LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2020-10-24

Course module: TEN2

Date & Time: 2020-10-24 14:00–18:00

Location: TERE & TER3 (midterm), TER1 (final exam)

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The examiner will make one visit, approximately one hour after the exam has started, and will be available on the phone the rest of the time.

Material & aids: You are allowed to use any reference material you like (books, print-outs, your own notes, ...), as long as it does not require an electronic device.

Scoring: The maximum score on the exam is 36 points, split evenly between parts one and two. Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points.

Instructions: Justify your solutions and answers with clear and concise arguments. All solutions and answers should be written on dedicated sheets of paper (i.e. not on the printed exam). Write your AID-number and the exam date on all sheets of paper that you hand in. Start each numbered problem on a new sheet. Right before you hand in, sort your solutions in consecutive order and add page numbers in the upper right corner.

Part I: Geometry and Estimation

1. Let $\mathbf{x} = (-2, 1, 3, 1)$ be homogeneous coordinates of a point in the extended Euclidean space (3D), and let $\mathbf{p}_1 = (1, -2, -2, 3)$ and $\mathbf{p}_2 = (2, 0, -1, -2)$ be dual homogeneous coordinates of two planes.

(a) Which of the planes $\mathbf{p}_1$ and $\mathbf{p}_2$ does the point $\mathbf{x}$ lie the closest to? (2 p)

(b) Compute the dual Plücker coordinates of the *horizon line* of $\mathbf{p}_1$, i.e. the line consisting of all ideal points (points at infinity) that lie on $\mathbf{p}_1$. (1 p)

(c) Show that all planes parallel to $\mathbf{p}_1$ have the same horizon line as $\mathbf{p}_1$. (2 p)

2. Consider a 3D *rotation matrix* $\mathbf{R}$ which satisfies

$$\mathbf{R} \sim \begin{pmatrix} 2 & -1 & a \\ 2 & 2 & b \\ -1 & 2 & c \end{pmatrix} \iff \mathbf{R} = \lambda \begin{pmatrix} 2 & -1 & a \\ 2 & 2 & b \\ -1 & 2 & c \end{pmatrix}$$

for some scalars $a, b, c, \lambda \in \mathbb{R}$.

(a) Determine all possible $a, b, c, \lambda \in \mathbb{R}$ which make $\mathbf{R}$ a valid rotation matrix. (2 p)

(b) Choose any of the rotation matrices $\mathbf{R}$ that satisfy the above, and compute an axis–angle representation of the rotation represented by $\mathbf{R}$. (3 p)

3. Let $\mathscr{G}$ be the set of rigid transformations in 2D for which the rotation angle is $\frac{k\pi}{2}$ for some $k \in \mathbb{Z}$.

(a) Express a general element $\mathbf{T} \in \mathscr{G}$ in matrix form. (1 p)

(b) Find all $\mathbf{T} \in \mathscr{G}$ that are their own inverses, i.e. all $\mathbf{T}$ for which $\mathbf{T}^2 = \mathbf{I}$. (1 p)

(c) Show that $\mathscr{G}$ is a *group* with respect to composition. (2 p)

4. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^4$ be homogeneous coordinates of points in the extended Euclidean space (3D), and let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^3$ be homogeneous coordinates of points in the extended Euclidean plane (2D). Suppose that the points are known, and that we seek a camera matrix $\mathbf{C} \in \mathbb{R}^{3\times4}$ that maps each $\mathbf{X}_k$ to $\mathbf{x}_k$, i.e. $\mathbf{x}_k \sim \mathbf{C}\mathbf{X}_k$. If this cannot be achieved *exactly*, we can still estimate $\mathbf{C}$ by minimising either an algebraic or a geometric cost function (error).

(a) Construct a suitable data matrix $\mathbf{A}$ that can be used to estimate $\mathbf{C}$ from the point correspondences $\mathbf{x}_k \leftrightarrow \mathbf{X}_k$. What is the smallest number of correspondences needed to uniquely (up to scale) determine $\mathbf{C}$? (3 p)

(b) The method based on (a) minimises an algebraic error, which lacks geometric interpretation. Suggest a geometric error $\varepsilon_G(\mathbf{C})$ for this estimation problem. (1 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Consider three functions $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, given by

$$f_1(\mathbf{u},\mathbf{v}) = \mathbf{v}^\top \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{u},$$
$$f_2(\mathbf{u},\mathbf{v}) = \mathbf{a}^\top(\mathbf{u} \times \mathbf{v}), \quad \text{for a fixed non-zero } \mathbf{a} \in \mathbb{R}^3,$$
$$f_3(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_2.$$

(a) Explain why f_1 is a valid scalar product on $\mathbb{R}^3$, and explain why f_2 and f_3 are *not* valid scalar products. (2 p)

(b) Determine the *Gram matrix* $\mathbf{G}$ for the scalar product f_1 with respect to the standard basis of $\mathbb{R}^3$, i.e. the basis

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Is this basis orthonormal with respect to the chosen scalar product? (2 p)

(c) Let $\tilde{\mathbf{b}}_k$, for $k = 1, 2, 3$, be the *dual basis* vectors corresponding to the basis in (b), and compute $\tilde{\mathbf{b}}_1$. (1 p)

6. Let $\mathbf{B} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$.

(Let $\mathbb{R}^2$ be equipped with the standard scalar product, i.e. $\mathbf{G}_0 = \mathbf{I}$.)

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(b) Compute the *lower frame bound* L and the *upper frame bound* U for the frame. (2 p)

(c) Determine whether this is a *tight frame* or not. (1 p)

7. Recall the two-channel filter bank conditions for discrete-time signals, expressed in the Fourier domain as

$$H_1(u)H_0(u) + G_1(u)G_0(u) = 2 \qquad \text{(FB1)}$$
$$H_1(u)H_0(u+\pi) + G_1(u)G_0(u+\pi) = 0 \qquad \text{(FB2)}.$$

(a) Show that a *conjugate mirror filter*, defined by

$$H_0(u) = H_1^*(u),$$
$$G_1(u) = e^{-iu}H_1^*(u+\pi),$$
$$G_0(u) = e^{iu}H_1(u+\pi) = G_1^*(u),$$

always satisfies the second filter bank condition (FB2). (2 p)

(b) If we use conjugate mirror filters as described in (a), how can the first filter bank condition (FB1) be written in terms of the absolute value of H_1 (in two points)? (2 p)

8. Consider two camera views, which observe the following points:

$$\begin{aligned} &\text{View 1}: \quad \mathbf{x}_1 = (1,-1,1), \quad \mathbf{x}_2 = (0,2,1), \quad \mathbf{x}_3 = (-1,2,1), \\ &\text{View 2}: \quad \mathbf{x}'_1 = (3,-6,1), \quad \mathbf{x}'_2 = (1,-2,1), \quad \mathbf{x}'_3 = (0,0,1). \end{aligned}$$

Let $\mathbf{F}$ be the *fundamental matrix* that maps points in the second view (primed coordinates) to *epipolar lines* in the first view (coordinates without prime), given by

$$\mathbf{F} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

(a) Which pairs of points $\mathbf{x}_k \leftrightarrow \mathbf{x}'_k$ above satisfy the *epipolar constraint* with this $\mathbf{F}$? (2 p)

(b) Let $\mathbf{l}'_1$ be the epipolar line in the second (primed) view, generated by the point $\mathbf{x}_1$. Compute $\mathbf{l}'_1$. (1 p)

(c) Compute the epipoles, $\mathbf{e}$ (in the first view) and $\mathbf{e}'$ (in the second view).
Hint: In this particular case, $\mathbf{F}$ is of a (hopefully) very familiar form! (1 p)