Answers to the Midterm Examination 2019-10-21 in TSBB06 Multi-Dimensional Signal Analysis

Note: The answers given here are not an authoritative description of how answers to the questions must be given in order to pass the exam. Some explanations given here may not have to be included in the answer, unless explicitly called for.

Scoring: is in terms of half-points.
Type A problems are scored with one of [0p, 0.5p, 1p], while
Type B problems are scored with one of [0p, 0.5p, 1p, 1.5p, 2p].

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PART I, Geometry

Answer 1 (A, 1p) The line should satisfy $\mathbf{l}^\top \mathbf{x}_1 = 0$ and $\mathbf{l}^\top \mathbf{x}_2 = 0$, i.e. the vector $\mathbf{l}$ should be orthogonal to both $\mathbf{x}_1$ and $\mathbf{x}_2$. We can use $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = (2, 3, -9)$.

Answer 2 (A, 1p) We can transform the points $\{(2, 2),\ (0, 3),\ (2, -2)\}$ and see where they end up. We get

$$\mathbf{H}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}5\\1\\-1\end{pmatrix} \sim \begin{pmatrix}-5\\-1\\1\end{pmatrix}, \qquad \mathbf{H}\begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}4\\2\\1\end{pmatrix},$$

$$\text{and} \qquad \mathbf{H}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\-1\end{pmatrix} \sim \begin{pmatrix}-1\\3\\1\end{pmatrix}.$$

It is clear (already after transforming the first point) that $(x, y) = (-5, -1)$.

Answer 3 (A, 1p) There are many such notions to choose from, including (but not limited to) *length*, *area*, *volume* (if in 3D), and *handedness*.

Answer 4 (B, 2p) First, recall that $[\,\hat{\mathbf{n}}\,]_\times$ is skew-symmetric, i.e. $[\,\hat{\mathbf{n}}\,]_\times^\top = -[\,\hat{\mathbf{n}}\,]_\times$. Secondly, note that

$$[\,\hat{\mathbf{n}}\,]_\times^2 = \begin{pmatrix}-n_2^2 - n_3^2 & n_1 n_2 & n_1 n_3\\ n_1 n_2 & -n_1^2 - n_3^2 & n_2 n_3\\ n_1 n_3 & n_2 n_3 & -n_1^2 - n_1 2^2\end{pmatrix} = ([\,\hat{\mathbf{n}}\,]_\times^2)^\top$$

is symmetric.

Using Rodrigues' formula, we see that $\mathbf{R} - \mathbf{R}^\top = 2\sin\alpha\,[\,\hat{\mathbf{n}}\,]_\times$. Since we know the structure of $[\,\hat{\mathbf{n}}\,]_\times$, and since $\|\hat{\mathbf{n}}\| = 1$, we can now read off both $\hat{\mathbf{n}}$ and $\sin\alpha$. Unfortunately, since $\sin\alpha = \sin(\pi - \alpha)$, we still do not know what the angle is.

This can be resolved by also computing

$$\begin{aligned}\operatorname{trace}(\mathbf{R}) &= \operatorname{trace}(\mathbf{I} + \sin\alpha\,[\,\hat{\mathbf{n}}\,]_\times + (1 - \cos\alpha)[\,\hat{\mathbf{n}}\,]_\times^2) = \\ &= \underbrace{\operatorname{trace}(\mathbf{I})}_{=3} + \sin\alpha\,\underbrace{\operatorname{trace}([\,\hat{\mathbf{n}}\,]_\times)}_{=0} + (1 - \cos\alpha)\,\underbrace{\operatorname{trace}([\,\hat{\mathbf{n}}\,]_\times^2)}_{=-2\|\hat{\mathbf{n}}\|^2=-2} = \\ &= 3 + (1 - \cos\alpha)(-2) = 1 + 2\cos\alpha,\end{aligned}$$

giving $\cos\alpha = \frac{1}{2}(\operatorname{trace}\mathbf{R} - 1)$.

Answer 5 (B, 2p) All points on the line through $\mathbf{x}_1$ and $\mathbf{x}_2$ are of the form $s_1\mathbf{x}_1 + s_2\mathbf{x}_2$, while all points on the line through $\mathbf{y}_1$ and $\mathbf{y}_2$ are of the form $t_1\mathbf{y}_1 + t_2\mathbf{y}_2$.

If the two lines contain a common point $\mathbf{z}$, then there exist $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $\mathbf{z} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 = t_1\mathbf{y}_1 + t_2\mathbf{y}_2$. If the two lines do not coincide, there will exist precisely one intersection point. We can solve the homogeneous system of equations

$$\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & -\mathbf{y}_1 & -\mathbf{y}_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ t_1 \\ t_2 \end{pmatrix} = \mathbf{0}$$

by computing an SVD, $\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & -\mathbf{y}_1 & -\mathbf{y}_2 \end{pmatrix} = \mathbf{USV}^\top$. If only one singular value is zero, that means the lines are distinct and intersect at precisely one point. This is found by noting that $(s_1, s_2, t_1, t_2) \sim \mathbf{v}_4$, where $\mathbf{v}_4$ denotes the rightmost column in $\mathbf{V}$, corresponding to the smallest singular value. In this case, the point is $\mathbf{z} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2$.

If two singular values are zero, this means that the lines coincide, and the intersection is then the whole line.

PART I, Estimation

Answer 6 (A, 1p) If the error is zero, this typically means either that the chosen model is too flexible and contains too many free parameters, or that we are using too little data in the estimation (the latter case was ruled out here). A too flexible model is bad because it (over)fits to the noise, and will generalise poorly to new data.

Answer 7 (A, 1p) In the orthogonal Procrustes problem we are given some 3D (2D also works) point correspondences $\mathbf{a}_k \leftrightarrow \mathbf{b}_k$, $k = 1, \dots, m$, and the objective is to determine an orthogonal matrix $\mathbf{R}$ minimising

$$\varepsilon_{\text{rot}} = \sum_{k=1}^{m} \|\mathbf{b}_k - \mathbf{R}\mathbf{a}_k\|^2.$$

This is done by first forming $\mathbf{A} = \begin{pmatrix}\mathbf{a}_1 & \dots & \mathbf{a}_m\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}\mathbf{b}_1 & \dots & \mathbf{b}_m\end{pmatrix}$, then computing an SVD $\mathbf{B}\mathbf{A}^\top = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, and finally setting $\mathbf{R} = \mathbf{U}\mathbf{V}^\top$. (This is not guaranteed to be a rotation, but that is only required in the strict version.)

Answer 8 (A, 1p) In this case, a 'good' set of singular values is characterised by having one singular value which is much smaller than the rest. This means that that the data matrix has an unambiguous one-dimensional null space (approximate null space), which holds the vectorised version of the homography matrix.

Answer 9 (B, 2p)

a) For every point $\mathbf{x}_k$, we get an equation

$$\underbrace{\begin{pmatrix} x_k^2 & 2x_k y_k & 2x_k & y_k^2 & 2y_k & 1 \end{pmatrix}}_{\mathbf{A}_k} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{22} \\ c_{23} \\ c_{33} \end{pmatrix} = 0.$$

Now stack the $\mathbf{A}_k$ to make a data matrix

$$\mathbf{A} = \begin{pmatrix} x_1^2 & 2x_1 y_1 & 2x_1 & y_1^2 & 2y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_m^2 & 2x_m y_m & 2x_m & y_m^2 & 2y_m & 1 \end{pmatrix}, \tag{1}$$

and compute its (right) null space. This can be done e.g. by computing an SVD $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$. Then the rightmost column in $\mathbf{V}$ will be an approximation to the null space $(c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33})$. Note: $\mathbf{C}$ can only be determined up to a (non-zero) scalar multiple, and it is irrelevant which multiple is found.

b) The data matrix need to have a unique one-dimensional null space (representing $\mathbf{C}$). In general, this happens if we use five points, since then the data matrix will be of size 5×6.

Answer 10 (B, 2p) One approach is to use the constraint $\|\mathbf{z}\| = 1$ (the *homogeneous method*). The solution to this constrained minimisation problem is found by computing an SVD of the data matrix, $\mathbf{A} = \mathbf{USV}^\top$. Then an optimal $\mathbf{z}$ is given by the rightmost column in $\mathbf{V}$.

Another approach is to arbitrarily set one element in $\mathbf{z}$ to something non-zero, and then solve for the other elements as an inhomogeneous least-squares problem. If for example the final element in $\mathbf{z}$ is set to one, we may write $\mathbf{z} = (\mathbf{z}_0, 1)$. Partitioning the data matrix as $\mathbf{A} = \begin{pmatrix}\mathbf{A}_0 & \mathbf{b}\end{pmatrix}$, where $\mathbf{b}$ is the rightmost column of $\mathbf{A}$, we have

$$\|\mathbf{Az}\| = \left\| \begin{pmatrix}\mathbf{A}_0 & \mathbf{b}\end{pmatrix} \begin{pmatrix}\mathbf{z}_0 \\ 1\end{pmatrix} \right\| = \|\mathbf{A}_0\mathbf{z}_0 + \mathbf{b}\|.$$

If $\mathbf{A}$ is a 'tall' matrix of full rank, this norm is minimised when $\mathbf{z}_0 = -(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$. This gives a non-zero minimiser $\mathbf{z}$ to the original problem.