LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2021-01-12

Course module: TEN2

Date & Time: 2021-01-12 14:00–18:00

Location: *Distance mode due to the ongoing COVID-19 pandemic*

Examiner: Mårten Wadenbäck
Phone: +46 13 28 27 75
Zoom: `https://liu-se.zoom.us/j/68533465654`

Material & aids: All non-human resources are allowed. This means that during the exam *you are not allowed to discuss the problems with anyone or to ask someone for help*, but you are allowed to use any material (written, video, ...) or software you wish in order to solve the problems.

Scoring: The maximum score on the exam is 36 points, split evenly between Part I and Part II. Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points.

Instructions: Justify your solutions and answers with clear and concise arguments. All solutions and answers should be written on dedicated pages (i.e., not on the printed exam). Write your AID-number and the exam date on all pages that you hand in. Start each numbered problem on a new page. Right before you hand in, sort your solutions in consecutive order and add page numbers in the upper right corner.

Your solutions must be submitted in Lisam, as a *single* PDF file. *Only handwritten solutions will be considered!* It is acceptable, however, to use a tablet (or similar) as input. If you wish to write your solutions on paper, you can digitise them using the (free) mobile app *Office Lens* from Microsoft.

If you passed the voluntary midterm exam (i.e., obtained at least 10 points), you can use your midterm score instead of solving Part I. This is done by *not handing in solutions to any problem from Part I*.

Good luck!

Part I: Geometry and Estimation

1. Consider four points in the extended Euclidean plane (2D), with homogeneous coordinates given by

$$\mathbf{x}_1 = (1, 2, 1), \qquad \mathbf{x}_2 = (2, -1, 1), \qquad \mathbf{x}_3 = (-1, 5, 1), \qquad \mathbf{x}_4 = (1, 2, 0).$$

(a) Determine which of these points lies closest to the line with dual homogeneous coordinates $\mathbf{l} = (12, -5, 26)$. (2 p)

(b) Compute the *ideal point* on the line through $\mathbf{x}_1$ and $\mathbf{x}_2$. (1 p)

(c) Suggest a simple test that determines whether three points (in the extended Euclidean plane), given as homogeneous coordinates, lie on the same line or not. Explain why it works, and then use your test on $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. (2 p)

2. Let $\mathscr{A}$ be the set of affine transformations on the extended Euclidean space (3D).

(a) Express a general element $\mathbf{T} \in \mathscr{A}$ in matrix form. How many *degrees of freedom* does $\mathbf{T}$ have? (1 p)

(b) Show that the *plane at infinity*, $\mathbf{p}_\infty = (0, 0, 0, 1)$, remains unchanged by a general transformation $\mathbf{T} \in \mathscr{A}$. (2 p)

(c) What is the smallest number of point correspondences $\mathbf{x}_j \leftrightarrow \mathbf{x}'_j$ needed to uniquely determine a transformation in $\mathscr{A}$? (2 p)

3. Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be homogeneous coordinates for points in the extended Euclidean plane. Suppose that we wish to estimate dual homogeneous coordinates $\mathbf{l}$ for a line that best fits these points.

(a) Suggest an *geometric cost function* that achieves this. (2 p)

(b) This cost function can actually be minimised by computing the *singular value decomposition* (SVD) of a suitable (symmetric) matrix constructed from $\mathbf{x}_1, \dots, \mathbf{x}_n$. Show why and how this works. (2 p)

4. A symmetric 4×4 matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix}$$

can be used to represent a *quadric surface*. The quadric surface then consists of all $\mathbf{x} \sim (x, y, z, 1)$ satisfying the equation $\mathbf{x}^\top \mathbf{C} \mathbf{x} = 0$, or, written out explicitly,

$$c_{11}x^2 + 2c_{12}xy + 2c_{13}xz + 2x_{14}x + c_{22}y^2 + 2c_{23}yz + 2c_{24}y + c_{33}z^2 + 2c_{34}z + c_{44} = 0.$$

Suppose that we want to estimate $\mathbf{C}$ from a number of known points $\mathbf{x}_1, \dots, \mathbf{x}_m$.

(a) The equation of the quadric surface is linear in the entries c_{ij}. Construct a suitable data matrix, and explain how it can be used to estimate $\mathbf{C}$ from $\mathbf{x}_1, \dots, \mathbf{x}_m$. (3 p)

(b) What is the smallest number of points needed to determine $\mathbf{C}$? (1 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Let $\mathbb{R}^3$ be equipped with a scalar product given by $\langle \mathbf{u} \mid \mathbf{v} \rangle = \mathbf{v}^\top \mathbf{G}_0 \mathbf{u}$, and suppose that

$$\mathbf{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & a \\ 0 & a & 2 \end{pmatrix}$$

for some constant $a \in \mathbb{R}$.

(a) Which real numbers a result in $\mathbf{G}_0$ defining a scalar product? (1 p)
(Hint: Scalar products satisfy $\langle \mathbf{u} \mid \mathbf{u} \rangle \geq 0$, with equality if and only if $\mathbf{u} = \mathbf{0}$.)

(b) Suppose henceforth that $a = 1$, and consider a basis of $\mathbb{R}^3$ given by

$$\mathbf{b}_1 = (1, 0, 0), \qquad \mathbf{b}_2 = (0, 1, 1), \qquad \mathbf{b}_3 = (0, 2, -1).$$

Determine the *Gram matrix* $\mathbf{G}$ with respect to this basis. (1 p)

(c) Let $\mathbf{v} = (1, 2, 3)$, and let U be the subspace spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$. Find the vector $\mathbf{u} \in U$ that is closest to $\mathbf{v}$ (as measured by the norm induced by the scalar product). (3 p)

6. Let $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$, and let $\mathbb{R}^2$ be equipped with the scalar product given by $\mathbf{G}_0 = \begin{pmatrix} 6 & -1 \\ -1 & 1 \end{pmatrix}$.

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(b) Compute the *lower frame bound* L and the *upper frame bound* U for the frame. (1 p)

(c) Compute the dual frame vectors corresponding to the frame vectors in $\mathbf{B}$. (2 p)

7. Consider two camera views, which observe the following points:

$$\text{View 1}: \quad \mathbf{x}_1 = (1, 1, 1), \qquad \mathbf{x}_2 = (0, 3, 1),$$
$$\text{View 2}: \quad \mathbf{x}_1' = (1, 2, 1), \qquad \mathbf{x}_2' = (1, 4, 1).$$

Let $\mathbf{F}$ be the *fundamental matrix* that maps points in the second view (primed coordinates) to *epipolar lines* in the first view (coordinates without prime), given by

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & -1 \\ a & 2 & b \\ a & 1 & 3 \end{pmatrix}$$

for some constants $a, b \in \mathbb{R}$.

(a) For which values of a and b is $\mathbf{F}$ a valid fundamental matrix? (1 p)

(b) Assume that $\mathbf{x}_1$ and $\mathbf{x}_1'$ satisfy the *epipolar constraint*. What is the distance between $\mathbf{x}_2$ and the *epipolar line* $\mathbf{l}_2 \sim \mathbf{F}\mathbf{x}_2'$? (3 p)

8. In *principal component analysis* (PCA), we want to find an orthogonal matrix $\mathbf{B}$ that minimises

$$\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\right\|^2\right],$$

where $\mathbb{E}$ means taking the expectation over all vectors, and where we have made the simplifying assumption that $\mathbf{G}_0 = \mathbf{I}$.

(a) Show that minimising ε is equivalent to maximising $\varepsilon_1 = \mathbb{E}[\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}]$. (2 p)

(b) How would the expression for ε change in case we decided to use a general scalar product, i.e. not necessarily using $\mathbf{G}_0 = \mathbf{I}$? (2 p)