Answers to the Final Examination 2020-01-17 in TSBB06 Multi-Dimensional Signal Analysis

Note: The answers given here are not an authoritative description of how answers to the questions must be given in order to pass the exam. Some explanations given here may not have to be included in the answer, unless explicitly called for.

Scoring: is in terms of half-points.
Type A problems are scored with one of [0p, 0.5p, 1p], while
Type B problems are scored with one of [0p, 0.5p, 1p, 1.5p, 2p].

Mårten Wadenbäck

PART I, Geometry

Answer 1 (A, 1p) The ideal point is of the form $\mathbf{x}_\infty = (x_\infty, y_\infty, 0)$ and satisfies $\mathbf{l}^\top \mathbf{x}_\infty = 0$, since it lies on $\mathbf{l}$. We find $\mathbf{l}$ as $\mathbf{l} \sim \mathbf{x}_1 \times \mathbf{x}_2 = (3, 1, -1)$. This means that $\mathbf{x}_\infty \sim (1, -3, 0)$.

Answer 2 (A, 1p) The Plücker coordinates can be written as $\mathbf{L} = \mathbf{x}\mathbf{x}_0^\top - \mathbf{x}_0\mathbf{x}^\top$, where $\mathbf{x}_0$ and $\mathbf{x}$ are two distinct points on the line. If $\mathbf{x}$ also lies in the plane, then $\mathbf{p}^\top\mathbf{x} = 0$, and

$$\mathbf{L}\mathbf{p} = (\mathbf{x}\mathbf{x}_0^\top - \mathbf{x}_0\mathbf{x}^\top)\mathbf{p} = \mathbf{x}(\mathbf{x}_0^\top\mathbf{p}) - \mathbf{x}_0\underbrace{(\mathbf{x}^\top\mathbf{p})}_{=0}.$$

If $\mathbf{x}_0$ does not lie on the plane, then $\mathbf{x}_0^\top\mathbf{p} \neq 0$, and then we get the point of intersection as $\mathbf{x} \sim \mathbf{L}\mathbf{p}$.

If $\mathbf{L}\mathbf{p} = \mathbf{0}$, this means that both $\mathbf{x}_0$ and $\mathbf{x}$ lie on the plane, and thus that the plane contains the whole line.

Answer 3 (A, 1p) The points $\mathbf{x}$ on the line are characterised by $\mathbf{l}^\top\mathbf{x} = 0$. If $\hat{\mathbf{x}}$ are homogeneous coordinates of the transformed point and $\hat{\mathbf{l}}$ are dual homogeneous coordinates of the transformed line, then $\hat{\mathbf{l}}^\top\hat{\mathbf{x}} = 0$. With $\hat{\mathbf{x}} \sim \mathbf{H}\mathbf{x}$ this becomes $\hat{\mathbf{l}}^\top\mathbf{H}\mathbf{x} = 0$, which holds if $\hat{\mathbf{l}} \sim \mathbf{H}^{-\top}\mathbf{l}$.

Answer 4 (B, 2p) Affine transforations are represented in matrix form as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0}^\top & 1 \end{pmatrix}, \qquad \text{where } \mathbf{A} \text{ is square and non-singular.}$$

1. *Closure*: Clearly,

   $$\mathbf{T}_2\mathbf{T}_1 = \begin{pmatrix} \mathbf{A}_2 & \mathbf{b}_2 \\ \mathbf{0}^\top & 1 \end{pmatrix}\begin{pmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{0}^\top & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_2\mathbf{A}_1 & \mathbf{A}_2\mathbf{b}_1 + \mathbf{b}_2 \\ \mathbf{0}^\top & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_3 & \mathbf{b}_3 \\ \mathbf{0}^\top & 1 \end{pmatrix} = \mathbf{T}_3,$$

   also represents an affine transforation.

2. *Associativity*: This follows immediately from matrix multiplication.

3. *Identity*: The identity transforation is $\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}$.

4. *Inverse*: The inverse of $\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0}^\top & 1 \end{pmatrix}$ is given by

   $$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{b} \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Answer 5 (B, 2p)

(a) The intersection can be either empty, a point, a line, or a plane.

(b) The intersection is the set of points $\mathbf{X}$ satisfying

$$\underbrace{\begin{pmatrix} \mathbf{p}_1^\top \\ \vdots \\ \mathbf{p}_n^\top \end{pmatrix}}_{\mathbf{M}} \mathbf{X} = \mathbf{0}$$

By computing the rank of the coefficient matrix $\mathbf{M}$, we can tell what geometric object the intersection is:

- $\operatorname{rank} \mathbf{M} = 4$ means that the intersection is empty
- $\operatorname{rank} \mathbf{M} = 3$ means that the intersection is a point
- $\operatorname{rank} \mathbf{M} = 2$ means that the intersection is a line
- $\operatorname{rank} \mathbf{M} = 1$ means that the intersection is a plane (i.e. all planes $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are the same)

PART I, Estimation

Answer 6 (A, 1p) The purpose of Hartley normalisation is to make the SVD profile of the data matrix less ambiguous (i.e. it should be easier to decide which singular values are essentially zero and which are not).

A set of points that has been Hartley normalised has mean value zero in both the x- and y-direction, and the mean distance to the origin is $\sqrt{2}$.

Answer 7 (A, 1p) The formula is

$$f(\mathbf{l}) = \sum_{k=1}^{n} \big((\mathrm{norm}_{\mathrm{D}}\, \mathbf{l})^{\top} (\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_k) \big)^2.$$

Answer 8 (A, 1p) The advantage of the geometric error is that it measures something that has a meaningful interpretation, which the algebraic error often does not have. The disadvantage of the geometric error is that it is often much more difficult to minimise that the geometric error.

Answer 9 (B, 2p) Suppose that $\mathbf{a}_k \leftrightarrow \mathbf{b}_k$, $k = 1, \ldots, p$ are the correspondences. The goal is to find a rotation $\mathbf{R}$ and a translation $\mathbf{t}$ which minimise the error

$$\epsilon(\mathbf{R}, \mathbf{t}) = \sum_{k=1}^{p} \|\mathbf{b}_k - \mathbf{R}\mathbf{a}_k - \mathbf{t}\|^2.$$

First, subtract the mean ($\mathbf{a}_0$ and $\mathbf{b}_0$) from each point set, giving $\mathbf{a}'_k = \mathbf{a}_k - \mathbf{a}_0$ and $\mathbf{b}'_k = \mathbf{b}_k - \mathbf{b}_0$. Form matrices $\mathbf{A} = \begin{pmatrix} \mathbf{a}'_1 & \ldots & \mathbf{a}'_p \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{b}'_1 & \ldots & \mathbf{b}'_p \end{pmatrix}$.

Secondly, compute the optimal rotation $\mathbf{R}$ using OPP on the centered data. This is done by computing an svd of $\mathbf{B}\mathbf{A}^{\top}$, i.e. $\mathbf{B}\mathbf{A}^{\top} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\top}$, and setting $\mathbf{R} = \mathbf{U}\mathbf{V}^{\top}$.

Finally, the optimal translation is computed as $\mathbf{t} = \mathbf{b}_0 - \mathbf{R}\mathbf{a}_0$.

Answer 10 (B, 2p)

(a) We want to find a matrix $\mathbf{H}$ satisfying $\hat{\mathbf{x}}_k \times \mathbf{H}\mathbf{x}_k = \mathbf{0}$, or, in matrix form, $[\hat{\mathbf{x}}_k]_{\times}\mathbf{H}\mathbf{x}_k = \mathbf{0}$. At most two of the three rows are linearly independent, and we can safely[1] omit the last row, giving us two linear equations in the nine unknowns of $\mathbf{H}$. Let $\mathbf{A}_k$ be a matrix holding the coefficients of the two linear equations thus created from the correspondence $\hat{\mathbf{x}}_k \leftrightarrow \mathbf{x}_k$. Then, the data matrix created from all correspondences will be a vertically stacked

[1]This is true as long as no $\hat{\mathbf{x}}_k$ is an ideal point, which measured image points never are. Thus, the points can be assumed to be P-normalised, and then $\hat{z}_k = z_k = 1$.

matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}.$$

(b) Each point correspondence contributes with two rows to the data matrix, and can thus at most increase the rank by two. Since the data matrix has dimensions $2n \times 9$, the rank can never exceed nine. Thus, the largest rank the data matrix can have is $\min(2n, 9)$.

PART II, Linear Signal Representations

Answer 11 (A, 1p) The *dual basis* is a set of vectors $\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_n$ satisfying

$$\hat{\mathbf{b}}_j^\top \mathbf{G}_0 \mathbf{b}_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Answer 12 (A, 1p) The closest vector $\mathbf{u}$ will be the orthogonal projection of $\mathbf{v}$ on U, i.e. $\mathbf{u} = \mathbf{B}(\mathbf{B}^\top \mathbf{G}_0 \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{v}$.

This can be derived in the following way (as in the lecture):

Thus, $\mathbf{v} - \mathbf{u}$ must be orthogonal to U, i.e. $\langle \mathbf{v} - \mathbf{u} \mid \mathbf{b}_k \rangle = \mathbf{b}_k^\top \mathbf{G}_0 (\mathbf{v} - \mathbf{u}) = 0$ for all k. Now, let $\mathbf{B} \in \mathbb{R}^{n \times r}$ be a matrix with the subspace basis $\mathbf{b}_1, \ldots, \mathbf{b}_r$ as its columns. The orthogonality $\mathbf{v} - \mathbf{u} \perp U$ can then be expressed for all $\mathbf{b}_k$ simultaneously as

$$\mathbf{B}^\top \mathbf{G}_0 (\mathbf{v} - \mathbf{u}) = \mathbf{0} \iff \mathbf{B}^\top \mathbf{G}_0 \mathbf{v} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{u} \iff \mathbf{B}^\top \mathbf{G}_0 \mathbf{v} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{B} \mathbf{c},$$

where the $\mathbf{c}$ that appeared at the end are the coefficients of $\mathbf{u}$ with respect to the subspace basis in $\mathbf{B}$. Solving for $\mathbf{c}$, we obtain $\mathbf{c} = (\mathbf{B}^\top \mathbf{G}_0 \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{G}_0 \mathbf{v}$. Finally, $\mathbf{u} = \mathbf{B}\mathbf{c} = \mathbf{B}(\mathbf{B}^\top \mathbf{G}_0 \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{G}_0 \mathbf{v}$.

Answer 13 (A, 1p) If $\mathbf{c}_1 = \tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{v}$, then $\mathbf{B}\mathbf{c}_1 = \mathbf{v}$. For any $\mathbf{c}_0$ in the null space of $\mathbf{B}$, the vector $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_0$ will give valid reconstructing coefficients.

Answer 14 (B, 2p)

(a) The frame condition is that there should exist constants A and B with $0 < A \leq B$, such that

$$A \|\mathbf{v}\|^2 \leq \sum_k \left| \langle \mathbf{v} \mid \mathbf{b}_k \rangle \right|^2 \leq B \|\mathbf{v}\|^2.$$

In general, a more compact way of writing it is (note the Frobenius norm in the middle!)

$$A \|\mathbf{v}\|^2 \leq \|\mathbf{F}\mathbf{v}\|_{\mathrm{F}}^2 \leq B \|\mathbf{v}\|^2,$$

which in our particular case becomes (according to (b))

$$A \|\mathbf{v}\|^2 \leq \|\mathbf{B}\mathbf{B}^\top \mathbf{v}\|^2 \leq B \|\mathbf{v}\|^2.$$

(b) The frame operator can be written as $\mathbf{F} = \mathbf{B}\mathbf{B}^\top \mathbf{G}_0 = \mathbf{B}\mathbf{B}^\top$ (since $\mathbf{G}_0 = \mathbf{I}$).

Answer 15 (B, 2p)

(a) The coordinates can in this case be interpreted as the derivatives of the signal.

(b) See lecture 2C. The Gramian at signal position k will be

$$\begin{pmatrix} a[0]c[k] & & & \\ & a[1]c[k+1] & & \\ & & \ddots & \\ & & & a[\ell-1]c[k+\ell-1] \end{pmatrix},$$

where a is the *applicability function* (a sequence of length ℓ) and c is the *signal certainty*.

PART II, Signal Processing Applications

Answer 16 (A, 1p) The conditions for $\phi(t)$ to be a scaling function are:

1. the functions $f_k(t) = \phi(t-k)$, $k \in \mathbb{Z}$, is an orthogonal basis for some function space, and
2. $\phi(t)$ can be written as a linear combination of the functions $\phi(2t-k)$, $k \in \mathbb{Z}$.

Answer 17 (A, 1p) The assumptions are:

1. the sampling noise is unbiased (zero mean), and
2. the sampling noise is independent between samples (but has a constant variance σ^2 over time).

Answer 18 (A, 1p) Let $\Psi(v) = \mathcal{F}\{\psi(t)\}$. The criterion for invertibility of the continuous wavelet transform will then be

$$0 < \int_{-\infty}^{\infty} \frac{|\Psi(v)|^2}{|v|}\, dv < \infty.$$

Answer 19 (B, 2p)

(a) The basis is determined in a way which minimises the expected squared difference between the data and its projection on the subspace spanned by the basis, i.e.

$$\epsilon = \mathbb{E}\Big[\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\|^2\Big].$$

(b) Given the data $\mathbf{f}_1, \ldots, \mathbf{f}_p$, we can compute the eigenvalues and eigenvectors of the *correlation matrix* $\mathbf{C} = \frac{1}{p}\sum_{k=1}^{p} \mathbf{f}_k\mathbf{f}_k^\top$ (sorted in decreasing order of the eigenvalues). The eigenvectors will then be the principal components, and we can choose how many we want to use.

Alternatively (usually), we can obtain the principal components by computing an SVD of the data matrix $\mathbf{F} = \begin{pmatrix}\mathbf{f}_1 & \ldots & \mathbf{f}_p\end{pmatrix}$, i.e. $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$. In this case the pincipal components will be the columns of $\mathbf{V}$ (already sorted).

Answer 20 (B, 2p)

(a) An epipole is the image point to which another camera centre is projected, i.e. the image of another camera centre. In two-view geometry, there will be one epipole in each of the two images.

The epipoles can be computed by first computing the camera centres $\mathbf{n}$ and $\hat{\mathbf{n}}$ from the camera matrices (i.e. solving for the null spaces of $\mathbf{C}$ and $\hat{\mathbf{C}}$), and then projecting into the two views as $\mathbf{e} = \mathbf{C}\hat{\mathbf{n}}$ and $\hat{\mathbf{e}} = \hat{\mathbf{C}}\mathbf{n}$.

(b) In this case, it is assumed that $\mathbf{x} \sim \mathbf{C}\mathbf{X}$ and $\hat{\mathbf{x}} \sim \hat{\mathbf{C}}\mathbf{X}$. This can be expressed using the cross product as

$$\begin{cases} \mathbf{x} \times \mathbf{C}\mathbf{X} = \mathbf{0} \\ \hat{\mathbf{x}} \times \hat{\mathbf{C}}\mathbf{X} = \mathbf{0} \end{cases} \iff \begin{cases} [\mathbf{x}]_\times \mathbf{C}\mathbf{X} = \mathbf{0} \\ [\hat{\mathbf{x}}]_\times \hat{\mathbf{C}}\mathbf{X} = \mathbf{0} \end{cases} \iff \begin{pmatrix} [\mathbf{x}]_\times \mathbf{C} \\ [\hat{\mathbf{x}}]_\times \hat{\mathbf{C}} \end{pmatrix} \mathbf{X} = \mathbf{0}.$$

This is a linear system of equations, which we know how to solve.