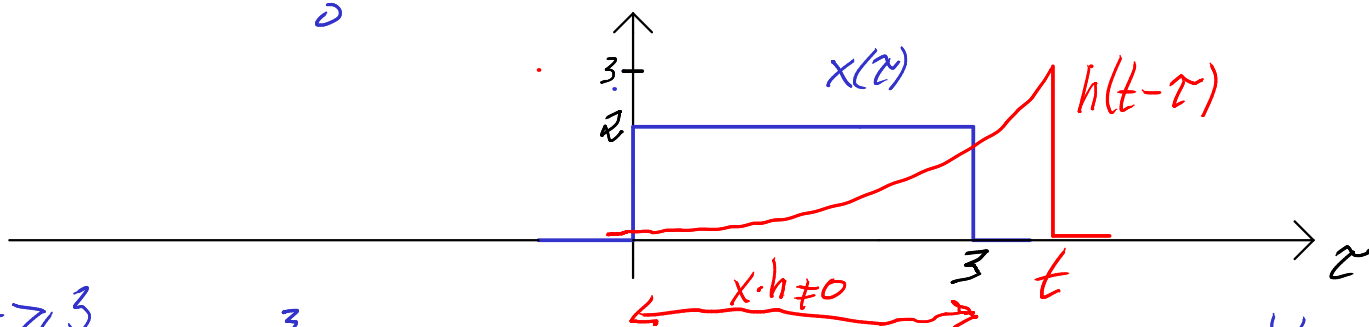
$t < 0$

$$x(\tau) h(t-\tau) = 0 \quad \forall \tau \Rightarrow y_{zs}(t) = \int_{-\infty}^{\infty} 0 d\tau = \underline{0}$$

$0 \leq t < 3$



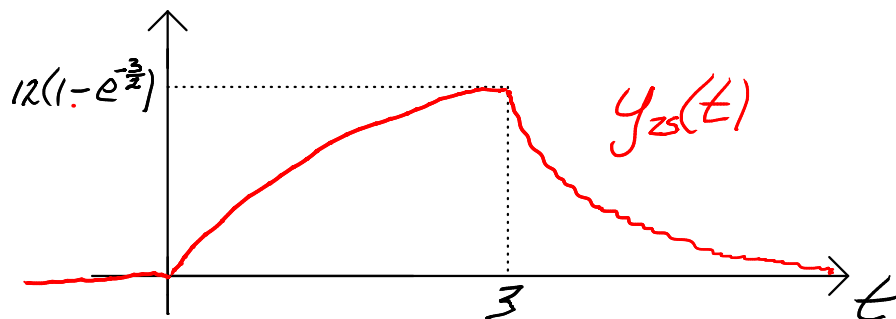
$$\begin{aligned}
 \underline{y_{zs}(t)} &= \int_{-\infty}^0 0 d\tau + \int_0^t 2 \cdot 3 e^{-\frac{1}{2}(t-\tau)} d\tau + \int_t^{\infty} 0 d\tau \\
 &= 6 e^{-\frac{1}{2}t} \int_0^t e^{\frac{1}{2}\tau} d\tau = \dots = \underline{12(1 - e^{-\frac{1}{2}t})}
 \end{aligned}$$



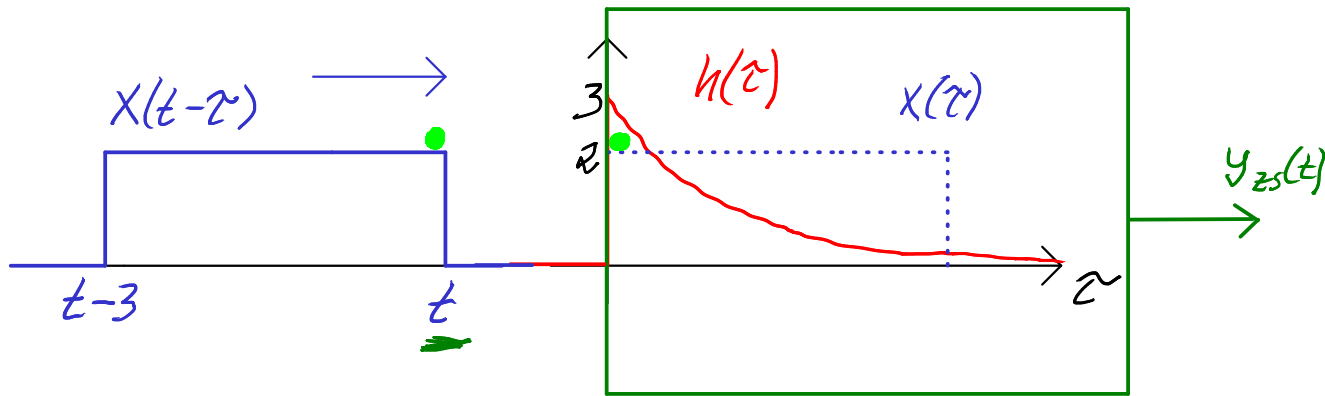
$t \geq 3$

$$\underline{y_{zs}(t)} = \int_0^3 2 \cdot 3 e^{-\frac{1}{2}(t-\tau)} d\tau = \dots = \underline{12(e^{\frac{3}{2}} - 1)e^{-\frac{1}{2}t}}$$

$$\underline{y_{zs}(t)} = \begin{cases} 0; & t < 0 \\ 12(1 - e^{-\frac{1}{2}t}); & 0 \leq t < 3 \\ 12(e^{\frac{3}{2}} - 1)e^{-\frac{1}{2}t}; & t \geq 3 \end{cases}$$



Testa gärna själv att beräkna $y_{zs}(t) = \int_{-\infty}^{\infty} \underline{x(t-\tau)} h(\tau) d\tau$!



Impulssvaret $h(t)$ och kausalitet

OM $x(t < t_0) = 0 \implies y_{zs}(t < t_0) = 0$
Kausalt system

Ex: $\delta(t < 0) = 0 \implies \span style="border: 1px solid red; padding: 2px;">h(t < 0) = 0$

$h_1(t) = 3e^{-\frac{1}{2}(t+2)} u(t+2)$ ← Ikke-kausalt

$h_2(t) = 3e^{-\frac{1}{2}t} u(t)$

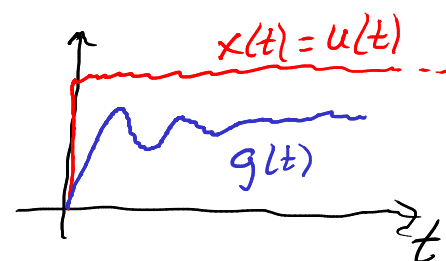
Anti-kausalt system: $h(t) = 0 ; t \geq 0$

Stegsvar för LTI-system

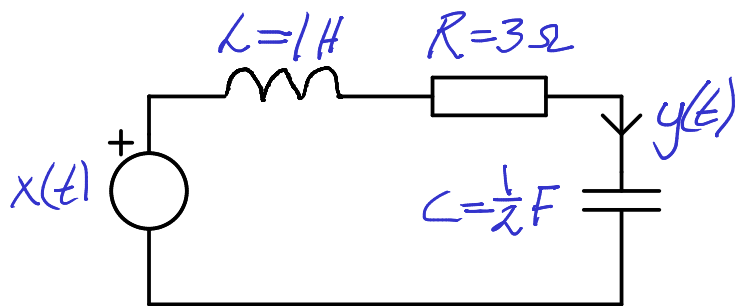
$$\underline{g(t)} = \mathcal{H}\{u(t)\} = (u * h)(t) = \int_{-\infty}^{\infty} u(t-\tau) h(\tau) d\tau =$$

$\tau = 0; t - \tau < 0 \implies \tau > t$

$$= \int_{-\infty}^t h(\tau) d\tau \implies \underline{h(t) = \frac{dg(t)}{dt}}$$



Tidigare exempel:



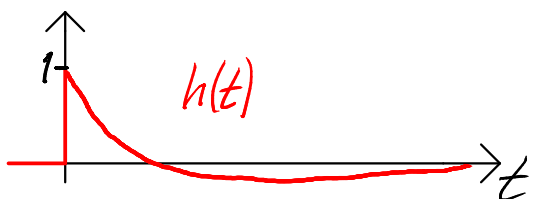
$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$y(0) = 0, \quad y'(0) = -5$$

$$\Rightarrow \underline{y_{zi}(t) = -5e^{-t} + 5e^{-2t}; \quad t \geq 0}$$

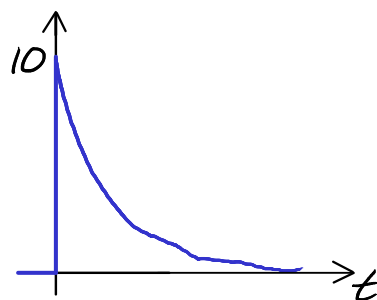
Man kan visa att detta LTI-system

har impulssvar $\underline{h(t) = (2e^{-2t} - e^{-t})u(t)}$

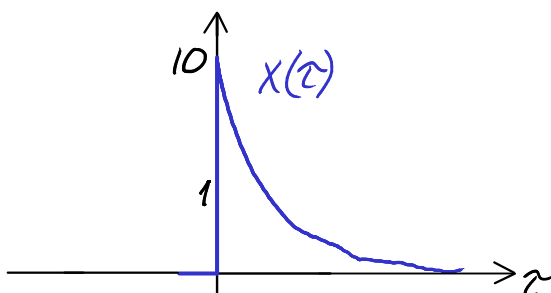


Låt insignalen vara

$$\underline{x(t) = 10e^{-3t} \cdot u(t)}$$



Beräkna den totala utsignalen $y(t) = y_{zi}(t) + y_{zs}(t)$



$$y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$