LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2021-03-16

Course module: TEN2

Date & Time: 2021-03-16 08:00–12:00

Location: *Distance mode due to the ongoing COVID-19 pandemic*

Examiner: Mårten Wadenbäck
Phone: +46 13 28 27 75 (safest option!)
Zoom: `https://liu-se.zoom.us/j/68533465654`

Material & aids: All non-human resources are allowed. This means that during the exam *you are not allowed to discuss the problems with anyone or to ask someone for help*, but you are allowed to use any material (written, video, ...) or software you wish in order to solve the problems.

Scoring: The maximum score on the exam is 36 points, split evenly between Part I and Part II. Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points.

Instructions: Justify your solutions and answers with clear and concise arguments. All solutions and answers should be written on dedicated pages (i.e., not on the printed exam). Write your AID-number and the exam date on all pages that you hand in. Start each numbered problem on a new page. Right before you hand in, sort your solutions in consecutive order and add page numbers in the upper right corner. Please include a scoring table, similar to this one, as your first page:

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Solved								
Score								

Your solutions must be submitted in Lisam, as a *single* PDF file. *Only handwritten solutions will be considered!* It is acceptable, however, to use a tablet (or similar) as input. If you wish to write your solutions on paper, you can digitise them using the (free) mobile app *Office Lens* from Microsoft.

Good luck!

Part I: Geometry and Estimation

1. Consider four points in the extended Euclidean plane (2D), with homogeneous coordinates given by

$$\mathbf{x}_1 = (3, 1, 1), \qquad \mathbf{x}_2 = (1, -1, 0), \qquad \mathbf{x}_3 = (0, 1, 1), \qquad \mathbf{x}_4 = (1, 2, 1).$$

(a) Compute (dual) homogeneous coordinates $\mathbf{l}_{12}$ for the line through $\mathbf{x}_1$ and $\mathbf{x}_2$, and find homogeneous coordinates for the *ideal point* on this line. (2 p)

(b) Determine (dual) homogeneous coordinates $\mathbf{l}$ for the line through $\mathbf{x}_3$ that is parallel to the line $\mathbf{l}_{12}$ defined in (a). Additionally, find homogeneous coordinates $\mathbf{x}_\infty$ for the ideal point on the line $\mathbf{l}$. (1 p)

(c) Determine which of the lines $\mathbf{l}_{12}$ and $\mathbf{l}$, defined in (a) and (b), that is the closest to the point $\mathbf{x}_4$. (2 p)

2. Let $\mathbf{R}_x$ be a matrix representing a rotation the angle $-\pi/4$ (i.e., *clockwise*) around the x-axis. Similarly, let $\mathbf{R}_z$ be a matrix representing a rotation the angle $\pi/4$ (i.e., *anti-clockwise*) around the z-axis.

(a) Write down the two matrices $\mathbf{R}_x$ and $\mathbf{R}_z$ explicitly. (1 p)

(b) Let $\mathbf{R}$ be the rotation resulting from performing the rotation $\mathbf{R}_x$ followed by the rotation $\mathbf{R}_z$. Compute the rotation axis for the rotation $\mathbf{R}$. (2 p)

(c) Show that the rotation axis will be different if the rotations $\mathbf{R}_x$ and $\mathbf{R}_z$ are performed in the opposite order. (2 p)

3. *Hartley normalisation* can be used to improve the numerical properties in some geometric estimation problems, e.g. DLT-based homography estimation. When we perform Hartley normalisation to a set of points $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, we compute a new set of *Hartley normalised* points $\{\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n\}$ along with a transformation matrix $\mathbf{T}$ relating the two point sets as $\hat{\mathbf{x}}_j = \mathbf{T}\mathbf{x}_j$ (for $j = 1, \dots, n$).

(a) Suppose we perform Hartley normalisation one more time, i.e. we Hartley normalise the set of points $\{\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n\}$, resulting in a new set of points $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ and a transformation matrix $\hat{\mathbf{T}}$. Write down a mathematical relation between the point $\mathbf{y}_j$ and the point $\mathbf{x}_j$. (2 p)

(b) Suppose we wish to estimate a homography matrix $\mathbf{H}$ from a number of point correspondences $\mathbf{x}_j \leftrightarrow \mathbf{x}'_j$, using DLT and Hartley normalisation. We then compute the Hartley normalised point correspondences $\hat{\mathbf{x}}_j \leftrightarrow \hat{\mathbf{x}}'_j$, and estimate a homography matrix $\hat{\mathbf{H}}$ for these correspondences using DLT. Explain how $\mathbf{H}$ can be obtained from $\hat{\mathbf{H}}$. (2 p)

4. Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^4$ be homogeneous coordinates of points in the extended Euclidean space (3D). Suppose that we wish to estimate a plane $\mathbf{p}$ to these points, but with the *strict requirement* that $\mathbf{p}$ is a plane through the origin.

(a) Construct a suitable data matrix $\mathbf{A}$ and explain how it can be used to estimate $\mathbf{p}$ such that we minimise some (relevant) algebraic error. (2 p)

(b) Formulate a geometric cost function for the estimation problem. (You do not need to show how to compute a $\mathbf{p}$ that minimises this cost function.) (2 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Consider the two integral expressions (in this case, *functionals*)

$$I_1(u,v) = \int_0^1 \big(u(x) + v(x)\big)\,dx,$$
$$I_2(u,v) = \int_0^1 u(x)\,v(x)\,dx,$$

where u and v belong to some space V (which we shall not specify in detail) of functions defined on the interval $[0,1]$, and such that the integrals are defined. We can define a scalar product $\langle u \mid v\rangle = I_2(u,v)$ on V.

(a) Explain why $I_1(u,v)$ *cannot* be a valid scalar product on V. (1 p)

(b) Show that the functions $f_1(x) = 2x$ and $f_2(x) = 1 - 2x^2$ are *orthogonal* with respect to the scalar product $\langle u \mid v\rangle = I_2(u,v)$. (2 p)

(c) The function $f_3(x) \equiv 1$ *cannot* be written as a linear combination of $f_1(x)$ and $f_2(x)$. Find the function $g(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x)$ that minimises $\|f_3 - g\|^2 = \langle f_3 - g \mid f_3 - g\rangle$.
Hint: Recall from (b) that f_1 and f_2 are orthogonal (but maybe not normalised)! (3 p)

6. Let $\mathbf{B} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$, and assume that $\mathbb{R}^2$ is equipped with the standard scalar product (i.e. $\mathbf{G}_0 = \mathbf{I}$).

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (2 p)

(b) Compute the dual frame vectors corresponding to the frame vectors in $\mathbf{B}$. (1 p)

(c) Let $\mathbf{v} = (3,1)$, and find reconstructing coefficients $\mathbf{c}$ such that $\mathbf{v} = \mathbf{Bc}$ and such that $\|\mathbf{c}\|$ is as small as possible. (2 p)

7. Let $f(t)$ be a continuous signal that is band-limited to $|\omega| < \frac{\pi}{3}$. Assume we sample $f(t)$ at integer times, giving rise to a sampled signal $s[k] = f[k] + n[k]$, where the sampling noise has variance σ^2.

(a) Since we have oversampled the signal, we can reconstruct the signal in a way such that the noise energy is lower than σ^2, if we make some suitable assumptions about the sampling noise. What are these assumptions, and what noise energy can we achieve under those assumptions? (2 p)

(b) Explain briefly the main idea behind the noise reduction in (a). (2 p)

8. (a) Let $\mathbf{C}$ be a camera matrix, and let ℓ be a 3D line that does not pass through the camera centre of $\mathbf{C}$. Show that the projection of ℓ into the view of $\mathbf{C}$ is a 2D line. (2 p)

(b) Perhaps surprisingly, the image of a sphere when projected into the view of a perspective pinhole camera does not always have a circle as its contour. What type of curve *is* the shape of its contour, in general? (1 p)
Hint: Make a simple drawing, and think about how the contour arises!