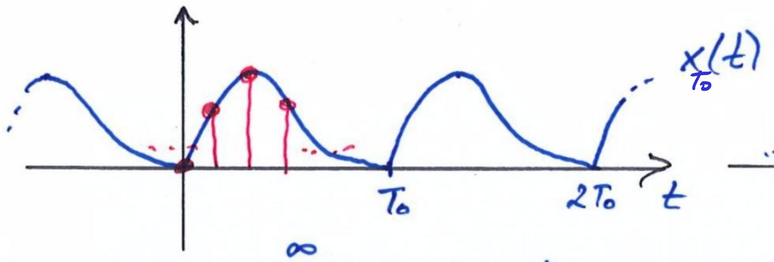
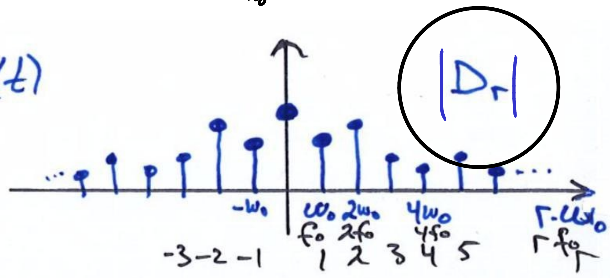
Signaler & System – Föreläsning 13: Fouriertransformanalys av tidsdiskreta signaler & system

Frekvensegenskap för en tidsdiskret N_0-periodisk signal $x_{N_0}[n]$:



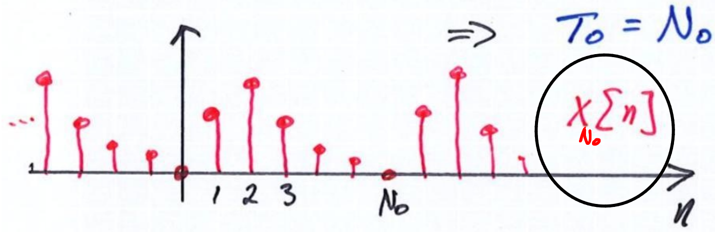


$$x_{T_0}(t) = \sum_{r=-\infty}^{\infty} D_r e^{jr\omega_0 t} \quad \text{där} \quad D_r = \frac{1}{T_0}\int_{T_0} x_{T_0}(t)\, e^{-jr\omega_0 t}\, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

Sample $x(t)$: $x_{N_0}[n] = x_{T_0}(nT)$ i N_0 punkter / period

$$\Rightarrow T_0 = N_0 \cdot T$$



Från Föreläsning 7 – Diskreta FourierTransformen, DFT:

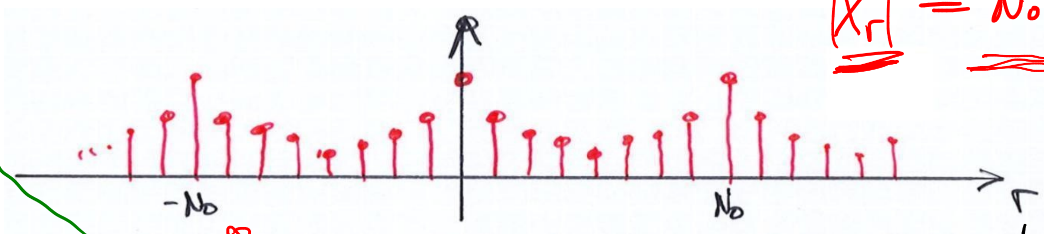
$T_0 = T \cdot N_0$

$$x_n = x_{N_0}[n] = x_{T_0}(nT) = \sum_{r=-\infty}^{\infty} D_r e^{jr\omega_0 nT} = \Big/ \Omega_0 = \omega_0 T = \frac{2\pi}{T_0}\cdot T = \frac{2\pi}{N_0} \Big/ = \dots$$

$$= \sum_{r=0}^{N_0-1}\left(\sum_{m=-\infty}^{\infty} D_{r-mN_0}\right) e^{jr\Omega_0 n} = \Big/ X_r = N_0 \sum_{m=-\infty}^{\infty} D_{r-mN_0} \Big/ X_r = N_0 \cdot \mathcal{D}_r \quad \circledast$$

$$= \boxed{\frac{1}{N_0}\sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n} = IDFT\{X_r\}} \quad \text{där} \quad \boxed{X_r = DFT\{x_n\} = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}}$$

$$|X_r| = N_0 |\mathcal{D}_r|$$



$\mathcal{D}_r = \sum_{m=-\infty}^{\infty} D_{r-mN_0}$ = komplexa fourierseriekoeff. till $x_{N_0}[n]$

$\Rightarrow x_{N_0}[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n}$ / = komplexa Fourserien för $x_{N_0}[n]$ / $= N_0 \cdot \frac{1}{N_0}\sum_{N_0} \mathcal{D}_r e^{jr\Omega_0 n} = N_0 \cdot IDFT\{\mathcal{D}_r\}$

$\circledast \Rightarrow \mathcal{D}_r = \frac{1}{N_0}\cdot X_r = \frac{1}{N_0} DFT\{\underbrace{x_{N_0}[n]}_{x_n}\}$

Fourieranalys av N_0-periodiska signaler (kap. 9.1)

Om $x_{No}[n]$ är N_0-periodisk, dvs. $x_{No}[n] = x_{No}[n+N_0]$

$\Longrightarrow$ **fourierserieutveckla** (**DTFS**, Discrete-Time Fourier Series):

$$x_{N_0}[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \quad \text{där} \quad \mathcal{D}_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x_{N_0}[n]e^{-jr\Omega_0 n}, \quad \Omega_0 = \frac{2\pi}{N_0}$$

(N_0-periodiska komplexa fourierseriekoefficienter)

$$\boxed{X_r = N_0 \cdot \mathcal{D}_r} \Rightarrow \boxed{x_{N_0}[n] = N_0 \cdot \text{IDFT}\{\mathcal{D}_r\}} \boxed{\mathcal{D}_r = \frac{1}{N_0}\cdot \text{DFT}\{x_{N_0}[n]\}}$$

$$\left(\Rightarrow \boxed{x_n = x_{N_0}[n]}\right)$$

$$\left(\text{Notera kursbokens def: } \boxed{\mathcal{D}_r = X_r} \Rightarrow \boxed{x_{N_0}[n] = N_0 x_n}\right)$$

$\Longrightarrow$ Använd **DFT** för frekvensanalys av *tidsdiskreta periodiska signaler*!

(*som beräknas med någon lämplig FFT-algoritm*)

$\Longrightarrow$ *I kursen tar vi därför **inte** upp fourierserier för tidsdiskreta signaler…*

$$\mathcal{D}_r = \frac{1}{N_0} X_r = \frac{1}{N_0}\cdot \underbrace{\bar{X}(\omega)}_{\mathcal{F}\{\bar{x}(t)\}}\Big|_{\omega = r\cdot\omega_0}$$

Jämför med $\underbrace{D_n}_{x_{T_0}(t)} = \frac{1}{T_0} \underbrace{X(\omega)}_{x(t) \text{ icke-per}}\Big|_{\omega = n\cdot\omega_0}$

VIDEON, Del 1

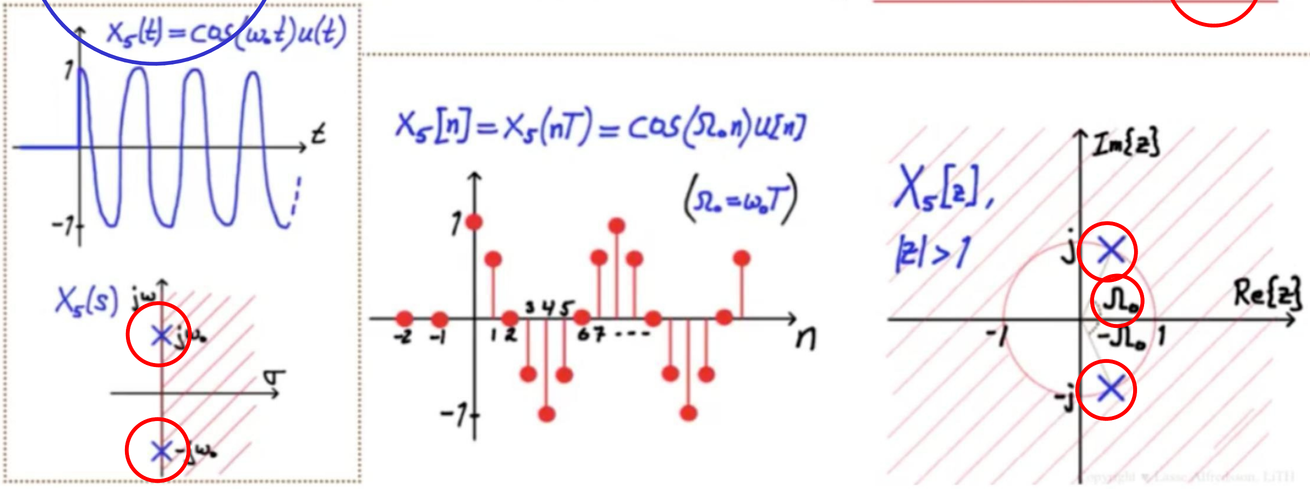
Tidskontinuerliga vs. tidsdiskreta frekvenssignaler

$$\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s}{s^2+\omega_0^2};\quad \mathrm{Re}\{s\}>0$$

$$\left(s^2+\omega_0^2=0 \Rightarrow \underline{s=\pm j\omega_0}\right)$$

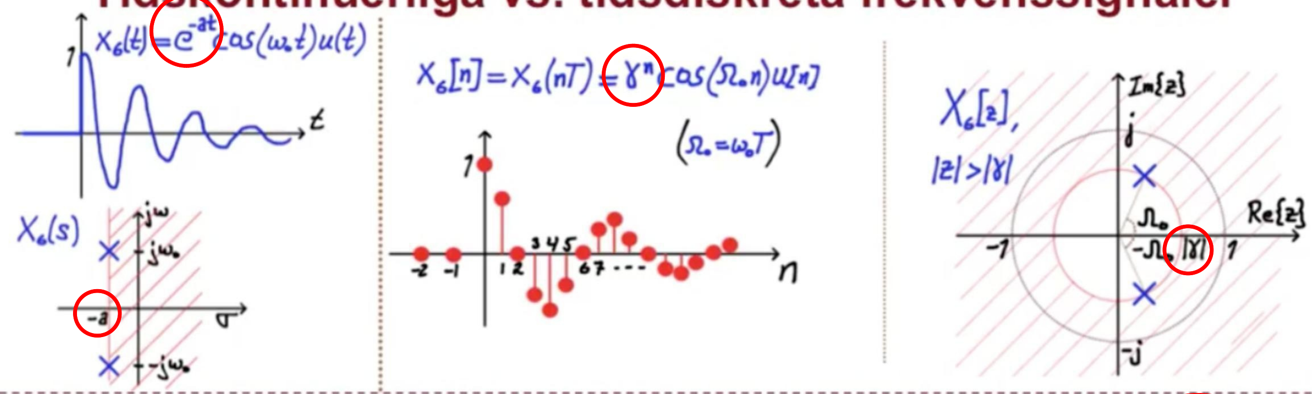
$$\cos(\Omega_0 t)u[n] \Leftrightarrow \frac{z(z-\cos\Omega_0)}{z^2-2(\cos\Omega_0)\cdot z+1};\quad |z|>1$$

$$\left(\begin{array}{l} z^2-2(\cos\Omega_0)\cdot z+1=0 \\ \Rightarrow \underline{z=\cos\Omega_0 \pm j\sin\Omega_0 = e^{\pm j\Omega_0}} \end{array}\right)$$



VIDEON, Del 2

Tidskontinuerliga vs. tidsdiskreta frekvenssignaler



$$e^{-at}\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2};\quad \mathrm{Re}\{s\}>-a$$

$$\left((s+a)^2+\omega_0^2=0 \Rightarrow \underline{s=-a\pm j\omega_0}\right)$$

$$\gamma^n\cos(\Omega_0 n)u[n] \Leftrightarrow \frac{z(z-\gamma\cos\Omega_0)}{z^2-2\gamma(\cos\Omega_0)z+\gamma^2};\quad |z|>|\gamma|$$

$$\left(\begin{array}{l} z^2-2\gamma(\cos\Omega_0)\cdot z+\gamma^2=0 \\ \Rightarrow \underline{z=|\gamma|e^{\pm j\Omega_0}} \end{array}\right)$$

VIDEON, Del 3

Fouriertransformen till tidsdiskret signal

- Om $j\omega$ **-axeln** ligger i konvergensområdet för $X(s)$

$$\Rightarrow \quad X(\omega) = X(s)\Big|_{s=j\omega} = X(j\omega) \qquad \left(\mathcal{F}\{x(t)\} = \mathcal{L}\{x(t)\}\Big|_{s=j\omega}\right)$$

- Om **enhetscirkeln** ($|z|=1$) ligger i konvergensområdet för $X[z]$

$$\Rightarrow \quad X[\Omega] = X[z]\Big|_{z=e^{j\Omega}} = X\left[e^{j\Omega}\right] \qquad \left(\mathcal{F}\{x[n]\} = \mathcal{Z}\{x[n]\}\Big|_{z=e^{j\Omega}}\right)$$

Fouriertransformen till $x[n]$

(*Eng*: *"DTFT, Discrete-Time Fourier Transform"*): $\neq$ DFT

$$\boxed{X[\Omega] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}}$$

Boken: $X(\Omega) = \mathrm{DTFT}\{x[n]\}$

Existensvillkor:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

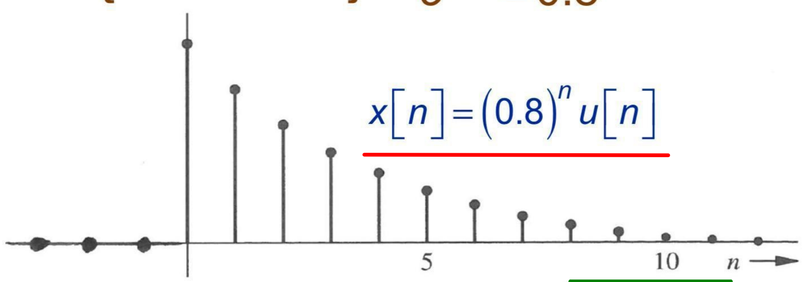
$$\sum_n |x[n]|^2 < \infty$$

VIDEON, Del 4

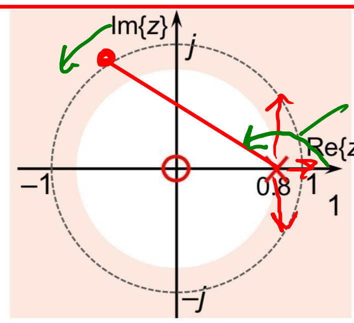
$$\mathcal{F}\left\{(0.8)^n u[n]\right\} = \frac{e^{j\Omega}}{e^{j\Omega} - 0.8}$$

$z = e^{j\Omega}$

$$\boxed{X[z] = \frac{z}{z-0.8}, \quad |z| > 0.8}$$

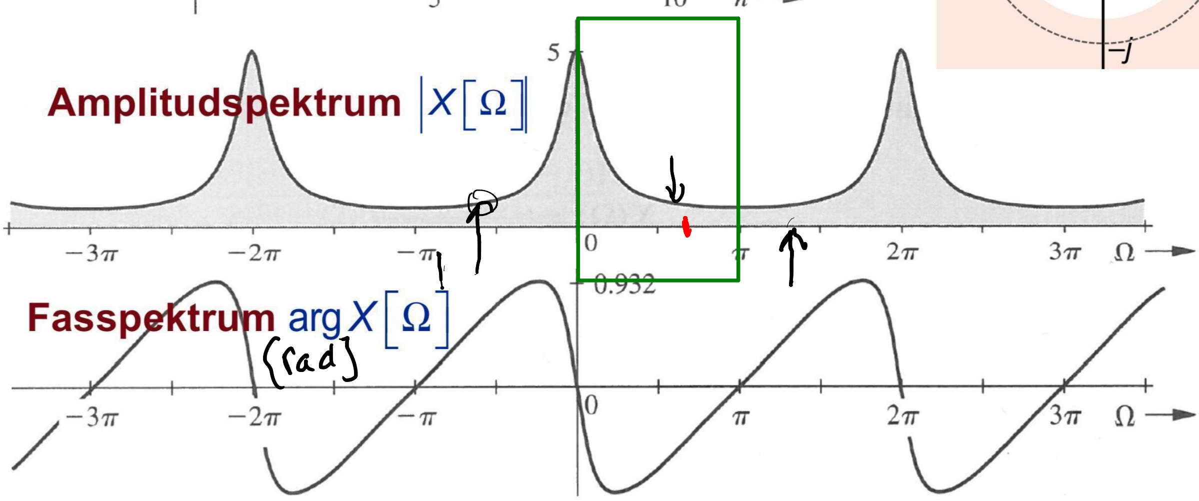


$$x[n] = (0.8)^n u[n]$$



Amplitudspektrum $|X[\Omega]|$

Fasspektrum $\arg X[\Omega]$ {rad}



Fouriertransformen (DTFT, Discrete-Time Fourier Transform)

- Fouriertransformen till energisignal $x[n]$:

$$\boxed{\mathcal{F}\{x[n]\} = X[\Omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}}$$

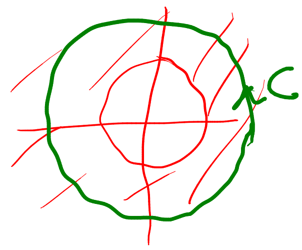
Boken:
$X(\Omega) = \text{DTFT}\{x[n]\}$

- Inversa fouriertransformen till $X[\Omega]$ ($X(\Omega)$):

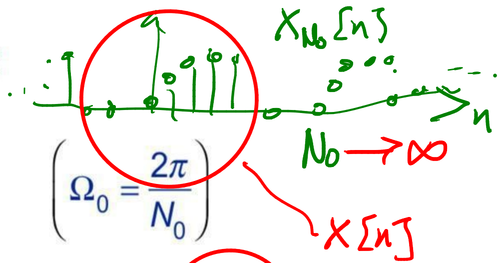
$$\boxed{\mathcal{F}^{-1}\{X[\Omega]\} = x[n] = \frac{1}{2\pi}\int_{2\pi} X[\Omega]e^{j\Omega n}\,d\Omega}$$

Boken:
$x[n] = \text{IDTFT}\{X(\Omega)\}$

Existensvillkor: $\boxed{\mathcal{F}\{x[n]\} \exists \text{ om } \sum_{n=-\infty}^{\infty} |x[n]| < \infty}$



Fouriertransformen kan erhållas/härledas:



1. Genom utvidgning av $x_{N_0}[n]$ = en N_0-periodisk upprepning av $x[n]$ $\Rightarrow N_0\mathcal{D}_r = X[r\cdot\Omega_0]$ $\left(\Omega_0 = \dfrac{2\pi}{N_0}\right)$

   Låt $N_0 \to \infty \Rightarrow x_{N_0}[n] \to x[n], \quad r\cdot\Omega_0 \to \Omega, \quad \underline{\underline{N_0\mathcal{D}_r \to X[\Omega]}}$

   $\Omega_0 = \dfrac{2\pi}{N_0}$

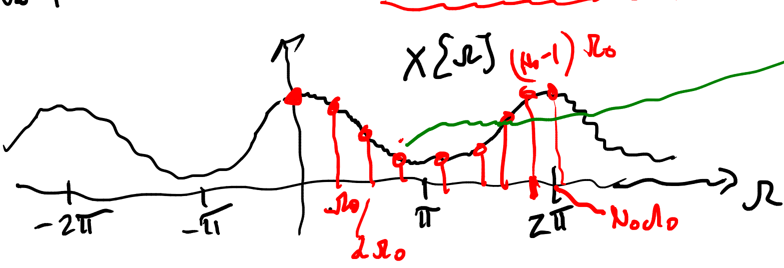
2. Som z-transformen längs enhetscirkeln (*då den ligger i konvergensområdet!*):

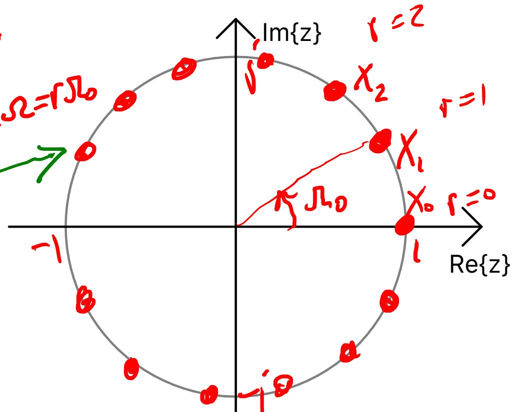
$$X\left[e^{j\Omega}\right] = X[z]\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X[\Omega]$$

$$x[n] = \frac{1}{2\pi j}\oint_C X[z]z^{n-1}dz = \left/ \begin{array}{l} z = e^{j\Omega} \Rightarrow \dfrac{dz}{d\Omega} = j\cdot e^{j\Omega} \\ C = e^{j\Omega},\quad \Omega: \alpha \text{ till } \alpha+2\pi \end{array} \right/ = \frac{1}{2\pi}\int_{2\pi} X\left[e^{j\Omega}\right]e^{j\Omega n}\,d\Omega$$

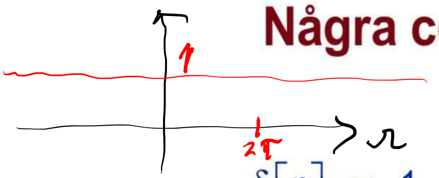
$$\text{DFT}\{x[n]\} = X_r = \sum_{N_0} x[n]\,e^{-j r\Omega_0 n} = X[\Omega]\Big|_{\Omega = r\Omega_0}$$

N_0 punkter





Några centrala fouriertransformpar



$$\boxed{\mathcal{F}\{x[n]\} \text{ är } 2\pi\text{-periodisk!}}$$

$$\delta[n] \Leftrightarrow 1$$

$X(t) = 1$

$X(\omega) = 2\pi\delta(\omega)$

$$\delta[n-k] \Leftrightarrow e^{-jk\Omega}$$

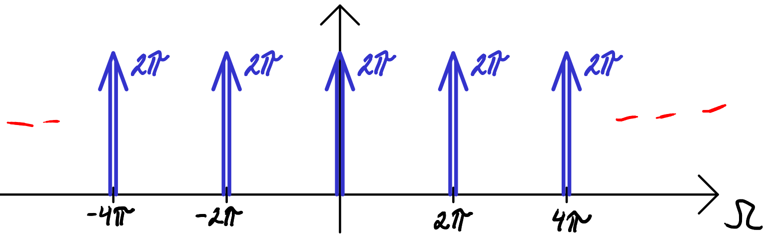
$$1 \Leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k \cdot 2\pi)$$

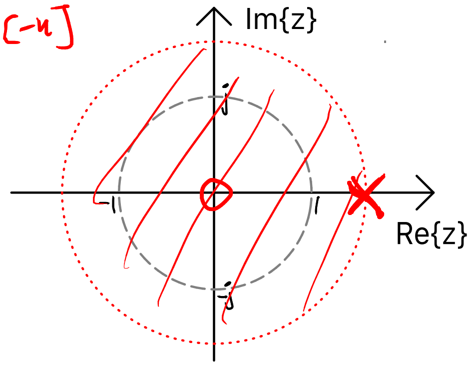
$$\gamma^n u[n] \Leftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma}; \quad |\gamma|<1$$

$$-\gamma^n \underbrace{u[-n-1]}_{=u_0[-n]} \Leftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma}; \quad |\gamma|>1$$

$n \leq -1$

$-(1{,}5)^n u_0[-n]$



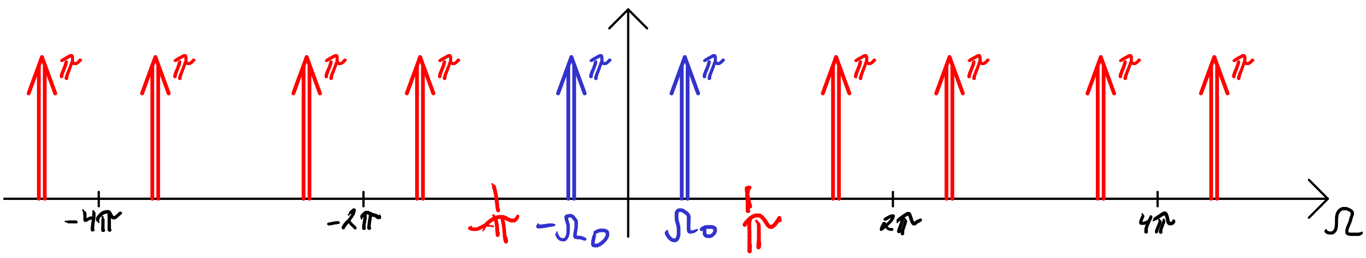


$$u[n] \Leftrightarrow \mathrm{vp}\left\{\frac{e^{j\Omega}}{e^{j\Omega}-1}\right\} + \boxed{\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k \cdot 2\pi)}$$

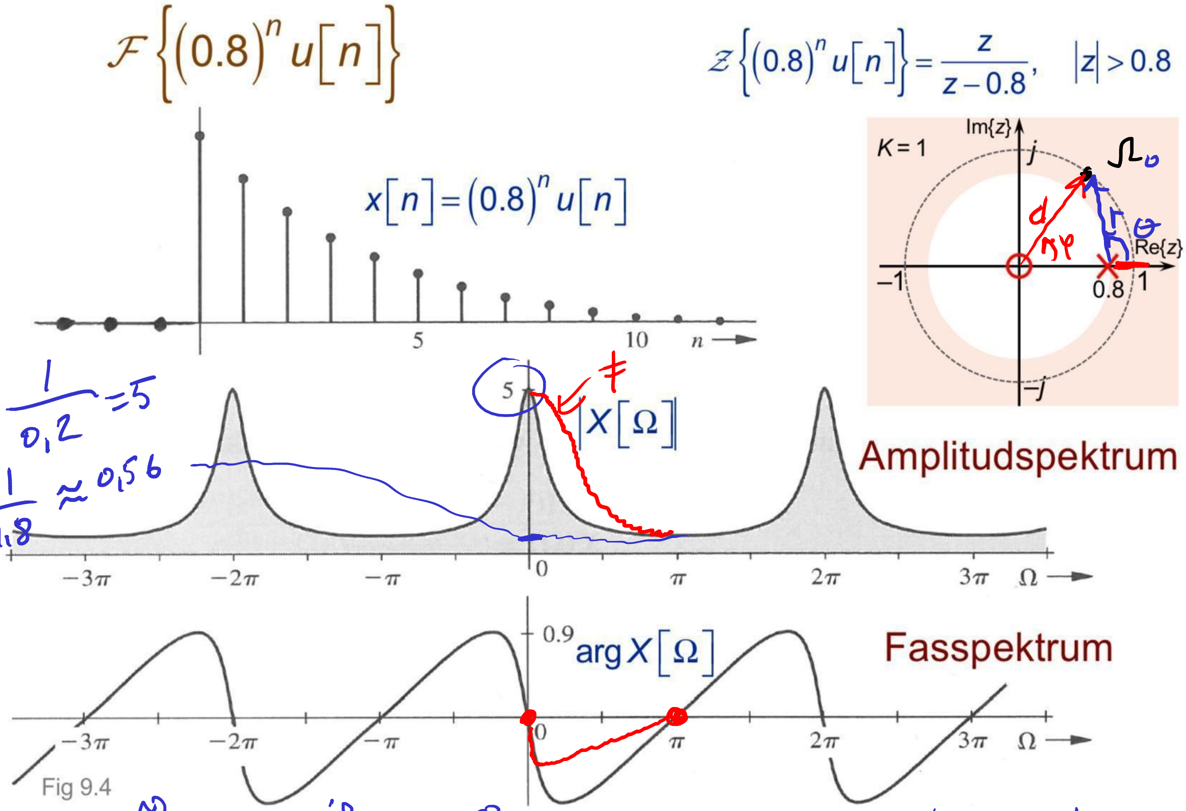
$\omega_0 t$

$$\cos(\Omega_0 n) \Leftrightarrow \pi \sum_{k=-\infty}^{\infty} \left(\delta(\Omega + \Omega_0 - k \cdot 2\pi) + \delta(\Omega - \Omega_0 - k \cdot 2\pi)\right)$$

$$\cos(\Omega_0 n) u[n] \Leftrightarrow \mathrm{vp}\left\{\frac{e^{j2\Omega} - e^{j\Omega}\cos\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1}\right\} + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \left(\delta(\Omega + \Omega_0 - k \cdot 2\pi) + \delta(\Omega - \Omega_0 - k \cdot 2\pi)\right)$$



Exemplet i den förberedande video:



$$|X[0]| = \frac{1}{0,2} = 5$$

$$|X[\pi]| = \frac{1}{1,8} \approx 0,56$$

$$X[\Omega] = \sum_{n=0}^{\infty} 0,8^n \cdot e^{-j\Omega n} = \sum_{n=0}^{\infty} (0,8 \cdot e^{-j\Omega})^n = \Big/ \; |0,8 \cdot e^{-j\Omega}| < 1 \text{ ok här} \; \Big/$$

$$= \frac{1}{1 - 0,8 e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - 0,8} = \frac{1 \cdot e^{j\Omega}}{\cos\Omega - 0,8 + j\sin\Omega}$$

$$= \frac{1}{\sqrt{1,64 - 1,6\cos\Omega}} \, e^{j\left(\Omega - \arctan\frac{\sin\Omega}{\cos\Omega - 0,8}\right)} \quad (\pm\pi \text{ om } \cos\Omega - 0,8 < 0)$$

$$= |X[\Omega]| \cdot e^{j \arg X[\Omega]}$$

Poissons summationsformel – *igen*

$$x(t) \xrightarrow{\text{Sampling}} x[n] = x(nT)$$

$$\bar{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \quad (1)$$

$\bar{X}(\omega) = \mathcal{F}\{\bar{x}(t)\}$

Kap. 8 $\Rightarrow$
Poissons summationsformel:

$$\boxed{\bar{X}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n \cdot \underbrace{\omega_s}_{=\frac{2\pi}{T}})}$$

$\omega = \frac{\Omega}{T}$

Från DFT-videon inför föreläsning 7: $\bar{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT}$

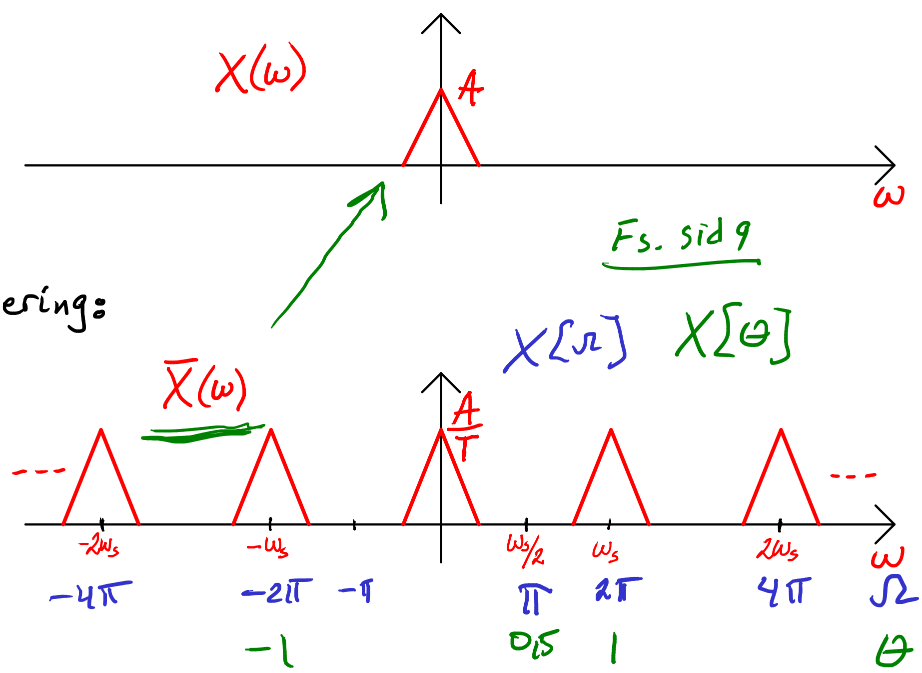
Låt $\Omega = \omega T \Rightarrow \quad \bar{X}\left(\frac{\Omega}{T}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X[\Omega]$

$\Rightarrow$ Alternativ Poissons summationsformel: $\boxed{X[\Omega] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\frac{\Omega - n \cdot 2\pi}{T}\right)}$

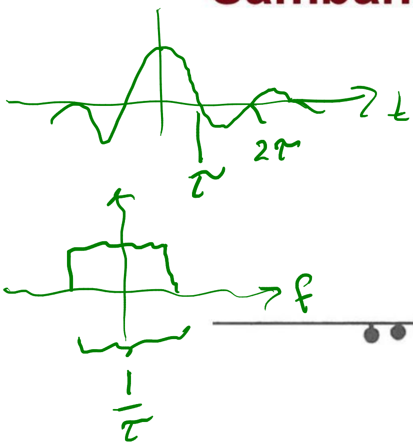
$x(t) \Rightarrow$

Sampla $\Rightarrow$ Periodisering:

$$\begin{cases} \bar{x}(t) = x(t)\cdot\delta_T(t) \\ x[n] = x(nT) \end{cases}$$



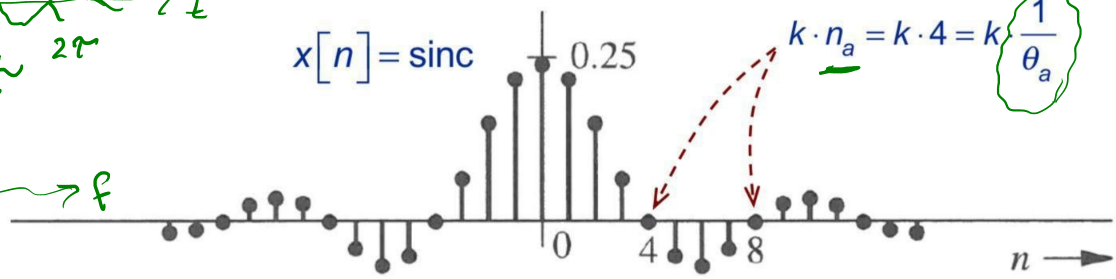
Samband, sinc ⇔ rect



Nollgenomgångar:

$k \cdot n_a = k \cdot 4 = k \cdot \frac{1}{\theta_a}$

$x[n] = \text{sinc}$



$X[\Omega] = \sum \text{rect}$

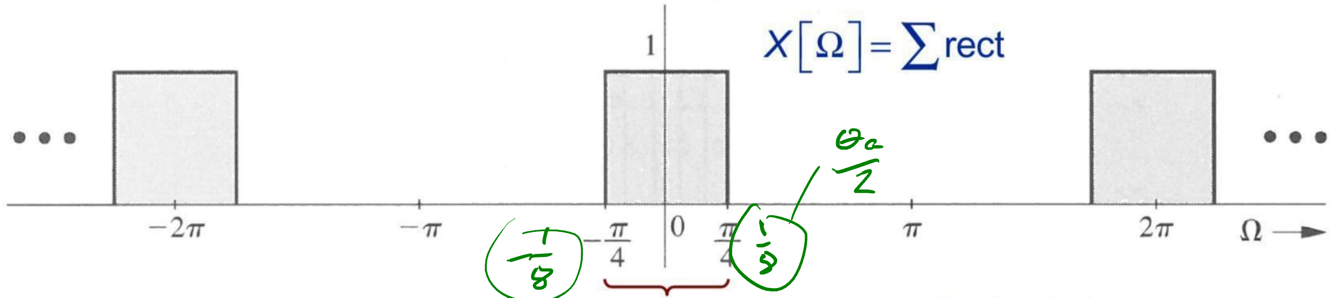


Fig 9.7

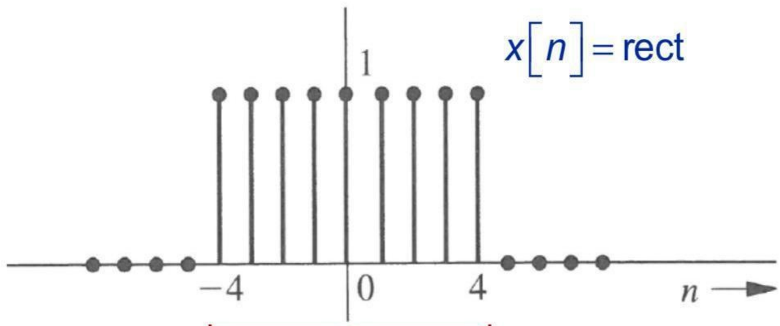
Bredd $\Omega_a = \pi/2 \Rightarrow \theta_a = 1/4 \quad (= 1/n_a)$

$\theta_a = \frac{1}{4}$

θ

Samband, rect ⇔ sinc

$x[n] = \text{rect}$



Bredd (längd) $n_b = 9 = \frac{1}{\theta_b}$

Nollgenomgångar:

$k \cdot \theta_b = k \cdot \frac{1}{n_b} = k \cdot \frac{1}{9}$, dvs

$k \cdot \Omega_b = k \cdot \frac{2\pi}{9}$

$X[\Omega] = \sum \text{sinc}$

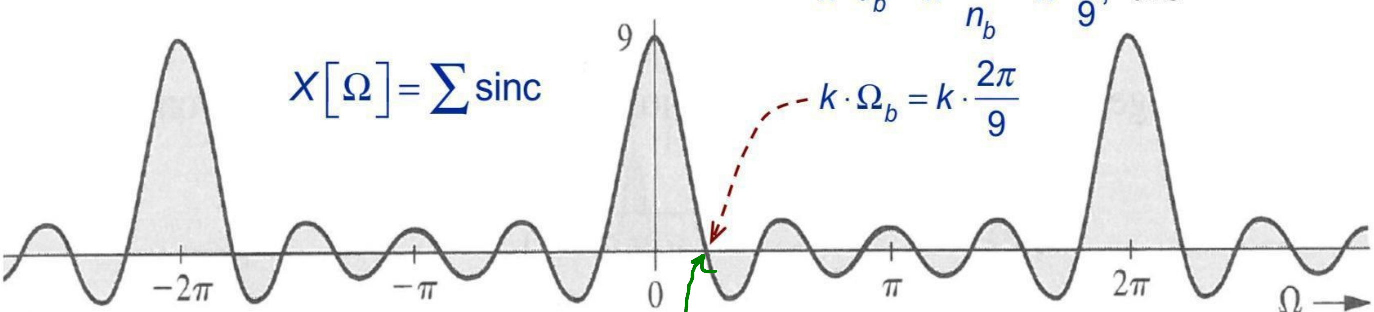


Fig 9.6

$\theta_b = \frac{1}{9}$

θ

Centrala fouriertransformegenskaper (F.s. Tab. 7)

Tidsskiftning: $x[n-n_0] \Leftrightarrow X[\Omega]e^{-j\Omega n_0}$

Frekvensskiftning: $x[n]e^{j\Omega_0 n} \Leftrightarrow X[\Omega-\Omega_0]$

spec.fall: $x[n](-1)^n \Leftrightarrow X[\Omega-\pi]$

Spegling: $x[-n] \Leftrightarrow X[-\Omega]$

Mult. med n: $n \cdot x[n] \Leftrightarrow j\dfrac{dX[\Omega]}{d\Omega}$

Symmetri, $x[n]$ reell: $X[-\Omega] = X^*[\Omega]$

Faltning: $x_1[n] * x_2[n] \Leftrightarrow X_1[\Omega]X_2[\Omega]$

Multiplikation: $x_1[n]x_2[n] \Leftrightarrow \dfrac{1}{2\pi}X_1[\Omega]\otimes X_2[\Omega]$

mult

Cirkulär Faltning

2π-periodisk faltning

Boken: P

Fs. sid 8:

Parsevals formel:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]| = \frac{1}{2\pi}\int_{2\pi} \underbrace{|X[\Omega]|^2}_{\text{En.spektrum}} d\Omega$$

Systemanalys & utsignalsberäkning

$H[\Omega]$ ∃ ← Stabilt Energifritt LTI-system

$x[n]$ → $h[n]$ → $\underbrace{y[n]}_{=y_{zs}[n]} = x[n] * h[n]$

$\Rightarrow Y[\Omega] = X[\Omega]H[\Omega]$

$Y[z] = X[z]H[z]$

Frekvensfunktionen: $H[\Omega] = \mathcal{F}\{h[n]\} = H[z]\big|_{z=e^{j\Omega}} = H[e^{j\Omega}]$

Frekvensspektrum:

$X[\Omega] = \mathcal{F}\{x[n]\} = |X[\Omega]|e^{j\arg X[\Omega]}$, $\quad Y[\Omega] = \mathcal{F}\{y[n]\} = |Y[\Omega]|e^{j\arg Y[\Omega]}$

⟹ Utsignalens frekvensspektrum:

- Amplitudspektrum: $|Y[\Omega]| = |X[\Omega]| \cdot |H[\Omega]|$
- Fasspektrum: $\arg Y[\Omega] = \arg X[\Omega] + \arg H[\Omega]$

Exempel:

Ett stabilt energifritt LTI-system med insignal $x[n]$ och utsignal $y[n]$

beskrivs av följande differensekvation: $y[n]-0.5y[n-1]=x[n]$

Beräkna systemets frekvensfunktion $H[\Omega]$ och impulssvar $h[n]$

samt utsignalen för insignalen $x[n]=0.8^n u[n]$

Lösningsgång:

$$\begin{tikzcd}
\text{Differensekv.} \arrow[r, "\mathcal{F}", "1"'] & H[\Omega] \arrow[r, "\mathcal{F}^{-1}"] \arrow[dr] & h[n] & \\
x[n] \arrow[r, "\mathcal{F}"] & X[\Omega] \arrow[r] & Y[\Omega]=X[\Omega]H[\Omega] \arrow[r, "\mathcal{F}^{-1}"] & y[n]
\end{tikzcd}$$

1 $\mathcal{F}\{\text{diff.-ekv}\}$ $\quad Y[\Omega]\left(1-0{,}5e^{-j\Omega}\right)=X[\Omega]$

$$H[\Omega]=\frac{Y[\Omega]}{X[\Omega]}=\frac{1}{1-0{,}5e^{-j\Omega}}=\frac{e^{j\Omega}}{e^{j\Omega}-0{,}5}$$

Tab. 8:5 $\Rightarrow$ $h[n]=0{,}5^n\cdot u[n]$

$$Y[\Omega]=X[\Omega]\,H[\Omega]=\frac{e^{j\Omega}}{e^{j\Omega}-0{,}8}\cdot\frac{e^{j\Omega}}{e^{j\Omega}-0{,}5}$$

$= \text{P.B.U} = \ldots \Rightarrow$ $y[n]=\left(\frac{8}{3}0{,}8^n-\frac{5}{3}0{,}5^n\right)u[n]$

Inte så rolig...

$$\frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}-0{,}8}-\frac{5}{3}\frac{e^{j\Omega}}{e^{j\Omega}-0{,}5}$$

Vanligare lösningsgång:

$$\begin{tikzcd}
 & & h[n] \\
\text{Differensekv.} \arrow[r, "\mathcal{Z}"] & H[z] \arrow[ur, "\mathcal{Z}^{-1}"] \arrow[r, "z=e^{j\Omega}", "\text{Om stabilt system}"'] \arrow[dr] & H[\Omega] \\
x[n] \arrow[r, "\mathcal{Z}"] & X[z] \arrow[r] & Y[z]=X[z]H[z] \arrow[r, "\mathcal{Z}^{-1}"] & y[n]
\end{tikzcd}$$

Efter-kommentar:
Som jag sa i slutet av föreläsningen, så använder man gärna z-transformen så långt man kan (när man får det) och går främst över till fouriertransformen när man behöver det, som t.ex. när man vill ha signalens frekvensspektrum eller systemets frekvenskaraktäristik.

GOD JUL

OCH LYCKA TILL PÅ TENTAN!