LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2021-10-25

Course module: TEN2

Date & Time: 2021-10-25 14:00–18:00

Location: TER3

Examiner: Mårten Wadenbäck
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The examiner will make one visit, approximately one hour after the exam has started, and will be available on the phone the rest of the time.

Material & aids: You are allowed to use any non-electronic reference material you like, as long as you *do not bring solutions to previous exams or midterms*. This means that you may bring books, printouts, your own notes, etc.

Scoring: The maximum score on the exam is 36 points, split evenly between Part I and Part II. Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points.

Instructions: Justify your solutions and answers with clear and concise arguments. All solutions and answers should be written on dedicated pages (i.e., not on the printed exam). Write your AID-number and the exam date on all pages that you hand in. Start each numbered problem on a new page. Right before you hand in, sort your solutions in consecutive order and add page numbers in the upper right corner.

Good luck!

Part I: Geometry and Estimation

1. Consider a plane in the extended Euclidean space (3D), with dual homogeneous coordinates $\mathbf{p} = (2, -1, -2, 6)$. Additionally, let $\mathbf{x}_1 = (3, 1, 1, 1)$ and $\mathbf{x}_2 = (1, -1, 0, 1)$ be homogeneous coordinates of two points.

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same distance to the plane $\mathbf{p}$. What is the distance? (1 p)

(b) Let $\mathbf{p}'$ be the plane that is parallel to $\mathbf{p}$ and contains $\mathbf{x}_1$. Determine the dual homogeneous coordinates of $\mathbf{p}'$. (1 p)

(c) Let ℓ be the line through $\mathbf{x}_1$ and $\mathbf{x}_2$. Find the intersection $\mathbf{x}_0$ between ℓ and $\mathbf{p}$. (2 p)

(d) Show that the point $\mathbf{x}_0$ defined in (c) also lies on $\mathbf{p}'$. (1 p)

2. Three points in the extended Euclidean plane (2D), with homogeneous coordinates

$$\mathbf{x}_1 = (5, 1, 1), \qquad \mathbf{x}_2 = (0, 0, 1), \qquad \mathbf{x}_3 = (12, 5, 1)$$

are transformed into new points $\mathbf{x}'_k = \mathscr{T}(\mathbf{x}_k)$, for an unspecified transformation $\mathscr{T}$, such that

$$\mathbf{x}'_1 = (-1, 2, 1), \qquad \mathbf{x}'_2 = (-2, -3, 1), \qquad \mathbf{x}'_3 = (-2, 10, 1).$$

(a) Is it possible that the transformation $\mathscr{T}$ is a *rigid* transformation? An *affine* transformation? A *homography* transformation? (2 p)

(b) If, additionally, $\mathbf{x}_4 = (1, -1, 1)$ is transformed by $\mathscr{T}$ into $\mathbf{x}'_4 = (1, 0, 1)$, is it *then* possible that the transformation $\mathscr{T}$ is a *rigid* transformation? An *affine* transformation? A *homography* transformation? (2 p)

3. Let $\mathscr{H}_x$ be the set of all homography matrices of the form

$$\mathbf{H} = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & a \end{pmatrix}, \quad \text{with } a \neq 0 \text{ and } c \neq 0.$$

(a) Show that every point on the x-axis is mapped to itself by any $\mathbf{H} \in \mathscr{H}_x$. (1 p)

(b) Is it true that $\mathbf{H}_1, \mathbf{H}_2 \in \mathscr{H}_x \implies \mathbf{H}_2\mathbf{H}_1 \in \mathscr{H}_x$? (In other words, is $\mathscr{H}_x$ closed under composition of the transformations?) (1 p)

(c) Suppose we want to fit $\mathbf{H} \in \mathscr{H}_x$ to point correspondences $(x'_k, y'_k, 1) \leftrightarrow (x_k, y_k, 1)$. Construct a data matrix $\mathbf{A}$ such that the parameter vector (a, b, c, d) can be found as the (approximate) null space of $\mathbf{A}$.
Hint: Use the general DLT constraint $(\mathbf{x}_k^\top \otimes [\mathbf{x}'_k]_\times) \operatorname{vec} \mathbf{H} = \mathbf{0}$ to determine what the coefficients should be for the parameters. (3 p)

4. Homogeneous representations of three distinct points in the extended Euclidean space (3D) are given as the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^4$. Assume that at least one of the points is a proper point (i.e., not an ideal point), and let π be the plane spanned by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

(a) Describe a method for finding the *horizon line* of π, i.e. all ideal points on π. (3 p)

(b) Define an *algebraic* cost function as

$$\varepsilon_A(\mathbf{p}) = ||\mathbf{A}\mathbf{p}||^2, \quad \text{where } \mathbf{A} = \begin{pmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{pmatrix}.$$

How can this be changed into a *geometric* cost function ε_G? (1 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Consider three functions $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, given by

$$
\begin{aligned}
f_1(\mathbf{u},\mathbf{v}) &= u_1 u_2 + v_1 v_2,\\
f_2(\mathbf{u},\mathbf{v}) &= u_1 v_2 + u_2 v_1,\\
f_3(\mathbf{u},\mathbf{v}) &= \mathbf{v}^\top \mathbf{u} + (v_1 + v_2)(u_1 + u_2).
\end{aligned}
$$

(a) Only one of the three functions above is a valid scalar product. Which one? Explain why the other two are not valid scalar products. (2 p)

(b) Determine the *Gram matrix* $\mathbf{G}$ for the valid scalar product in (a) with respect to the basis

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Is this basis orthonormal with respect to the chosen scalar product? (2 p)

(c) Determine the *dual basis vectors* $\tilde{\mathbf{b}}_1$ and $\tilde{\mathbf{b}}_2$ corresponding to the basis in (b), with respect to the scalar product used in (b). (2 p)

6. Let $\mathbf{B} = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & -3 & 1 & -2 \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$, and assume that $\mathbb{R}^2$ is equipped with at scalar product defined by $\mathbf{G}_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(b) Compute the *lower frame bound* L and the *upper frame bound* U for the frame. (1 p)

(c) Show that (in the general case) a frame operator is self-adjoint. (2 p)

7. Consider two camera views, which observe the following points:

$$
\begin{aligned}
&\text{View 1}: \quad \mathbf{x}_1 = (5,1,1), \qquad \mathbf{x}_2 = (-2,-3,1),\\
&\text{View 2}: \quad \mathbf{x}'_1 = (-1,2,1), \qquad \mathbf{x}'_2 = (0,0,1).
\end{aligned}
$$

Let $\mathbf{F}$ be the *fundamental matrix* that maps points in the second view (primed coordinates) to *epipolar lines* in the first view (coordinates without prime), given by

$$\mathbf{F} = \begin{pmatrix} a & b & c \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

for some constants $a, b, c \in \mathbb{R}$.

(a) Determine the scalars a, b, c such that $\mathbf{F}$ is a valid fundamental matrix, and such that $\mathbf{x}_1 \leftrightarrow \mathbf{x}'_1$ and $\mathbf{x}_2 \leftrightarrow \mathbf{x}'_2$ satisfy the *epipolar constraint*. (2 p)

(b) If $\mathbf{x}_1 \leftrightarrow \mathbf{x}'_1$ satisfy the epipolar constraint, what does that tell us about $\mathbf{x}_1$ and $\mathbf{x}'_1$? (1 p)

8. In *principal component analysis* (PCA), we want to find an orthogonal matrix $\mathbf{B}$ that minimises

$$\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\right\|^2\right],$$

where $\mathbb{E}$ means taking the expectation over all vectors, and where we have made the simplifying assumption that $\mathbf{G}_0 = \mathbf{I}$.

(a) Show that minimising ε is equivalent to maximising $\varepsilon_1 = \mathbb{E}[\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}]$. (2 p)

(b) How would the expression for ε change in case we decided to use a general scalar product, i.e. not necessarily using $\mathbf{G}_0 = \mathbf{I}$? (2 p)