LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2020-08-28

Course module: TEN2

Date & Time: 2020-08-28 14:00–18:00

Location: *Distance mode due to the ongoing COVID-19 pandemic*

Examiner: Mårten Wadenbäck
Phone: +46 13 28 27 75
Zoom: `https://liu-se.zoom.us/j/64899181404`

Material & aids: All non-human resources are allowed. This means that during the exam *you are not allowed to discuss the problems with anyone or to ask someone for help*, but you are allowed to use any material (written, video, ...) or software you wish in order to solve the problems.

Scoring: The maximum score on the exam is 36 points, split evenly between parts one and two (this time only approximately evenly!). Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points. (These limits may be lowered, but not raised.)

Part I: Geometry and Estimation

1. Consider four points in the extended Euclidean space (3D), with homogeneous coordinates given by

$$\mathbf{x}_1 = (2, -3, -1, 1), \qquad \mathbf{x}_2 = (1, 3, 4, 2),$$
$$\mathbf{x}_3 = (-1, 0, 2, 1), \qquad \mathbf{x}_4 = (1, 2, -1, 1).$$

(a) The plane with dual homogeneous coordinates $\mathbf{p} = (-5, -1, -1, 6)$ contains three of these points. Determine which of these points it is that *does not* lie in the plane, and then compute its (shortest) distance to the plane. (2 p)

(b) Let ℓ be the line through $\mathbf{x}_2$ and $\mathbf{x}_3$. Compute the *ideal point* $\mathbf{x}_\infty$ on ℓ. (1 p)

(c) The ideal point $\mathbf{x}_\infty$ from (b) can in fact be read off directly from a particular place in the Plücker coordinates of the line ℓ. Show that this works in the general case. (2 p)

2. Let $\mathscr{H}_0$ be the set of (planar) homographies that map the origin to itself.

(a) Express a general element $\mathbf{H} \in \mathscr{H}_0$ in matrix form.
(Hint: Map the origin with a general homography and see what falls out.) (1 p)

(b) Suppose we wish to estimate a homography matrix $\mathbf{H} \in \mathscr{H}_0$ from a number of point correspondences $\mathbf{x}_j \leftrightarrow \mathbf{x}'_j$. Construct a suitable data matrix $\mathbf{A}$ for this problem. What is the smallest number of correspondences needed to uniquely determine $\mathbf{H}$? (3 p)

(c) Show that $\mathscr{H}_0$ is a *group* with respect to composition. (2 p)

3. Consider two points in the extended Euclidean plane (2D), with homogeneous coordinates given by $\mathbf{x}_1 = (2, 3, 1)$ and $\mathbf{x}_2 = (-4, 1, 1)$.

(a) Compute the *Hartley normalisation* of the point set $\{\mathbf{x}_1, \mathbf{x}_2\}$, i.e. compute points $\{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2\}$ that are appropriately scaled and translated versions of $\{\mathbf{x}_1, \mathbf{x}_2\}$. (2 p)

(b) Hartley normalisation can be achieved by applying a specific affine transformation to the points. What is the transformation matrix $\mathbf{T}$ that achieves the Hartley normalisation in this case? (1 p)

4. Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be homogeneous coordinates for points in the extended Euclidean plane, and let $\mathbf{l}_1$ be dual homogeneous coordinates of a line. Suppose we wish to estimate another line, $\mathbf{l}_2$, from the points, but that we do not care much about a point if it lies close to $\mathbf{l}_1$.

(a) Suggest an algebraic cost function $f_A(\mathbf{l}_2)$ that achieves this.
(Hint: When is $|\mathbf{l}_1^\top \mathbf{x}| \cdot |\mathbf{l}_2^\top \mathbf{x}| = 0$?) (2 p)

(b) Suggest a geometric cost function $f_G(\mathbf{l}_2)$ that achieves this. (1 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Let $\mathbb{R}^2$ be equipped with a scalar product given by $\langle \mathbf{u} \mid \mathbf{v} \rangle = \mathbf{v}^\top \mathbf{G}_0 \mathbf{u}$, and suppose that

$$\mathbf{G}_0 = \begin{pmatrix} a & -1 \\ -1 & a \end{pmatrix}$$

for some constant $a \in \mathbb{R}$.

(a) Which real numbers a result in $\mathbf{G}_0$ defining a scalar product?
(Hint: Scalar products satisfy $\langle \mathbf{u} \mid \mathbf{u} \rangle \geq 0$, with equality if and only if $\mathbf{u} = \mathbf{0}$.) (1 p)

(b) Suppose henceforth that $a = 7$, and let $\mathbf{b}_1 = (1, 1)$ and $\mathbf{b}_2 = (1, -1)$ be a basis in $\mathbb{R}^2$. Compute the corresponding dual basis vectors $\tilde{\mathbf{b}}_1$ and $\tilde{\mathbf{b}}_2$. (2 p)

(c) Let $\mathbf{v} = (2, 0)$. Find the vector $\mathbf{u}$ parallel to $\mathbf{b}_1$ that is closest to $\mathbf{v}$ (as measured by the norm induced by the scalar product). (3 p)

6. Let $\mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$.

(Let $\mathbb{R}^2$ be equipped with the standard scalar product, i.e. $\mathbf{G}_0 = \mathbf{I}$.)

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(b) Compute the dual frame vectors corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(c) Show that (in the general case) a frame operator is self-adjoint. (2 p)

7. By performing *principal component analysis* to a large number of signals in $\mathbb{R}^4$, it has been found that the main part of the signals always lies in the subspace spanned by $\mathbf{b}_1 = (1, 1, -1, -1)/2$ and $\mathbf{b}_2 = (1, -1, -1, 1)/2$.

(a) Suppose the the least significant principal component is $\mathbf{b}_4 = (1, 1, 1, 1)/2$. Find $\mathbf{b}_3$, i.e. the second-but-least significant principal component. (2 p)

(b) Compute the coefficients (with respect to the principal component vectors) of the signal $\mathbf{v} = (4, 1, -2, 0)$. How large is the error if $\mathbf{v}$ is only represented using the first two principal components? (2 p)

8. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be camera matrices given by

$$\mathbf{C}_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ -1 & 1 & -1 & 1 \\ 4 & -2 & -3 & 1 \end{pmatrix}, \qquad \mathbf{C}_2 = \begin{pmatrix} 0 & 2 & -2 & 0 \\ -1 & 1 & 2 & 1 \\ 2 & 0 & 3 & -2 \end{pmatrix}.$$

(a) Which of these cameras has its centre at the 3D point $(1, 0, 0)$? (1 p)

(b) For the camera in (a), compute the epipole $\mathbf{e}_?$ in this view of the other camera. (2 p)