Lösningsgångar till Lektionsuppgifter TSDT84 (period 1)

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1 Fourierserieutveckling

Notera att det är olika C_n och θ_n i de tre fourierserierna nedan!

1.1 Frekvensspektrum

1.1.1 **a)** Förekommande vinkelfrekvenser: $\omega_1 = 4$ rad/s, $\omega_2 = 7$ rad/s.

$$\frac{\omega_1}{\omega_2} = \frac{4}{7} \in \mathbb{Q} \Rightarrow \text{Periodisk signal, periodtid } T_0 = \frac{2\pi}{\omega_0} \text{ där}$$

$$\omega_0 = \text{SGD}(\omega_1, \omega_2) = \text{SGD}(2^2, 7) = 1 \text{ rad/s} \ \Rightarrow T_0 = 2\pi \text{ sek}$$

$\omega_1 = 4 \cdot \omega_0$ och $\omega_2 = 7 \cdot \omega_0$ $\Rightarrow$ deltonerna $n=4$ och $n=7$ ingår i fourierserien, dvs. $a(t) = \sum_{n=4,7} C_n \cdot \cos(n\omega_0 t + \theta_n)$

b) Vinkelfrekvenser: $\omega_1 = \pi$ rad/s, $\omega_2 = 2\pi$ rad/s.

$$\frac{\omega_1}{\omega_2} = \frac{\pi}{2\pi} = \frac{1}{2} \in \mathbb{Q} \Rightarrow \text{Periodisk med periodtid } T_0 = \frac{2\pi}{\omega_0} \text{ där}$$

$$\omega_0 = \text{SDG}(\omega_1, \omega_2) = \text{SGD}(\pi, 2\pi) = \pi \text{ rad/s} \ \Rightarrow T_0 = 2 \text{ sek}$$

$\omega_1 = 1 \cdot \omega_0$ och $\omega_2 = 2 \cdot \omega_0$ $\Rightarrow$ deltonerna $n=1$ (dvs. grundtonen) och $n=2$ ingår i fourierserien, dvs. $b(t) = \sum_{n=1,2} C_n \cdot \cos(n\omega_0 t + \theta_n)$

c) Vinkelfrekvenser: $\omega_1 = \sqrt{2}$ rad/s, $\omega_2 = 2$ rad/s.

$$\frac{\omega_1}{\omega_2} = \frac{\sqrt{2}}{2} \notin \mathbb{Q} \Rightarrow \text{Signalen är ej periodisk}$$

d) Vinkelfrekvenser: $\omega_1 = 3$ rad/s, $\omega_2 = \frac{15}{4}$ rad/s

$$\frac{\omega_1}{\omega_2} = \frac{3 \cdot 4}{15} \in \mathbb{Q} \Rightarrow \text{Periodisk med periodtid } T_0 = \frac{2\pi}{\omega_0}, \text{ där}$$

$$\omega_0 = \text{SGD}(\omega_1, \omega_2) = \text{SGD}\left(\frac{1}{4} \cdot 4 \cdot 3, \frac{1}{4} \cdot 3 \cdot 5\right) = \frac{3}{4} \text{ rad/s} \ \Rightarrow T_0 = \frac{8\pi}{3} \text{ sek}$$

$\omega_1 = 4\omega_0$ och $\omega_2 = 5\omega_0$ $\Rightarrow$ deltonerna $n=4$ och $n=5$ ingår i fourierserien, dvs. $c(t) = \sum_{n=4,5} C_n \cdot \cos(n\omega_0 t + \theta_n)$

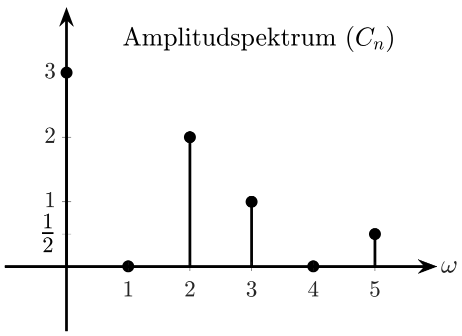
1.1.2 **a)**

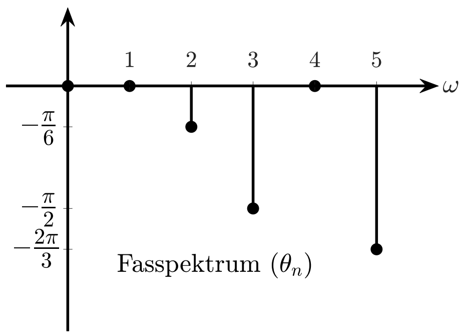
$$x(t) = 3 + \underbrace{\sqrt{3}\cos(2t) + \sin(2t)}_{\star} + \underbrace{\sin(3t)}_{=\cos\left(3t - \frac{\pi}{2}\right)} \underbrace{- \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right)}_{=\frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right)}$$

$$\begin{aligned}
\star &= \Big/ \sin(2t) = \cos\left(2t - \frac{\pi}{2}\right) \Big/ = \sqrt{3}\,\text{Re}\left\{e^{j2t}\right\} + \text{Re}\left\{e^{j\left(2t - \frac{\pi}{2}\right)}\right\} \\
&= \text{Re}\left\{\left(\sqrt{3} + e^{-j\frac{\pi}{2}}\right) e^{j2t}\right\} = \Big/ \sqrt{3} + e^{-j\frac{\pi}{2}} = \sqrt{3} - j = \sqrt{3+1}e^{j \arctan \frac{-1}{\sqrt{3}}} = 2e^{-j\frac{\pi}{6}} \Big/ \\
&= \text{Re}\left\{2e^{j\left(2t - \frac{\pi}{6}\right)}\right\} = 2\cos\left(2t - \frac{\pi}{6}\right)
\end{aligned}$$

Dvs. $x(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$

Enkelsidigt ("trigonometric") spektrum:

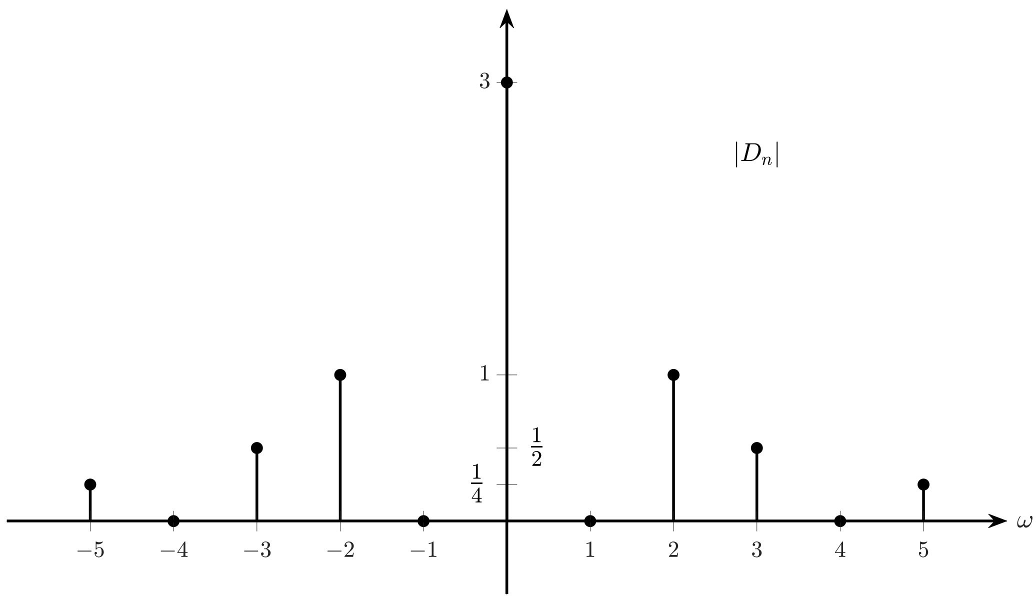


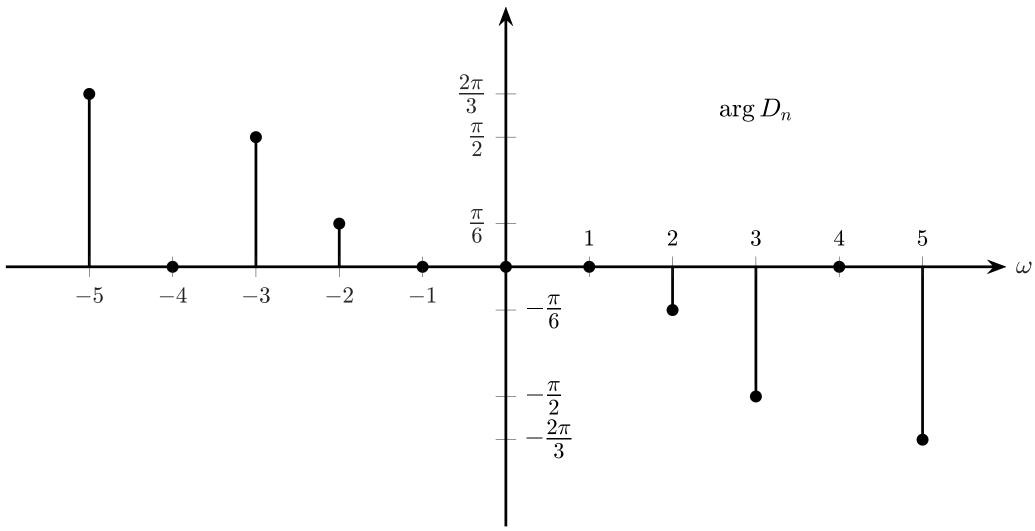


(Anm: Ej nödvändigt att ange spektrumkomponenterna vid $\omega = 1$ & 4 rad/s eftersom de inte finns med i $x(t)$)

b) $A \cdot \cos(\omega_0 t + \theta) = A \cdot \dfrac{e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)}}{2} = \dfrac{A}{2} e^{j\theta} \cdot e^{j\omega_0 t} + \dfrac{A}{2} e^{-j\theta} \cdot e^{-j\omega_0 t} \Rightarrow$

Dubbelsidigt ("exponential") spektrum:





c) Från spektrum i b) $\Rightarrow$

$$x(t) = \frac{1}{4}e^{j\frac{2\pi}{3}} \cdot e^{-j5t} + \frac{1}{2}e^{j\frac{\pi}{2}} \cdot e^{-j3t} + e^{j\frac{\pi}{6}} \cdot e^{-j2t} + 3 + \\ + \frac{1}{4}e^{-j\frac{2\pi}{3}} \cdot e^{j5t} + \frac{1}{2}e^{-j\frac{\pi}{2}} \cdot e^{j3t} + e^{-j\frac{\pi}{6}} \cdot e^{j2t}$$

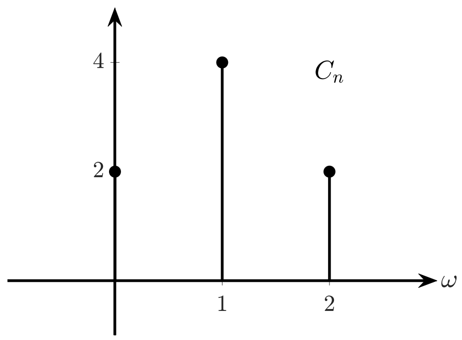
d) Omskrivning av $x(t)$ i c) $\Rightarrow$

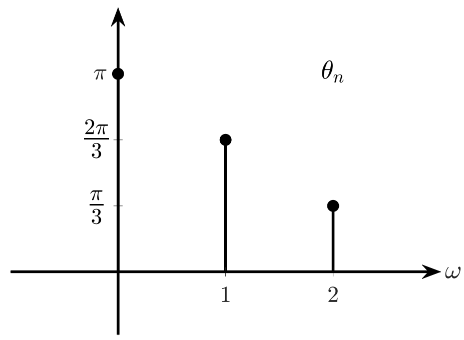
$$\begin{aligned} x(t) &= 3 + 2 \cdot \frac{e^{j\left(2t-\frac{\pi}{6}\right)} + e^{-j\left(2t-\frac{\pi}{6}\right)}}{2} + \frac{e^{j\left(3t-\frac{\pi}{2}\right)} + e^{-j\left(3t-\frac{\pi}{2}\right)}}{2} \\ &+ \frac{1}{2} \cdot \frac{e^{j\left(5t-\frac{2\pi}{3}\right)} + e^{-j\left(5t-\frac{2\pi}{3}\right)}}{2} \\ &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right) \\ &\left(= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \sin(3t) - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right) \right) \end{aligned}$$

1.1.3 **a)** Figuren $\Rightarrow$

$$\begin{aligned} x(t) &= 1 \cdot e^{-j\frac{\pi}{3}} \cdot e^{-j2t} + 2 \cdot e^{-j\frac{2\pi}{3}} \cdot e^{-jt} + 2 \cdot e^{\pm j\pi} + 2 \cdot e^{j\frac{2\pi}{3}} \cdot e^{jt} + 1 \cdot e^{j\frac{\pi}{3}} \cdot e^{j2t} \\ &= -2 + 4 \cdot \frac{e^{j\left(t+\frac{2\pi}{3}\right)} + e^{-j\left(t+\frac{2\pi}{3}\right)}}{2} + 2 \cdot \frac{e^{j\left(2t+\frac{\pi}{3}\right)} + e^{-j\left(2t+\frac{\pi}{3}\right)}}{2} \\ &= -2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right) \end{aligned}$$

b) $C_n = 2|D_n|; n > 0, \;\; C_0 = D_0, \;\; \theta_n = \arg D_n \; (\angle D_n) \quad \Rightarrow$





(Anm: $C_0 = -2 = 2 \cdot e^{j\pi}$)

c) Lösningen till (grafen i)deluppgift b) $\Rightarrow$

$$x(t) = -2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)$$

d) Trivialt – se $x(t)$ i deluppgift a) och c).

1.2 Fourierserieutveckling

1.2.1 **a)** Periodtid $T_0 = \frac{\pi}{2}$ sek $\Rightarrow$ Grundvinkelfrekv. $\omega_0 = \frac{2\pi}{T_0} = 4$ rad/s.

$$y(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\omega_0 t},\ \text{där}$$

$$\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} y(t)e^{-jn\omega_0 t}dt = \frac{2}{\pi}\int_0^{\pi/2} e^{-t}\cdot e^{-jn4t}dt = \frac{2}{\pi}\int_0^{\pi/2} e^{-(1+j4n)t}dt \\
&= \frac{2}{\pi}\left[\frac{e^{-(1+j4n)t}}{-(1+j4n)}\right]_0^{\pi/2} = \frac{2}{-\pi(1+j4n)}\left(e^{-(1+j4n)\frac{\pi}{2}} - e^0\right) \\
&= \Big/ e^{-(1+j4n)\frac{\pi}{2}} = e^{-\frac{\pi}{2}}\cdot e^{-jn2\pi} = e^{-\frac{\pi}{2}} \Big/ = \frac{2\left(1-e^{-\frac{\pi}{2}}\right)}{\pi(1+j4n)}
\end{aligned}$$

$$C_0 = D_0 = \frac{2}{\pi}\left(1 - e^{-\frac{\pi}{2}}\right) \approx 0,504$$

$$C_{n>0} = 2|D_n| = \frac{4\left(1-e^{-\frac{\pi}{2}}\right)}{\pi\sqrt{1^2+(4n)^2}} \approx \frac{1,01}{\sqrt{1+16n^2}}$$

$$\theta_n = \arg D_n = -\arctan\frac{4n}{1}$$

Svar: $y(t) \approx 0,504 + \sum_{n=1}^{\infty} \frac{1,01}{\sqrt{1+16n^2}} \cos(4nt - \arctan(4n))$

b) $y(t) = x\left(\frac{t}{2}\right) \Rightarrow x(t) = y(2t) = 0,504 + \sum_{n=1}^{\infty} \frac{1,01}{\sqrt{1+16n^2}} \cos(\underbrace{4n \cdot 2t}_{=8nt} - \arctan(4n))$

c) $x(t)$ har samma fourierseriekoefficienter C_0, C_n, θ_n som $y(t)$,
$\Rightarrow$ $x(t)$ har även samma komplexa fourierseriekoefficienter D_n – se uppgift a).
Doch är periodtiden för $x(t)$ hälften så stor som periodtiden för $y(t)$ $\Rightarrow$ dess grundvinkelfrekvens är dubbelt så stor; $\omega_{0,y} = 8$ rad/s.

d)

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$\Rightarrow x(at) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n(a\omega_0)t + \theta_n)$$

$$\Rightarrow C_0,\ C_n \text{ och } \theta_n \text{ är oförändrade}$$

Om $a > 1 \Rightarrow$ grundvinkelfrekvensen hos signalen $x(at)$ är en faktor a **högre** än grundvinkenfrekvensen hos $x(t) \Rightarrow x(at)$ är en **tidskomprimerad** version av $x(t)$; komprimerad en faktor a.

Om $a < 1 \Rightarrow$ grundvinkelfrekvensen $a\omega_0$ hos $x(at)$ är **lägre** än grundvinkelfrekvensen ω_0 hos $x(t) \Rightarrow x(at)$ är en **tidsexpanderad** version av $x(t)$ (expanderad en faktor $\frac{1}{a}$).

1.2.2 **a)** Figuren $\Rightarrow$ periodtid $T_0 = 4$ sek $\Rightarrow$ Grundvinkelfrekvens $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$ rad/s. Kalla signalen $x_a(t)$.

$$\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} x_a(t)e^{-jn\omega_0 t}dt = \frac{1}{4}\int_{-1}^{1} 1\cdot e^{-jn\frac{\pi}{2}t}dt + \frac{1}{4}\int_{1}^{3}(-1)e^{-jn\frac{\pi}{2}t}dt \\
&= /n \neq 0 \text{ ty division med } n/ = \frac{1}{4}\left[\frac{e^{-jn\frac{\pi}{2}t}}{-jn\frac{\pi}{2}}\right]_{-1}^{1} - \frac{1}{4}\left[\frac{e^{-jn\frac{\pi}{2}t}}{-jn\frac{\pi}{2}}\right]_{1}^{3} \\
&= \frac{2}{-4jn\pi}\left(e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}} - \left(\underbrace{e^{-jn\frac{\pi}{2}\cdot 3}}_{=e^{jn\frac{\pi}{2}}} - e^{-jn\frac{\pi}{2}}\right)\right) = \frac{2}{n\pi}\cdot\frac{e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}}{2j} \\
&= \frac{2}{n\pi}\sin\left(n\cdot\frac{\pi}{2}\right) \quad n \neq 0
\end{aligned}$$

$$D_0 = \frac{1}{T_0}\int_{T_0} x_a(t)\cdot\underbrace{e^{j0\cdot\omega_0 t}}_{=1}dt = \frac{1}{4}\int_{-1}^{1} 1dt + \frac{1}{4}\int_{1}^{3}(-1)dt = 0$$

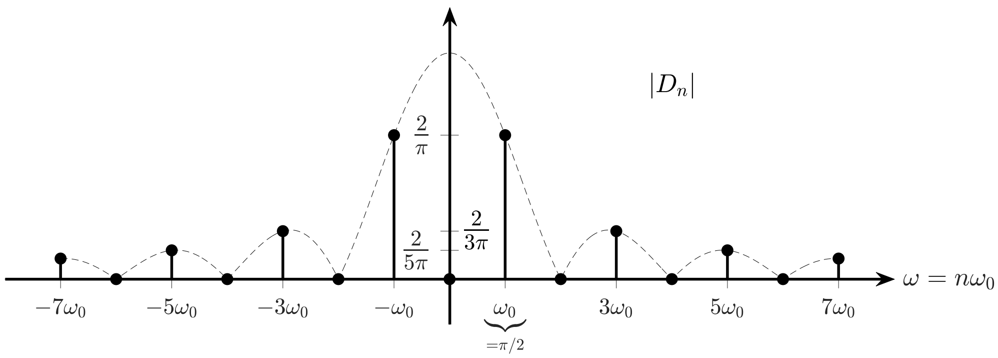
(vilket även ses direkt i figuren; D_0 = medelvärdet)

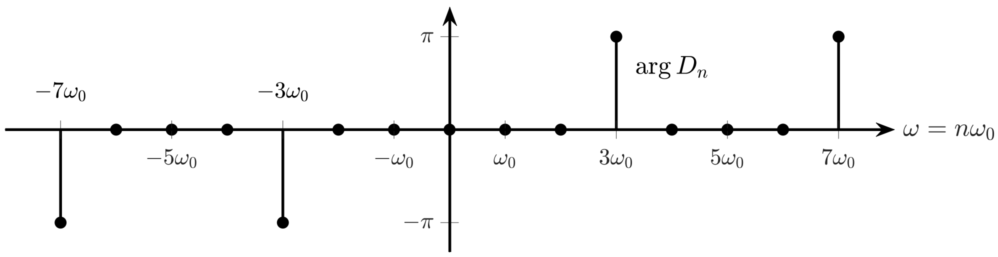
Svar:

$$D_n = \begin{cases} 0; & n = 0 \\ \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right); & n \neq 0 \ (\Rightarrow D_n = 0 \text{ för jämna } n) \end{cases}$$

$$D_n = \frac{2}{n\pi}\sin\left(n \cdot \frac{\pi}{2}\right) = \begin{cases} 0; & n = 0 \ (\text{även } n = 0) \\ \frac{2}{n\pi} = \frac{2}{n\pi}e^{j0}; & n = 1, 5, 9, \ldots \\ \frac{-2}{n\pi} = \frac{2}{n\pi} \cdot e^{j\pi}; & n = 3, 7, 11, \ldots \end{cases}$$

$$D_{n<0} = D_{-n}^* \Rightarrow D_{-1} = D_1^*, D_{-3} = D_3^*, \text{osv:}$$



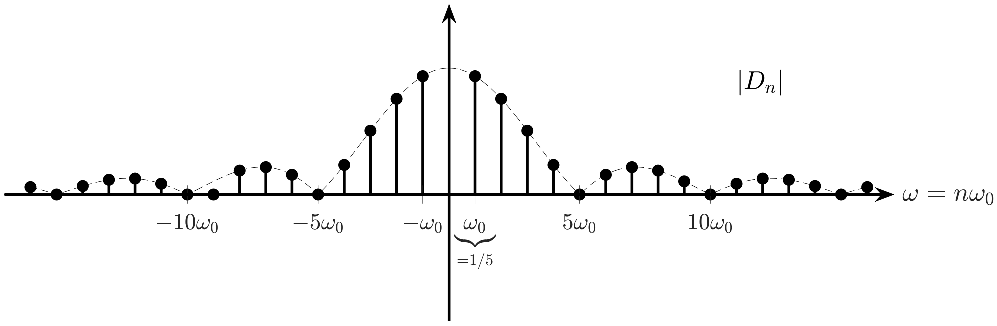


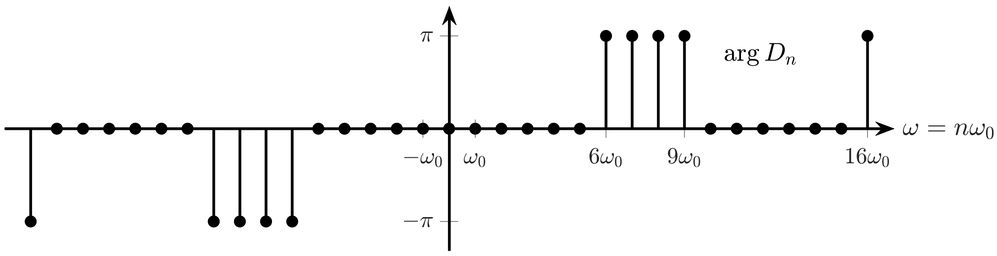
b) Låt $x_b(t)$ = signalen, med periodtid $T_0 = 10\pi$ sek $\Rightarrow$ Grundvinkelfrekvens $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$ rad/s. Dvs. $x_b(t) = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\omega_0 t}$, där

$$D_n = \frac{1}{T_0}\int_{T_0} x_b(t)e^{-jn\omega_0 t}dt = \frac{1}{10\pi}\int_{-\pi}^{\pi} 1 \cdot e^{-jn\frac{1}{5}t}dt = /n \neq 0 \text{ ty division med } n/$$

$$= \frac{1}{10\pi}\left[\frac{e^{-jn\frac{t}{5}}}{-jn\frac{1}{5}}\right]_{-\pi}^{\pi} = \frac{1}{-jn2\pi}\left(e^{-jn\frac{\pi}{5}} - e^{jn\frac{\pi}{5}}\right) = \frac{1}{n\pi}\frac{e^{jn\frac{\pi}{5}} - e^{-jn\frac{\pi}{5}}}{2j} = \frac{1}{n\pi} \cdot \sin\left(n\frac{\pi}{5}\right)$$

$$D_0 = \frac{1}{T_0}\int_{T_0} x_b(t)e^{-j0\omega_0 t}dt = \frac{1}{10\pi}\int_{-\pi}^{\pi} 1dt = \frac{1}{5}$$





c) $x_c(t)$ har period $T_0 = 2\pi$ sek $\Rightarrow \omega_0 = \dfrac{2\pi}{T_0} = 1$ rad/s

$x_c(t) = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\omega_0 t}$, där

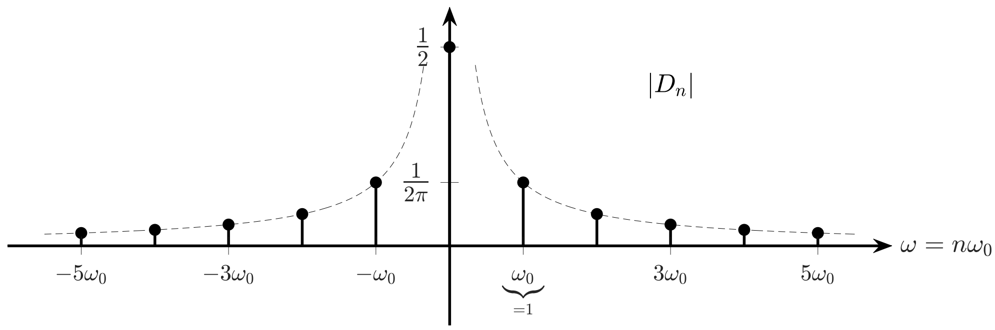
$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{T_0} x_c(t)e^{-jn\omega_0 t}dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}\cdot e^{-jnt}dt = /n \neq 0/ \\ &= \frac{1}{4\pi^2}\left[\frac{t\cdot e^{-jnt}}{-jn}\right]_0^{2\pi} - \frac{1}{4\pi^2}\int_0^{2\pi}\frac{e^{-jnt}}{-jn}dt \\ &= \frac{j}{4\pi^2 n}\left(2\pi\cdot e^{-jn2\pi} - 0\right) - \frac{1}{4\pi^2(-jn)^2}\underbrace{\left[e^{-jnt}\right]_0^{2\pi}}_{=e^{-jn2\pi}-e^0=1^n-1=0} = \frac{j}{2\pi n};\ n \neq 0 \end{aligned}$$

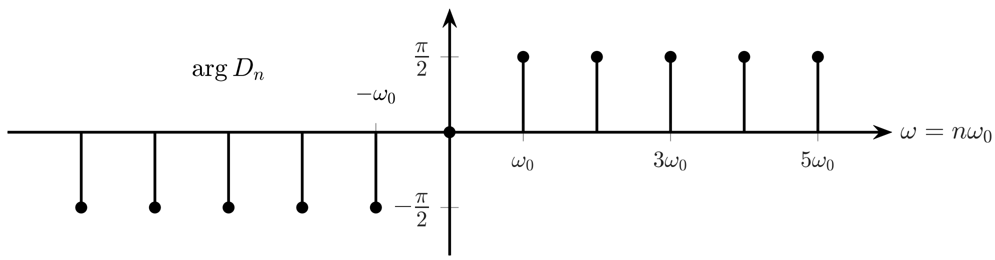
$$D_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}\cdot e^0 dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{2}$$

(vilket även ses direkt i figuren)

Svar:

$$D_n = \begin{cases} \frac{j}{2\pi n}; & n \neq 0 \\ \frac{1}{2}; & n = 0 \end{cases} \left(\Rightarrow \arg D_n = \begin{cases} \frac{\pi}{2}; & n > 0 \\ 0; & n = 0 \\ -\frac{\pi}{2}; & n < 0 \end{cases}\right)$$





1.2.3 **a)** $x(t)$ är periodisk med periodtid $T_0 = 8$ sek $\Rightarrow$ Grundvinkelfrekvensen är $\omega_0 = \dfrac{2\pi}{T_0} = \dfrac{\pi}{4}$ rad/s

$$x(t) = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\omega_0 t}, \text{ där}$$

$$\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{8}\int_{-4}^{0}\left(\frac{t}{2}+1\right)e^{-jn\frac{\pi}{4}t}dt + \frac{1}{8}\int_{0}^{4}\left(-\frac{t}{2}+1\right)e^{-jn\frac{\pi}{4}t}dt \\
&= /n \neq 0 \text{ ty division med } n/ \\
&= \frac{1}{8}\left[\left(\frac{t}{2}+1\right)\frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}}\right]_{-4}^{0} - \int_{-4}^{0}\frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}\cdot 8\cdot 2}dt \\
&+ \frac{1}{8}\left[\left(-\frac{t}{2}+1\right)\frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}}\right]_{0}^{4} - \int_{0}^{4}\frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}\cdot 8\cdot(-2)}dt \\
&= \frac{1}{8}\cdot\frac{e^0+e^{jn\pi}}{-jn\cdot\frac{\pi}{4}} - \left[\frac{e^{-jn\frac{\pi}{4}t}}{\left(-jn\frac{\pi}{4}\right)^2\cdot 16}\right]_{-4}^{0} + \frac{1}{8}\cdot\frac{-e^0-e^{jn\pi}}{-jn\cdot\frac{\pi}{4}} + \left[\frac{e^{jn\frac{\pi}{4}t}}{\left(-jn\frac{\pi}{4}\right)^2\cdot 16}\right]_{0}^{4} \\
&= \bigg/ \frac{e^0+e^{jn\pi}}{-jn\cdot\frac{\pi}{4}} + \frac{-e^0-e^{jn\pi}}{-jn\cdot\frac{\pi}{4}} = 0 \bigg/ = \frac{e^{jn\pi}-e^0}{-n^2\cdot\pi^2} + \frac{e^{-jn\pi}-e^0}{-n^2\cdot\pi^2} = \frac{2\left(1-(-1)^n\right)}{n^2\pi^2} \\
&= \begin{cases} 0; & \text{jämna } n \neq 0 \\ \frac{4}{n^2\pi^2}; & \text{udda } n \end{cases}
\end{aligned}$$

$$D_0 = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^0 dt = \text{/framgår direkt av figuren/} = 0$$

Svar:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{4}t} \text{ där } D_n = \begin{cases} 0; & \text{jämna } n \neq 0 \\ \frac{4}{n^2\pi^2}; & \text{udda } n \end{cases}$$

b) Vi ser att

$$\hat{x}(t) = x(t-2) = \text{/uppg. a)/} = \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{4}(t-2)} = \sum_{n=-\infty}^{\infty} D_n \cdot e^{-jn\frac{\pi}{2}} \cdot e^{jn\frac{\pi}{4}t}$$

$$= \sum_{n=-\infty}^{\infty} \hat{D}_n \cdot e^{jn\frac{\pi}{4}t}, \text{ där}$$

$$\hat{D}_n = D_n \cdot e^{-jn\frac{\pi}{2}} = \begin{cases} 0; & \text{jämna } n \\ \frac{4 \cdot e^{-jn\frac{\pi}{2}}}{n^2\pi^2}; & \text{udda } n \end{cases}$$

c) Figur $\Rightarrow$

$$\tilde{x}(t) = x(2t) = \text{/a)/} = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\frac{\pi}{4}\cdot 2t} = \sum_{n=-\infty}^{\infty} D_n \cdot e^{jn\frac{\pi}{2}t}, \text{ dvs.}$$

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \tilde{D}_n \cdot e^{jn\frac{\pi}{2}t}, \text{ där } \tilde{D}_n = D_n = \begin{cases} 0; & \text{jämna } n \\ \frac{4}{n^2\pi^2}; & \text{udda } n \end{cases}$$

Anm. grundvinkelfrekvensen för $\tilde{x}(t)$ är $\tilde{\omega}_0 = \frac{\pi}{2}$ rad/s, medan grundvinkelfrekvensen för $x(t)$ är $\omega_0 = \frac{\pi}{4}$ rad/s, dvs. $\tilde{x}(t) = x(2t) \Rightarrow \tilde{\omega}_0 = 2 \cdot \omega_0$