Guide* to answers for written examination in TSBB06 Multi-dimensional signal analysis, 2016-01-12

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PART I

Exercise 1 The dual homogeneous coordinates $\mathbf{l}$ must satisfy $\mathbf{y}\cdot\mathbf{l} = 0$. This relation gives many solutions for $\mathbf{l}$. Some examples are

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}.$$

Exercise 2 The translation matrix and the rotation matrix are given as

$$\mathbf{T}_{\text{trans}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_{\text{rot}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The combined transformation is

$$\mathbf{T}_{\text{rot}}\,\mathbf{T}_{\text{trans}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3 See the IREG compendium, figure 6.1.

Exercise 4 See the IREG compendium, section 9.2.2.

Exercise 5 Set $y = (u, v, 1)$ and define the elements of matrix $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}.$$

This means that we can expand $\mathbf{y}^\top \mathbf{Q}\,\mathbf{y} = 0$ as

$$\mathbf{y}^\top \mathbf{Q}\,\mathbf{y} = q_{11}u^2 + 2\,q_{12}u\,v + 2\,q_{13}u + q_{22}v^2 + 2\,q_{23}v + q_{33} = 0. \tag{1}$$

Similarly, the defining equation of the ellipse can be expanded as

$$4u^2 - 32\,u + 9v^2 - 36\,v + 99 = 0. \tag{2}$$

Using (1) and (2), we can identify the elements of $\mathbf{Q}$ (up to multiplication with a scalar):

$$q_{11} = 4, q_{12} = 0, q_{13} = -16, q_{22} = 9, q_{23} = -18, q_{33} = 99$$

*This guide is not an authoritative description of how answers to the questions must be given in order to pass the exam.

PART II

Exercise 6 To make the estimation result independent of which coordinate system is used for defining the coordinates of points. See the IREG compendium, section 11.2.2. Observation 52.

Exercise 7 See the IREG compendium, section 11.1.2.

Exercise 8 See the IREG compendium, section 11.2.1, the three cases on page 195.

Exercise 9 See the IREG compendium, sections 10.3 and 10.4.

Exercise 10 See the IREG compendium, section 13.2.

PART III

Exercise 11 $\tilde{\mathbf{B}}^* \mathbf{G}_0 \, \mathbf{B} = \mathbf{I}$. See lecture 2B.

Exercise 12 To assure the symmetry of the scalar product: $\langle \, \mathbf{u} \mid \mathbf{v} \, \rangle = \langle \, \mathbf{v} \mid \mathbf{u} \, \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. See lecture 2A, slide 5.

Exercise 13 The orthogonal projection of $\mathbf{v}$ onto the subspace spanned by $\mathbf{B}$ is given as $\mathbf{v}_1 = \mathbf{B}(\mathbf{B}^*\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^*\mathbf{G}_0\mathbf{v}$. The orthogonal distance from $\mathbf{v}$ to the space is then given as $\|\mathbf{v} - \mathbf{v}_1\| = \|\mathbf{v} - \mathbf{B}(\mathbf{B}^*\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^*\mathbf{G}_0\mathbf{v}\|$. See lecture 2C, slide 11.

Exercise 14 The dual frame is defined as $\tilde{\mathbf{b}}_k = \mathbf{F}^{-1}\mathbf{b}_k$, see lecture 2F, slide 11. The expansion of $\mathbf{v}$ as a linear combination of the frame vectors is discussed in slides 14 and 15.

Exercise 15 Using the scalar product between discrete functions of N samples:

$$\langle \, f[k] \mid g[k] \, \rangle = \sum_{k}^{N-1} f[k] \, g[k]^*,$$

the expression for discrete Fourier transform can be written:

$$F[l] = \sum_{k}^{N-1} f[k] \, e^{-i2\pi kl/N} = \langle \, f[k] \mid e^{2\pi ikl/N} \, \rangle$$

This implies that the functions, of the discrete variable k, $\tilde{b}_l[k] = e^{2\pi ikl/N}$ are the dual basis functions. The corresponding basis functions are then given as $b_l[k] = \frac{1}{N} \, e^{2\pi ikl/N}$ since

$$\langle \, b_n[k] \mid \tilde{b}_l[k] \, \rangle = \frac{1}{N} \sum_{k}^{N-1} e^{2\pi ikn} \, e^{-i2\pi kl/N} = \frac{1}{N} \sum_{k}^{N-1} e^{2\pi ik(n-l)/N} = \frac{1}{N} \, N \, \delta_{nl} = \delta_{nl}$$

PART IV

Exercise 16 The signal has a spectrum that normally is larger for lower frequencies, and decreases for higher frequencies. Since the spectrum describes the "probability of energy" in the frequency domain for a particular signal, it is reasonable to weight the optimization error with the (non-uniform) spectrum. See exercise 16.1.

Exercise 17 See lecture 2E, slide 26.

Exercise 18 W_0 consists of functions in V_1 that are orthogonal to the functions in V_0. This means that V_0 can be formulated as the direct sum of V_0 and W_0: $V_1 = W_0 \oplus V_0$. See lecture 2G, slide 44.

Exercise 19 The optimization problem is formulated in the frequency domain, for frequency functions corresponding to discrete signals, i.e., 2π-periodic functions. We have an ideal frequency function of the filter, denoted $F_I(u)$, and the actual frequency function of the filter, denoted $F(u)$, which is the Fourier transform of the filter coefficients $f[k]$, i.e., $F(u) = \mathcal{F}\{f\}(u)$, and we want to minimize the difference between the two frequency functions, i.e., minimize

$$\epsilon = \|F_I(u) - F(u)\|^2 = \|F_I(u) - \mathcal{F}\{f\}(u)\|^2,$$

over all choices of the filter coefficients f. The norm between the two frequency functions is defined based on

$$\epsilon = \|F_I(u) - F(u)\|^2 = \langle\, F_I(u) - F(u) \mid F_I(u) - F(u)\,\rangle,$$

where the scalar product between two frequency functions is defined as

$$\langle\, H(u) \mid G(u)\,\rangle = \int_{-\pi}^{\pi} H(u)\, W(u)\, \overline{G(u)}\, du.$$

Here, $W(u)$ is the weighting function in the frequency domain. It is sufficient to integrate from $-\pi$ to π, since the functions are 2π-periodic.

Exercise 20 In general, the filters are related to the basis functions $b_m[k]$ and the applicability function $a[n]$ as (see lecture 2C, slide 21):

$$f_m[k] = a[-k] \cdot b_m[-k].$$

Since there is only one basis function $b[k] = 1$, the corresponding filter function is

$$f[k] = a[-k] = a\_\mathrm{rev}[k].$$

In general, the local metric $G[k]$ at point k is given as a matrix with elements (see lecture 2C, slide 40):

$$\mathbf{G}_{ij}[k] = \langle\, b_j \mid b_i\,\rangle = \mathbf{b}_i^* \mathbf{G}_0[k]\, \mathbf{b}_j = \sum_n b_j[n]\, c[k+n]\, a[n]\, b_i^*[n],$$

where $\mathbf{G}_0[k] = \mathrm{diag}\big(a[n] \cdot c[k+n]\big)$. Since there is only one basis function $b[k] = 1$, the corresponding metric is a 1×1 matrix, with a single element

$$G[k] = \sum_n c[k+n]\, a[n] = \sum_n c[k-n]\, a[-n] = (c * a_{\mathrm{rev}})[k].$$