Guide* to answers for written examination in TSBB06 Multi-dimensional signal analysis, 2018-04-04

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PART I

Exercise 1 In order to determine the distance, we first need to normalize both vectors. The homogeneous coordinates of the point must be P-normalized:
$\mathbf{x} = (1, -3, 1, 2) \sim (1/2, -3/2, 1/2, 1)$
The dual homogeneous coordinates of the plane must be D-normalized:
$\mathbf{p} = (1, 2, 2, 1) \sim (-1/9, -2/9, -2/9, -1/9)$
The distance between the point and the plane is then given as the magnitude of the scalar product of these normalized vectors (the signed distance):
$d = |(1/2, -3/2, 1/2, 1) \cdot (-1/3, -2/3, -2/3, -1/3)| = |1/6| = 1/6 \approx 0.17$.

Exercise 2 Affine transformation must always map parallel line to parallel lines. This is not the case for homographies, parallel lines can become non-parallel after a homography transformation. This means that affine transformations preserves points at infinity, while homographies can transform proper points to points at infinity, or vice versa.

Exercise 3 The rotated 3D point is represented as the pure quaternion given by

$$\mathbf{q} \circ \mathbf{x}_q \circ \mathbf{q}^{-1} = \mathbf{q} \circ \mathbf{x}_q \circ \overline{\mathbf{q}},$$

where $\mathbf{x}_q$ is the pure quaternion representation of 3D point $\mathbf{x}$:

$$\mathbf{x}_q = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and $\mathbf{q}^{-1} = \overline{\mathbf{q}}$ is the inverse of the unit quaternion $\mathbf{q}$.

Exercise 4 Given two distinct parallel lines:

$$\mathbf{l}_1 = \begin{pmatrix} \hat{\mathbf{l}} \\ -\Delta_1 \end{pmatrix} \quad \mathbf{l}_2 = \begin{pmatrix} \hat{\mathbf{l}} \\ -\Delta_2 \end{pmatrix}$$

it follows that their cross product, that normally produces the point of intersection, is

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} \hat{\mathbf{l}} \\ -\Delta_1 \end{pmatrix} \times \begin{pmatrix} \hat{\mathbf{l}} \\ -\Delta_2 \end{pmatrix} = \begin{pmatrix} (\Delta_1 - \Delta_2)\, \hat{\mathbf{l}} \\ 0 \end{pmatrix}$$

*This guide is not an authoritative description of how answers to the questions must be given in order to pass the exam. Some explanations given here may not have to be included in the answer, unless explicitly called for.

This means that we can interpret $(a, b, 0)$ as the homogeneous coordinates of the point of intersection of two distinct and parallel lines, intuitively a point at infinite distance from any Euclidean point, a *point at infinity.*

Given two distinct points at infinity, with homogeneous coordinates:

$$\mathbf{x}_1 = \begin{pmatrix} \bar{\mathbf{x}}_1 \\ 0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} \bar{\mathbf{x}}_2 \\ 0 \end{pmatrix}$$

it follows that their cross product, that normally produces the line of intersection, is

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{pmatrix} \bar{\mathbf{x}}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \bar{\mathbf{x}}_2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

Times means that we can interpret $(0, 0, 1)$ as the dual homogeneous coordinates of the intersecting line of two distinct points at infinity, intuitively a line at infinity since it intersect with all points at infinity and only these points.

Exercise 5 Given two distinct 3D points, $\mathbf{x}_1$ and $\mathbf{x}_2$, the Plücker coordinates for the line that passes through the points is given as:

$$\mathbf{L} = \mathbf{x}_1 \mathbf{x}_2^\top - \mathbf{x}_2 \mathbf{x}_1^T.$$

Any point on this line has homogeneous coordinates that can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$, so if we pick two other points one the line, $\mathbf{x}_1'$ and $\mathbf{x}_2'$, their homogeneous coordinates are

$$\mathbf{x}_1' = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2, \quad \mathbf{x}_2' = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2.$$

The corresponding Plücker coordinates are:

$$\begin{aligned}
\mathbf{L}' &= \mathbf{x}_1' \mathbf{x}_2'^T - \mathbf{x}_2' \mathbf{x}_1'^\top = \\
&= (\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2)(\beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2)^\top - (\beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2)(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2)^\top = \\
&= \alpha_1 \beta_2 \mathbf{x}_1 \mathbf{x}_2^\top + \alpha_2 \beta_1 \mathbf{x}_2 \mathbf{x}_1^\top - \alpha_2 \beta_1 \mathbf{x}_1 \mathbf{x}_2^\top - \alpha_1 \beta_2 \mathbf{x}_2 \mathbf{x}_1^\top = \\
&= \left(\alpha_1 \beta_2 - \alpha_2 \beta_1\right)\left(\mathbf{x}_1 \mathbf{x}_2^\top - \mathbf{x}_2 \mathbf{x}_1^\top\right) = \left(\alpha_1 \beta_2 - \alpha_2 \beta_1\right) \mathbf{L}
\end{aligned}$$

Since we consider $\mathbf{L}$ as a projective element, multiplication by a scalar does not change the Plücker coordinates. This means that any pair of points on the same line will produce the same set of Plücker coordinates.

PART II

Exercise 6 The inhomogeneous method gives, at best, one single solution by solving a linear inhomogeneous equation. The homogeneous method gives the solution as a right singular vector of the data matrix, corresponding to a singular value equal to zero, in practice the smallest singular value. In addition to this solution, the SVD profile (= the set of singular values of the data matrix) characterizes the solution space. This means that the profile can tell if the solution is unique (the smallest singular value is well-defined), or if there exist multiple solutions (there are multiple singular values that are approximately equal to zero), or if a reasonable solution does not exist (there is no singular value approximately equal to zero).

Exercise 7 From the relation $\mathbf{y}' \sim \mathbf{H}\,\mathbf{y}$ can be form two equations (constraints), using DLT, in the form of linear combinations between the unknown elements of $\mathbf{H}$ and parameters that are combinations of elements in $\mathbf{y}'$ and in $\mathbf{y}$. These two equations can be represented as the matrix equation $\mathbf{A}\,\mathbf{z} = \mathbf{0}$, where $\mathbf{A}$ is a 2×9 data matrix and $\mathbf{z}$ is a vector that holds the 9 elements of $\mathbf{H}$. The algebraic error is given as $\epsilon = \|\mathbf{A}\,\mathbf{z}\|^2$.

Exercise 8 The SVD profile of the homogeneous solution to the minimization of algebraic errors depends on the choice of coordinate system in the Euclidean space where point of lines are defined. Consequently, the same geometric estimation problem may produce an SVD profile that describes the solution space either as having one dimension or multiple dimensions, just by changing from one coordinate system to another. Hartley normalization aims at choosing a data dependent coordinate system that reduces the dimensions of the solution space so that it properly reflects the geometric configuration, e.g., of a set of points. In addition, Hartley normalization makes algebraic errors more similar to geometric errors.

Exercise 9 See the IREG compendium, section 14.2, and Algorithm 14.4.

Exercise 10 The relation $\mathbf{y} \sim \mathbf{C}\,\mathbf{x}$ can be transformed to $\mathbf{0} = [\,\mathbf{y}\,]_\times \mathbf{C}\,\mathbf{x}$ using DLT. This is 3 linear homogeneous equations in the unknown elements of $\mathbf{C}$. However, since the cross product operator $[\,\mathbf{y}\,]_\times$ has rank 2, the three equations are linearly dependent. Thus, one of the two equations must be satisfied if the other two are true. In general, we can remove any of the three equations and keep the other two as constraints in the elements of $\mathbf{C}$.

PART III

Exercise 11 The coordinates of $\mathbf{v}$ are given as the scalar products between $\mathbf{v}$ the the dual basis vectors $\tilde{\mathbf{b}}_k$.

Exercise 12 $\mathbf{c}'$ must be a null vector of $\mathbf{B}$: $\mathbf{B}\,\mathbf{c}' = \mathbf{0}$. $\mathbf{c}$ has minimal norm of all vectors that can reconstruct $\mathbf{v}$, since $\mathbf{c}$ is orthogonal to $\mathbf{c}'$. See lecture 2F for more details and motivation.

Exercise 13 See lecture 2A, slides $28 - 33$.

Exercise 14 Only true when $\mathbf{v} \in U$, where U is the subspace spanned by $\mathbf{B}$. In general $\mathbf{B}^\star \mathbf{v}$ represents the orthogonal projection of $\mathbf{v}$ onto U. See lecture 2C for more details and motivation.

Exercise 15 Using the scalar product between discrete functions of N samples:

$$\langle\, f[k] \mid g[k]\,\rangle = \sum_k^{N-1} f[k]\; g[k]^*,$$

the expression for discrete Fourier transform can be written:

$$F[l] = \sum_k^{N-1} f[k]\; e^{-i2\pi kl/N} = \langle\, f[k] \mid e^{2\pi ikl/N}\,\rangle$$

This implies that the functions, of the discrete variable k, $\tilde{b}_l[k] = e^{2\pi ikl/N}$ are the dual basis functions. The corresponding basis functions are then given as $b_l[k] = \frac{1}{N}\, e^{2\pi ikl/N}$ since

$$\langle\, b_n[k] \mid \tilde{b}_l[k]\,\rangle = \frac{1}{N}\sum_k^{N-1} e^{2\pi ikn}\; e^{-i2\pi kl/N} = \frac{1}{N}\sum_k^{N-1} e^{2\pi ik(n-l)/N} = \frac{1}{N}\, N\, \delta_{nl} = \delta_{nl}$$

PART IV

Exercise 16 See lecture 2F, slide 31.

Exercise 17 See lecture 2C, slide 24.

Exercise 18 See lecture 2G, slide 24.

Exercise 19 The more coefficients the filter has, the higher is the cost of computing each filter result (in terms of time components, time, effect, etc.). From this perspective we want as few coefficients as possible. However, the more coefficients the filter has, the closer it can come to any ideal frequency function. Consequently, we would like to have as many coefficients as possible to come close to the ideal frequency function.

Exercise 20 We apply Principal Component Analysis (PCA) on the signal to determine an M-dimensional basis that can represent the signal optimally in the sense that it minimises the mean square of the norm of difference between the real signal and the one that can be represented by the PCA-basis. The PCA-basis is computed by making an SVD of the data matrix $\mathbf{A}$, consisting of the signal vectors $\mathbf{v}$ in its columns, i..e, $\mathbf{A} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$. The M-dimensional PCA-basis consists of the M leftmost column vectors in $\mathbf{U}$, i.e., the left singular vectors of $\mathbf{A}$ that corresponds to the M largest singular values. Alternatively and equivalently, the basis is given by an ON-basis of eigenvectors of the M largest eigenvalues of $\mathbf{A}\,\mathbf{A}^T$.

In this case, the coefficients that need to be stored for each signal vector $\mathbf{v}$ are its M coordinates relative to the PCA-basis. Since the PCA-basis is an ON-basis, the coordinates are given directly as the scalar product between $\mathbf{v}$ and each of the basis vectors. Also, the subspace reconstruction of $\mathbf{v}$ is given by a linear combination of the coordinates and the basis vectors.

The mean square of the reconstruction error is in this case given by the sum of the eigenvalues in $\mathbf{A}^T\mathbf{A}$ (or squared singular values in $\mathbf{A}$) whose eigenvectors/singular vectors are not part of the PCA-basis.