6 The Fourier transform

In this presentation we assume that the reader is already familiar with the Fourier transform. This means that we will not make a complete overview of its properties and applications. Instead, we focus on how it relates to multi-dimensional signal analysis. We begin by formally defining the Fourier transform, which is done for several classes of functions and illustrates that there is a whole class of transforms referred to as Fourier transforms. They differ only in what classes of functions they are applied to and in their details of definition, their general properties are very similar. Some of the more intricate issues of the Fourier transform are not discussed here, such as for which functions can we expect the Fourier transform to be well-defined.

6.1 Fourier transform of continuous variable functions

Let f be a function $f : \mathbb{R} \to \mathbb{C}$. We define its Fourier transform as the function F given as

$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt \quad \text{with inverse transform given as} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{iut}\, du \tag{73}$$

Maybe this way of defining the Fourier transform is not exactly what you are used to. Possibly you have instead seen something like

$$F'(u) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi iut}\, dt \quad \text{with inverse transform given as} \quad f(t) = \int_{-\infty}^{\infty} F'(u)\, e^{2\pi iut}\, du \tag{74}$$

Alternatively, you have seen the Fourier transform defined as

$$F''(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt \quad \text{with inverse transform given as} \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F''(u)\, e^{iut}\, du \tag{75}$$

Notice that for a fixed function f, the three different versions of the Fourier transform described above are related as

$$F(u) = F'\left(\frac{u}{2\pi}\right) = \sqrt{2\pi}\, F''(u) \tag{76}$$

Since they are so closely related, it should be clear that it does not matter which definition we use as long as we stick to that definition throughout the calculations where it is used. Personal preferences are usually the only reason for choosing a particular definition over the other. Here we will mainly use the first definition, given in (73), since it often leads to simpler expressions in this particular presentation. However, we will also have reasons to occasionally consider the third one, given in (75).

Regardless of which of the above definitions we use for the Fourier transform, it is the case that the set of functions f for which F is well-defined is a vector space, i.e., if f_1 and f_2 are two function with well-defined Fourier transforms, then $c_1 f_1 + c_2 f_2$ is also a function with well-defined Fourier transform. In the literature there are several proposals for how to formally define the vector space of Fourier transformable functions. These, in turn, depend on the fact that it is possible to interpret the integral in (73) in different ways. It turns out that from a practical point of view these different spaces are very similar, in that they contain a large common subset of functions that appear in many applications. By choosing the appropriate vector space it may be possible to transform particular functions at the expense of not being able to transform others. In the following presentation we will identify the space of Fourier transformable functions with the signal space V and not go into any detailed discussions about exactly which functions there are in V. An example of such a spaces is $\mathcal{L}^2$, but at this point we do not have to restrict ourselves to this space.

The set of functions F that are the Fourier transforms of the functions in V is again a vector space, which follows from the fact that the transform in invertible, and we refer to this space as $V_{\mathcal{F}}$. The apparent symmetry of the Fourier transform and its inverse maybe suggests that $V = V_{\mathcal{F}}$, but in this presentation this issue is not discussed in detail. It suffices to say that certain choices of V lead to $V = V_{\mathcal{F}}$, but other valid choices of V do not satisfy this equality. That said, we may still refer to the elements of $V_{\mathcal{F}}$ as signals, even though they are represented in another form than the signals in V.

To summarise the results so far, we see the Fourier transform in (73), here denoted $\mathcal{F}$, as a linear mapping $\mathcal{F} : V \to V_{\mathcal{F}}$. We write $\mathcal{F}f = F$ to denote that the Fourier transform applied to function f gives the function F.

In the literature, both the resulting function F and the transform $\mathcal{F}$ are referred to as a Fourier transform. It should be clear form the context which of the two we mean in the following presentation.

6.2 Fourier transform of discrete variable functions

In many practical applications, the signal is a function of a discrete variable, typically as a result of a sampling process of a continuous signal. In the following presentation, we assume that the resulting discrete function has N integer valued variables, e.g., $X = \mathbb{Z}^N$. Assuming that the continuous variable has been sampled as integer values is not necessary, but simplifies the presentation. In this context, however, we need to distinguish between the case that X is infinite or finite. We begin with the first case.

If we insert a function f of a discrete variable into (73) the result is not well-defined without additional tricks. For example, we can replace the integral with a summation and get the following definition of the Fourier transform

$$F(u) = \sum_{k=-\infty}^{\infty} f[k]\, e^{-iuk} \tag{77}$$

Notice that F in this case is not discrete even though f is discrete. Instead, it follows directly that F is 2π-periodic: $F(u + 2\pi p) = F(u)$ for $p \in \mathbb{Z}$. This means that it is sufficient to know F in an interval of length 2π, e.g., $u \in]-\pi, \pi]$. This property is reflected by the inverse transform, given as

$$f[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(u)\, e^{iuk}\, du \tag{78}$$

6.3 Fourier transform of a truncated discrete signal

In the case that f is both discrete and truncated, i.e., it consists of a finite number of samples that we can assume to be defined for $k = 0, \ldots, P-1$, the Fourier transform according to (77) is given as

$$F(u) = \sum_{k=0}^{P-1} f[k]\, e^{-iuk} \tag{79}$$

This function F is still 2π-periodic of the real variable u. However, since V in this case is P-dimensional it must be the case that also $V_{\mathcal{F}}$ is P-dimensional. In fact, if we sample F in the interval $]-\pi, \pi]$ at P points it is possible to reconstruct either F or f from these samples. In particular if we use the sample points given as

$$u_l = \frac{2\pi l}{P}, \quad l = 0, \ldots, P-1 \tag{80}$$

we can express the inverse transform as

$$f[k] = \frac{1}{P} \sum_{l=0}^{P-1} F(u_l)\, e^{iu_l k} \tag{81}$$

In this case we can formally define the Fourier transform as a function of a discrete variable:

$$F[l] = F(u_l) = \sum_{k=0}^{P-1} f[k]\, e^{-2\pi i k l / P} \tag{82}$$

which has an inverse transform given as

$$f[k] = \frac{1}{P} \sum_{l=0}^{P-1} F[l]\, e^{2\pi i k l / P} \tag{83}$$

6.4 Multi-variable Fourier transform

The transform defined in (73) can be extended also to functions of two or more real variables. In the two-variable case, the transform is defined as

$$F(u_1, u_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, e^{-i(u_1 x_1 + u_2 x_2)}\, dx_1\, dx_2 \tag{84}$$

with inverse transform given as

$$f(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u_1, u_2)\, e^{i(u_1 x_1 + u_2 x_2)}\, du_1\, du_2 \tag{85}$$

By using the notation $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u} = (u_1, u_2)$, the last two equations can also be expressed in a more compact form as

$$F(\mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-i(\mathbf{u}\cdot\mathbf{x})}\, d^2\mathbf{x} \tag{86}$$

with inverse transform given as

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F(\mathbf{u})\, e^{i(\mathbf{u}\cdot\mathbf{x})}\, d^2\mathbf{u} \tag{87}$$

where $\mathbf{u} \cdot \mathbf{x} = u_1 x_1 + u_2 x_2$.

Similarly, the Fourier transform for functions of two discrete variables can extended from (77) and (78) to

$$F(\mathbf{u}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} f[\mathbf{k}]\, e^{-i\mathbf{u}\cdot\mathbf{k}} \tag{88}$$

and

$$f[\mathbf{k}] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\mathbf{u})\, e^{i\mathbf{u}\cdot\mathbf{k}}\, d^2\mathbf{u} \tag{89}$$

where $\mathbf{k}$ denoted the two discrete variables $\mathbf{k} = (k_1, k_2)$ where $k_1, k_2 \in \mathbb{Z}$.

Finally, the Fourier transform for truncated functions of two discrete variables can be extended from (82) and (83) to

$$F[\mathbf{l}] = \sum_{k\in\Omega} f[\mathbf{k}]\, e^{-2\pi i \mathbf{k}\cdot\mathbf{l}/P} \tag{90}$$

and

$$f[\mathbf{k}] = \frac{1}{P^2} \sum_{l\in\Omega} F[\mathbf{l}]\, e^{2\pi i \mathbf{k}\cdot\mathbf{l}/P} \tag{91}$$

where $\Omega = [0, \ldots, P-1] \times [0, \ldots, P-1]$.

These types of generalisations of the Fourier transform extend straight-forward to arbitrary number of outer dimensions of the function f.

6.5 Relation to multi-dimensional signal analysis

Having reviewed the definitions of the various types of Fourier transforms that will be used in this presentation, we now turn to the properties of the transforms that relate to multi-dimensional signal analysis, some of which has already been illustrated in Section 3. The result of this discussion is that both the Fourier transform and its inverse can be regarded either as computing scalar products or linear combination depending on the application. Consequently, the Fourier transform and its inverse can be regarded either as an analysing or reconstructing operation, and we can choose the view that serves our purpose best.

For example, in the case of truncated functions of a discrete variable, a scalar product in the signal space can be defined as

$$\langle\, f \mid g\,\rangle = \sum_{k=0}^{P-1} f[k]\, \bar{g}[k], \tag{92}$$

and given the set of functions

$$b_l[k] = e^{2\pi ikl/P}, \quad l = 0, \dots, P-1 \tag{93}$$

we can write $F[l]$ in (82) as $F[l] = \langle f \mid b_l \rangle$. There are P such functions and they are linearly independent, which means that they form a basis for the P-dimensional space V. We refer to this basis as the "Fourier basis". The scalar products $F[l]$ correspond to the dual coordinates of f relative to the Fourier basis. By means of the inverse Fourier transform, (83), the signal f can be reconstructed from the dual coordinates $F[l]$ in a linear combination with the dual basis vectors given as

$$\tilde{b}_l[k] = \frac{1}{P}\, e^{2\pi ikl/P}, \quad l = 0, \dots, P-1 \tag{94}$$

Due to the symmetry of the Fourier transform and its inverse, we can also say that the inverse Fourier transform (83) computes $f[k]$ as the scalar products between $F[l]$ and the set of basis functions given as

$$a_k[l] = e^{-2\pi ikl/P}, \quad k = 0, \dots, P-1 \tag{95}$$

We refer to this set of functions as the "inverse Fourier basis". This means that $f[k]$ are the dual coordinates relative to the inverse Fourier basis, and that the function $F[l]$ can be reconstructed as a linear combination between $f[k]$ and the dual of the inverse Fourier basis, given by the functions

$$\tilde{a}_l[k] = \frac{1}{P}\, e^{-2\pi ikl/P}, \quad k = 0, \dots, P-1 \tag{96}$$

In this particular example V is of finite dimension, and it follows then that all four bases above are bases of $V = V_{\mathcal{F}}$.

Before we continue, we note that the critical issue in the current discussion is that the basis used for analysis and the basis used for reconstruction are dual relative to each other. The basis functions a_l and $\tilde{a}_l$ are dual relative to each other using the scalar product in (92):

$$\langle\, a_{l_1} \mid \tilde{a}_{l_2} \,\rangle = \frac{1}{P} \sum_{k=0}^{P-1} e^{2\pi ik(l_1 - l_2)/P} = \delta[l_1 - l_2] = \begin{cases} 1 & l_1 = l_2 \\ 0 & l_1 \neq l_2 \end{cases} \tag{97}$$

In the same way we can show that $\langle\, b_{k_1} \mid \tilde{b}_{k_2} \,\rangle = \delta[k_1 - k_2]$.

A similar approach can be applied to the discrete variable Fourier transform defined in Section 6.2. For example, the discrete variable signal f in (78) can interpreted as a linear combination of the set of functions

$$b_u[k] = \frac{e^{iuk}}{2\pi} \tag{98}$$

where $u \in]-\pi, \pi]$. The complication is that this is an infinite set, in fact even innumerable which is why they are integrated rather than added together. The signal space V, on the other hand, is in this case infinite but enumerable. Furthermore, each value of the Fourier transform function $F(u)$ can be seen as the scalar product between f and the functions

$$\tilde{b}_u[k] = e^{iuk} \tag{99}$$

for $u \in]-\pi, \pi]$ using the scalar product defined in (19). Again, there are innumerably infinitely many such functions. Consequently, it is not straight-forward to place this type of transform in the proposed framework of analysis and reconstruction. Again, however, it turns out that there is a dual relation between the analysing functions in (98) and the reconstructing functions in (99):

$$\langle\, b_{u_1} \mid \tilde{b}_{u_2} \,\rangle = \sum_{k=-\infty}^{\infty} \frac{e^{iu_1 k}}{2\pi}\, e^{-iu_2 k} = \delta(u_1 - u_2) \tag{100}$$

Intuitively, we see that the right hand side of this equation is either zero or some type of unit value, similar to what we have in (12). This works as a generalisation of the concept of dual sets to the case when they are not enumerable.

Returning to the continuous variable Fourier transform in Section 6.1, we can maintain the idea of dual bases using the scalar product in (20). In this case we notice that the two sets of functions

$$b_u(t) = \frac{e^{iut}}{2\pi} \quad \text{and} \quad \tilde{b}_u(t) = e^{iuk} \tag{101}$$

for $u \in \mathbb{R}$ are in a dual relation to each other:

$$\langle\, b_{u_1} \mid \tilde{b}_{u_2} \,\rangle = \int_{-\infty}^{\infty} \frac{e^{iu_1 t}}{2\pi}\, e^{-iu_2 t}\, dt = \delta(u_1 - u_2) \tag{102}$$

The difficulty of the last two examples, when V is infinite dimensional, is that it is not clear that the "basis functions" actually are elements of V. Depending on which formal definition of V we use, it may be the case that they are in fact not in V. In this case, the idea that they are a basis of V or that they can be used in a scalar product are not well-defined. We will leave these issues here, and merely point out that sets functions that are in a dual relation to each other seem to be useful.

7 Convolution

Convolution is a very common operation that we apply to signals. Before moving on to discuss convolution in relation to multi-dimensional signal analysis, we formally define the operation. Let g and f be two variables of the same domain X. The convolution between f and g, denoted $f * g$ is a function h defined as

$$h(x) = (f * g)(x) = \int_{x \in X} f(y)\, g(x - y)\, dy \tag{103}$$

This definition is very general and includes both various cases of outer dimensions of the functions f and g and also the possibility of the variables to be discrete. In the case that $X = \mathbb{R}$ we get the familiar expression for h:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x - y)\, dy \tag{104}$$

Similarly, for $X = \mathbb{R}^2$ we get

$$h(x_1, x_2) = (f * g)(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)\, g(x_1 - y_1, x_2 - y_2)\, dy_1\, dy_2 \tag{105}$$

which can be written in a somewhat more compact form as

$$h(\mathbf{x}) = (f * g)(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{y})\, g(\mathbf{x} - \mathbf{y})\, d^2\mathbf{y} \tag{106}$$

with $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. This generalises to higher outer dimensions of the functions f and g in the straight-forward way. Equation (103) also includes the case when f and g are functions of discrete variables. For example, with $X = \mathbb{Z}$ the integral in (103) simplifies to a sum:

$$h[k] = (f * g)[k] = \sum_{l=-\infty}^{\infty} f[l]\, g[k - l] \tag{107}$$

Again, this expression can be generalised to higher outer dimensions of the functions f and g in the straight-forward manner. Notice that (103) implies that $f * g = g * f$.

From the definition of the convolution operation it is not obvious why it so useful. There are, in fact, several motivations for why this operation is so common in signal processing. In the context of signal processing, it is often the case that one of the two functions is a signal and the other a filter. Here we will assume that f is a signal and g a filter. This means that the input the the convolution operation is a signal that we have obtained somehow, e.g., by measuring the air pressure with a microphone as a function of time, or by measuring the amount of light at an array of pixels in a digital camera, and a filter that has been chosen to transform f into h. The distinction of the two functions f and g into signal and

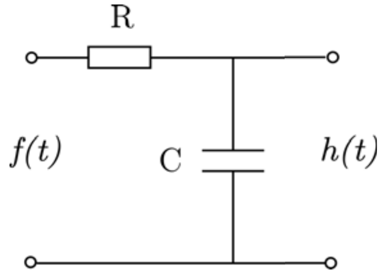

Figure 10: An RC-circuit.

filter means that we assume the signal f to be a particular observation of a signal vector that is drawn from some statistical model of the signal space. The filter, on the other hand, is assumed to be a fixed function that should work for any observation of the signal function.

Convolution appears naturally as the operation that produces an output signal from a in input signal that is fed into a linear circuit. For example, in Figure 10 we see a simple RC-filter. It has an input signal f and an output signal h. The relation between the two is given as

$$h(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \quad \text{with} \quad g(\tau) = \frac{1}{RC}\, e^{-\frac{\tau}{RC}}\, \sigma(\tau) \quad \text{and} \quad \sigma(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases} \tag{108}$$

As long as we represent output signal h as a function of time (t), it is given as the convolution between the input signal f and a function related to the parameters of the RC-filter, the so-called impulse response function g. This observation generalises to more complex types of linear circuits and also to many other systems that produces an output signal as a linear function of the input signal.

This brings us to a second motivation for the convolution operation. Let us consider set of all linear transformation on the signal space that are also *shift invariant*. This means that for every signal function $f \in V$ we consider linear transformations $T : V \to V$ such that

$$(T\, f)(x+y) = (T\, f(\cdot + y))(x) \tag{109}$$

Here $f(\cdot + y)$ means the function f shifted by y. By using a dot ("·") we avoid specifying a particular variable for the function. In intuitive terms, the shift invariance property means that T transforms f in the same way for all points in its domain. We can first apply T to the function and then shift it by adding y to its function variable, or we can shift the function by adding y to the function variable, and then apply T to the shifted function. The result is the same function when T is shift invariant.

The connection to convolution is that if f is our signal then any shift invariant linear transformation T can be expressed in terms of the convolution of f with some filter g. Vice versa, convolving f with g constitutes a linear and shift invariant transformation T on f. In order words, T is a function of the filter g: $T = T(g)$, and $T(g)$ is a linear mapping on V with the special property of being shift invariant. Why do we want shift invariance? At least from the outset, we usually have no reason to believe that a local model of our signal is different for one part of the signal than for some other part. This means that one part should be processed in exactly the same way as any other part, hence shift invariance. This assumption is in general not always applicable, and there are clearly cases where the shift invariance property of the filter is not reasonable. Those cases, however, are not common in the literature and are not treated here.

A third motivation for using the convolution operation is that it is straight-forward to analyse the result of the operation applied to a particular signal by means of the Fourier transform. In the Fourier domain, the convolution between f and g corresponding to the product of their transforms:

$$\mathcal{F}(f * g) = (\mathcal{F} f) \cdot (\mathcal{F} g) \tag{110}$$

This means that every frequency components of the signal f are simply multiplied by the value of the corresponding frequency component in g to give the resulting frequency component in the Fourier transform of $h = f * g$. By choosing a particular filter g we can then modify the frequency content of the signal f by amplifying or attenuating the magnitudes of its frequency components. Since the Fourier transform function $F = \mathcal{F}f$ in general is complex valued, it is also possible to change the argument of the frequency components of the signal by means of the convolution operation. By specifying which complex number should be multiplied onto each individual frequency component of the signal, we end up with a specification of the frequency components of the filter. In principle, by means of an inverse Fourier transform we then get a specification of the filter function g.

To summaries, the convolution operation can be seen as shift invariant linear transformation on the signal space which has a very simple representation in the Fourier domain. In the context of this presentation, however, it may even be more useful to consider its connection to linear algebra rather than to operations on functions or Fourier transforms in order to derive useful results. An example of this is the fact that it can be used to implement the two fundamental operations of multi-dimensional signal analysis: analysis and reconstruction. Depending on what problem we are dealing with, the convolution can be used to implement either the computation of scalar products (analysis) or linear combinations (reconstruction).

In the following, we will discuss these two ideas, and apply them to the case of discrete signals of a single variable (outer dimension = 1). The results can, however, be generalised to signals of continuous variables and to arbitrary outer dimensions in a straight-forward manner.

7.1 Convolution as a linear combination

Convolution can be seen as forming linear combinations, either of shifted versions of the signal combined with scalars from the filter, or of shifted versions of the filter combined with scalars from the signal. To see this, we define a set of auxiliary functions

$$p_l[k] = g[k - l], \quad l \in \mathbb{Z} \tag{111}$$

This allows us to rewrite (107) as

$$h[k] = \sum_{l=-\infty}^{\infty} f[l]\, p_l[k] \tag{112}$$

In this case, we see the result of the convolution as a signal, an element of the signal space V. This signal is the linear combination of the signals p_l together with the elements of the input signal f. In the case that input signal has infinitely many elements, this linear combination has infinitely many terms, but if the signal has a truncated domain, the linear combination has a finite number of terms.

Using the fact that $h = f * g = g * f$, we can also define a second set of a auxiliary functions

$$q_l[k] = f[k - l], \quad l \in \mathbb{Z} \tag{113}$$

and rewrite (107) as

$$h[k] = \sum_{l=-\infty}^{\infty} g[l]\, q_l[k] \tag{114}$$

In this case, the resulting signal is a linear combination of the signals q_l together with elements in the filter g. In the case that the filter has infinite support (an IIR-filter), this linear combination has infinitely many terms, but if the filter is of finite support (an FIR-filter), the linear combination has a finite number of terms.

Before proceeding, we note that the different types of linear combinations that have been presented here extend in the straight-forward way to functions of continuous variables and to functions of multiple variables.

7.2 Convolution as scalar products

Consider the convolution expression in (107). The result is a function of the discrete variable k, but for a fixed value of $k = k_0$ the left hand side of

$$h[k_0] = (f * g)[k_0] = \sum_{l=-\infty}^{\infty} f[l]\, g[k_0 - l] \tag{115}$$

is a scalar rather than a function. This scalar is produced as a sum of products between the elements in f and in g. Using the scalar product defined in (19), however, we cannot write $h[k_0]$ as the scalar product between f and g right from the box. Instead, we can use the auxiliary functions in (111) to get

$$h[k_0] = \sum_{l=-\infty}^{\infty} f[l]\, p_{-k_0}[-l] = \sum_{l=-\infty}^{\infty} f[l]\, \overline{\overline{p_{-k_0}[-l]}} = \langle\, f \mid \overline{p_{-k_0}[-\cdot]}\, \rangle \tag{116}$$

where $p_{-k_0}[-\cdot]$ denotes the function p_{-k_0} but with reversed variable, without specifying any particular variable for the function.

This is not the only way that a scalar product emerges from $h[k_0]$. For example, the complex conjugation can instead be made on f, leading to

$$h[k_0] = \sum_{l=-\infty}^{\infty} f[l]\, p_{-k_0}[-l] = \sum_{l=-\infty}^{\infty} p_{-k_0}[-l]\, \overline{\overline{f[l]}} = \langle\, p_{-k_0}[-\cdot] \mid \overline{f}\, \rangle \tag{117}$$

Notice that this result also follows from property 4 of the scalar product described in Section 2.1, and that the complex conjugation operations used here are only necessary in the case that V is complex.

Using the fact that $f * g = g * f$ and the auxiliary functions in (113), we can also obtain

$$h[k_0] = \sum_{l=-\infty}^{\infty} g[l]\, q_{-k_0}[-l] = \sum_{l=-\infty}^{\infty} g[l]\, \overline{\overline{q_{-k_0}[-l]}} = \langle\, g \mid \overline{q_{-k_0}[-\cdot]}\, \rangle \tag{118}$$

Alternatively, the complex conjugation can be made on g and we get:

$$h[k_0] = \langle\, q_{-k_0}[-\cdot] \mid \overline{g}\, \rangle \tag{119}$$

These variations, however, do not end here. The summation in (115) is over l, from $-\infty$ to ∞. The result should be the same if we instead let the index l go from ∞ to $-\infty$:

$$h[k_0] = (f * g)[k] = \sum_{l=-\infty}^{\infty} f[-l]\, g[k_0 + l] \tag{120}$$

This means that $h[k_0]$ can be written as

$$h[k_0] = \langle\, f[-\cdot] \mid \overline{p_{-k_0}}\, \rangle \tag{121}$$

Using the same approaches as above, we can also write $h[k_0]$ like

$$h[k_0] = \langle\, p_{-k_0} \mid \overline{f[-\cdot]}\, \rangle = \langle\, g[-\cdot] \mid \overline{q_{-k_0}}\, \rangle = \langle\, q_{-k_0} \mid \overline{g[-\cdot]}\, \rangle \tag{122}$$

From these various examples of how to write the filter response at point k_0 in terms of scalar products, it should be clear that we can also find other, but perhaps less useful, ways of writing $h[k_0]$ as a scalar product between some function related to the signal f and some other function related to the filter g. Which view we choose is rather arbitrary and the choice should be made to facilitate the solution of some practical problem. Notice that in these examples, we have still not made use of the additional freedom of choosing a scalar product between two functions of discrete variables in a more general way than (19).

Similarly to what was said at the end of Section 7.1, we note that the different types of scalar products that have been presented here extend in the straight-forward way to functions of continuous variables and to functions of multiple variables.