Fouriertransformen (*härledning & inledande exempel – se videoklipp*)

- Fouriertransformen till $x(t)$:

$$\boxed{\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

Jfr. fourierserie:

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt$$

- Inversa fouriertransformen till $X(\omega)$:

$$\boxed{\mathcal{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

Jfr. fourierserie:

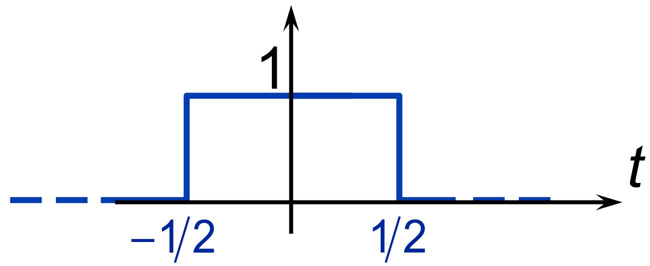
$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

Existensvillkor: $\boxed{\mathcal{F}\{x(t)\} \exists \text{ om } \int_{-\infty}^{\infty} |x(t)|\, dt < \infty}$

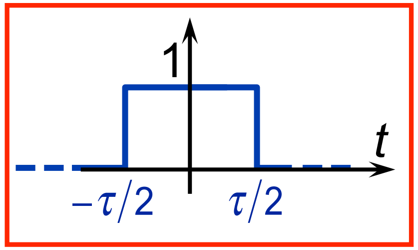
Ett viktigt fouriertransformpar: rect $\Leftrightarrow$ sinc

- Fyrkantpulsen rect(t) ("*unit gate function*")

$$\Pi(t) = \text{rect}(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$



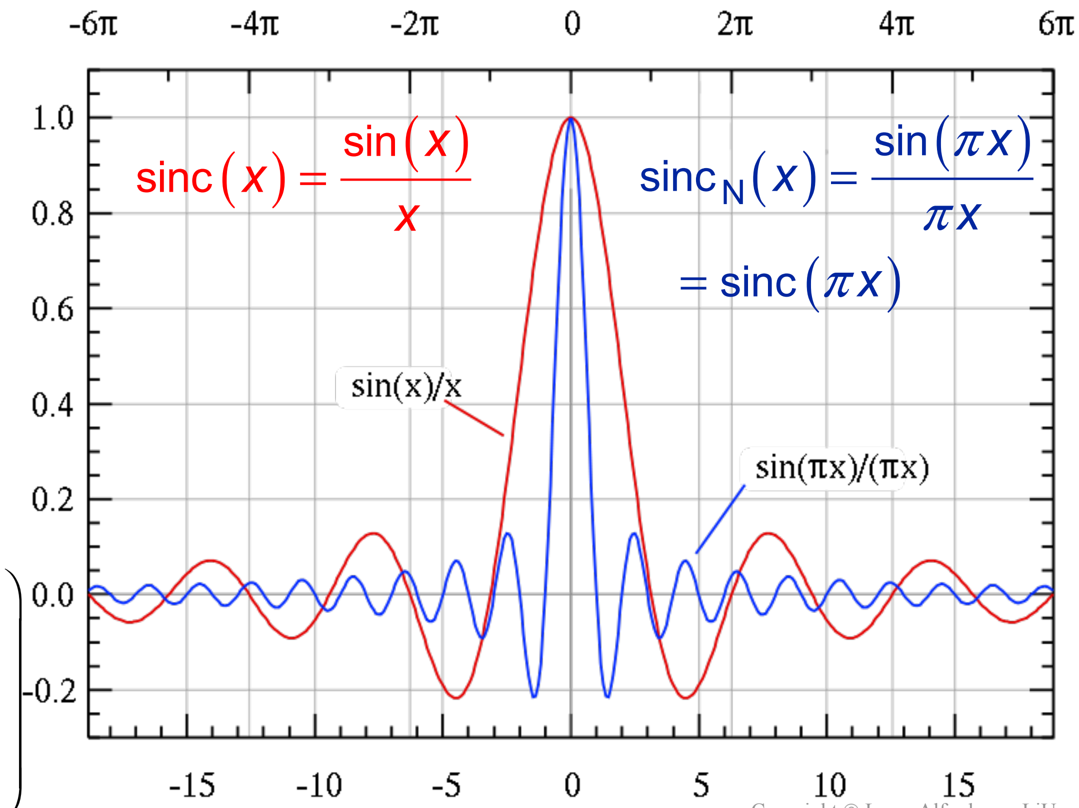
$$\mathcal{F}\left\{\text{rect}\left(\frac{t}{\tau}\right)\right\} = \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$$



$$= \tau \cdot \text{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)$$

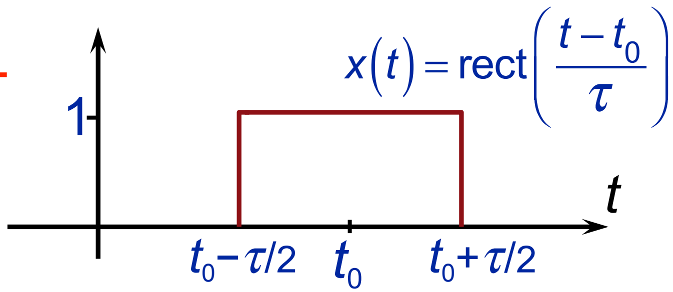
$$= \tau \cdot \text{sinc}_N(f\tau)$$

$$\left(= 0 \text{ då } \begin{cases} \omega = n \cdot \dfrac{2\pi}{\tau} \\ f = n \cdot \dfrac{1}{\tau} \end{cases}, \quad n = \pm 1, \pm 2, \pm 3, \ldots \right)$$

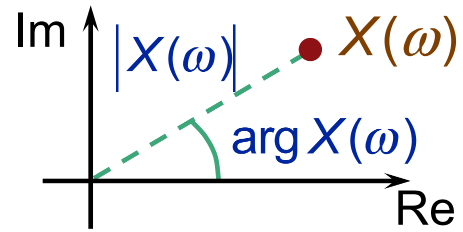


Ex. på frekvensspektrum $X(\omega)$

Exempel, **tidsförskjuten** fyrkantpuls med bredd τ & höjd 1:

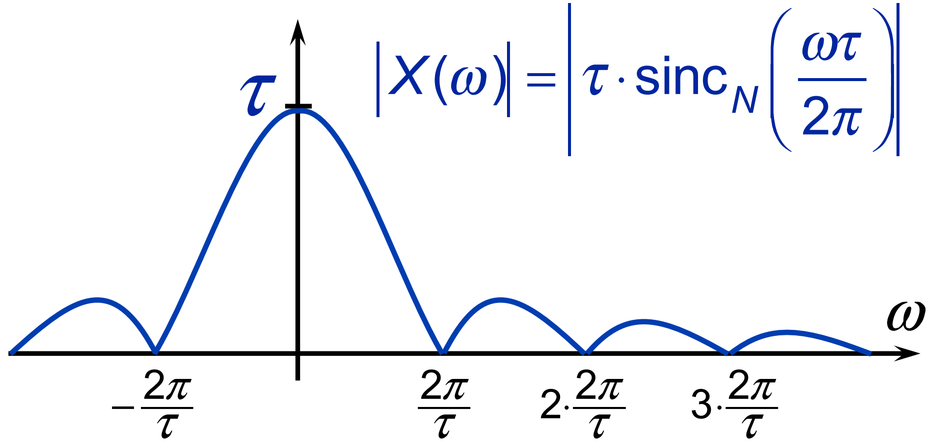


$$X(\omega) = \tau \cdot \mathrm{sinc}_N\left(\frac{\omega\tau}{2\pi}\right) \cdot e^{-j\omega t_0} = |X(\omega)|\, e^{j \cdot \arg X(\omega)}$$



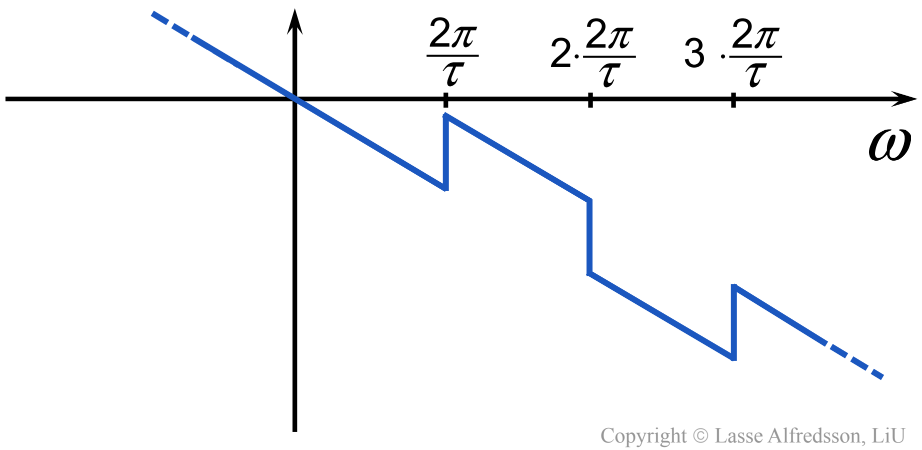
Amplitudspektrum:

$$|X(\omega)| = \left|\tau \cdot \mathrm{sinc}_N\left(\frac{\omega\tau}{2\pi}\right)\right|$$



Fasspektrum:

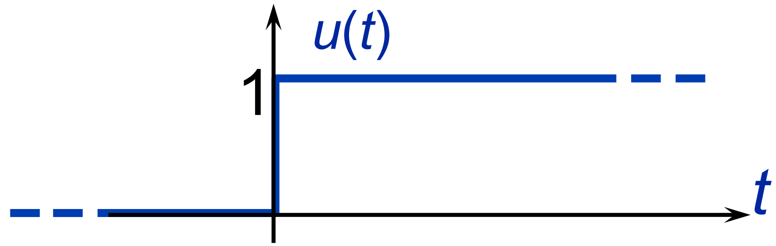
$$\arg X(\omega) = -\omega t_0 \;(\pm\pi)$$



Enhetssteget & diracimpulsen (Kap. 1.4-1 & 1.4-2)

- **Enhetssteget $u(t)$** (heavisidefunktionen, "*unit step function*")

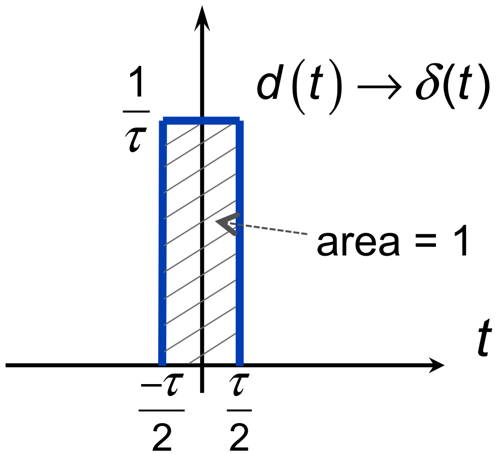
$$\boxed{u(t)=\begin{cases}1; & t\geq 0\\ 0; & t<0\end{cases}}$$



- **Diracimpulsen $\delta(t)$** ("*unit impulse function*")

$$\begin{cases}\delta(t)=0 & \forall\ t\neq 0\\ \int_{-\infty}^{\infty}\delta(t)\,dt=1\end{cases}$$

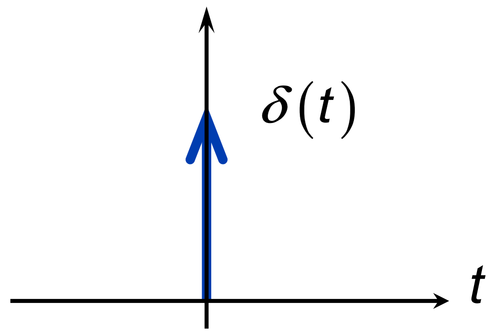
(*Paul Dirac:s egen "definition"*)



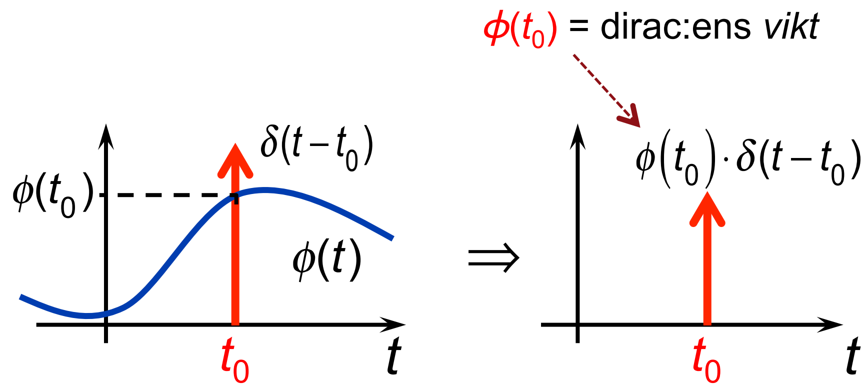
$d(t)\rightarrow\delta(t)$ då $\tau\rightarrow 0$

$$\overset{\tau\rightarrow 0}{\Longrightarrow}$$

(*Gränsvärdestolkning*)



Diracimpulsen (Kap. 1.4-2)



Egenskaper hos diracimpulsen:

1. $\phi(t)\cdot\delta(t-t_0)=\phi(t_0)\cdot\delta(t-t_0)$

1. $\Rightarrow$

2. $$\boxed{\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt=\phi(t_0)}$$

(”*The sampling/sifting property*”)

- $\delta(t)$ **definieras** av samband 2!
- $\delta(t)$ är en **distribution** (generaliserad funktion)

Distributioner definieras av sin *verkan*, via ett *integralsamband*, på andra (test-)funktioner (här är testfunktionen $\phi(t)$). (Distributionsteori ingår *inte* i kursen!)

3. $u(t)=\int_{-\infty}^{t}\delta(\tau)\,d\tau \quad\Rightarrow\quad \delta(t)=\dfrac{du(t)}{dt}$

Några centrala fouriertransformpar

$$\delta(t) \Leftrightarrow 1$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}; \quad a>0$$

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) \Leftrightarrow \pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right)$$

$$\sin(\omega_0 t) \Leftrightarrow j\pi\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)$$

$$\mathrm{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau\,\mathrm{sinc}_\mathrm{N}\left(\frac{\omega\tau}{2\pi}\right)=\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right), \quad \text{alt.} = \tau\,\mathrm{sinc}_\mathrm{N}(f\tau)$$

$$\mathrm{sinc}_\mathrm{N}(at)=\mathrm{sinc}(a\pi t) \Leftrightarrow \frac{1}{a}\mathrm{rect}\left(\frac{\omega}{2\pi a}\right)$$

$$\left(\text{Boken: } \frac{W}{\pi}\mathrm{sinc}(Wt) \Leftrightarrow \mathrm{rect}\left(\frac{\omega}{2W}\right)\right)$$

Några centrala fouriertransformegenskaper

Tidsförskjutning: $x(t-t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$

Frekvensförskjutning: $x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega-\omega_0)$

Tidsskalning: $x(a\cdot t) \Leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

Derivering: $\frac{d^n x(t)}{dt^n} \Leftrightarrow (j\omega)^n X(\omega)$

Dualitet: $X(t) \Leftrightarrow 2\pi x(-\omega)$

Konjugering: $x^*(t) \Leftrightarrow X^*(-\omega)$