Transformations for Consideration

We will discuss the following types of transformation in 2D:

- Rigid transformations
- Similarity transformations
- Affine transformations
- Homographies (‘What do we say to the god of death?’ ...)

Characterisation of a Transformation

We will characterise transformations in three equivalent ways:

- The geometric action of the transformation.
- The algebraic representation of the transformation.
- The geometric invariants of the transformation.

DoF = degrees of freedom =
= # parameters

Characterisation of a Transformation

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We are also interested in the number of *degrees of freedom* for each transformation, i.e. the smallest number of parameters needed in the algebraic description.

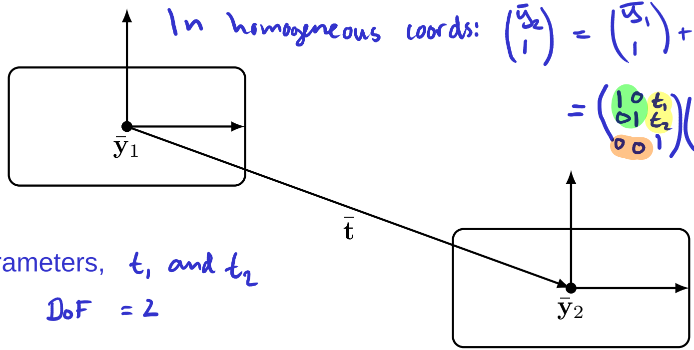
Translations

$\bar{y}_2 = \bar{y}_1 + \bar{t}$ not linear! no matrix mult!

In homogeneous coords: $\begin{pmatrix}\bar{y}_2\\1\end{pmatrix} = \begin{pmatrix}\bar{y}_1\\1\end{pmatrix} + \begin{pmatrix}\bar{t}\\0\end{pmatrix} =$

$$= \begin{pmatrix}1&0&t_1\\0&1&t_2\\0&0&1\end{pmatrix}\begin{pmatrix}y_{11}\\y_{12}\\1\end{pmatrix} = \begin{pmatrix}y_{11}+t_1\\y_{12}+t_2\\1\end{pmatrix}$$

$$= \begin{pmatrix}I & \bar{t}\\ 0^T & 1\end{pmatrix} \in \mathbb{R}^{3\times3}$$



We need two parameters, t_1 and t_2

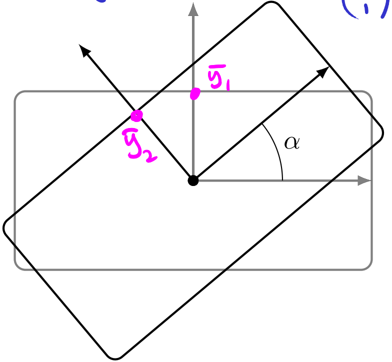
DoF $= 2$

Figure: A *translation* is obtained by adding a constant vector $\bar{t}$ to every point.

Rotations

$\bar{y}_2 = R\bar{y}_1$ with $R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ Note: $\det R = +1$

In homogeneous coordinates: $\begin{pmatrix} \bar{y}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} R\bar{y}_1 \\ 1 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ 1 \end{pmatrix}$



Needs 1 parameter

$\Rightarrow$ DoF $= 1$

Shear transformations

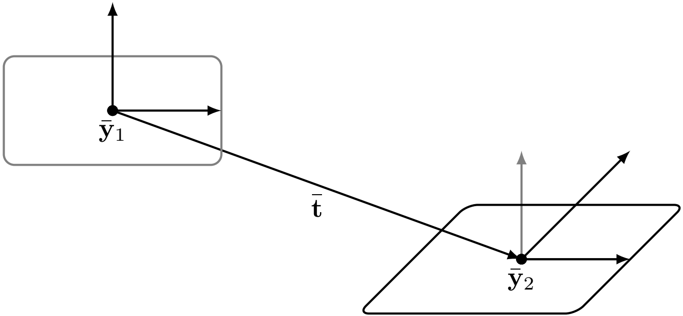


Figure: A *shear transformation* consists of a translation and a linear transformation with determinant one.

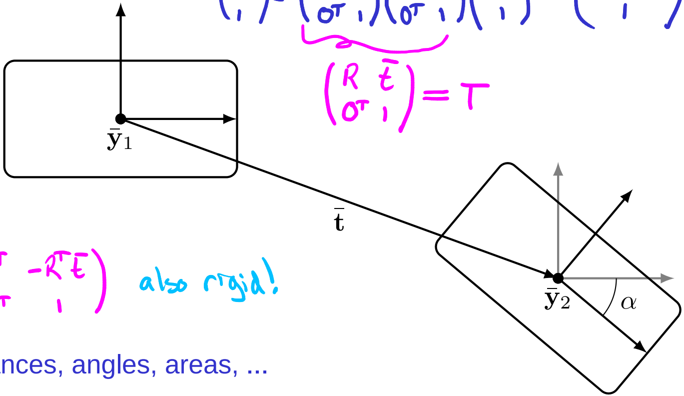
Rigid Transformations

$\bar{y}_2 = R\bar{y}_2 + \bar{t}$

$$\begin{pmatrix} \bar{y}_2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} I & \bar{t} \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0^T & 1 \end{pmatrix}} \begin{pmatrix} \bar{y}_1 \\ 1 \end{pmatrix} = \begin{pmatrix} R\bar{y}_2 + \bar{t} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} R & \bar{t} \\ 0^T & 1 \end{pmatrix} = T$$

DoF $= 3$



Inverse: $T^{-1} = \begin{pmatrix} R^T & -R^T\bar{t} \\ 0^T & 1 \end{pmatrix}$ also rigid!

Invariants? Distances, angles, areas, ...

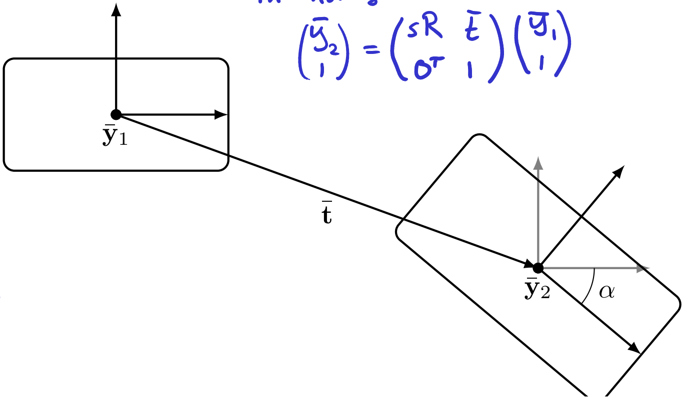
Similarity Transformations

$\bar{y}_2 = sR\bar{y}_1 + \bar{t}$

In homogeneous coords:

$$\begin{pmatrix} \bar{y}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} sR & \bar{t} \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ 1 \end{pmatrix}$$

DoF = 4



Invariants: Angles

Affine Transformations

$\bar{y}_2 = A\bar{y}_1 + \bar{t}$ with $\det A \neq 0$

In homogeneous coords:

$$\begin{pmatrix} \bar{y}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & \bar{t} \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ 1 \end{pmatrix} = T \begin{pmatrix} \bar{y}_1 \\ 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\bar{t} \\ 0^T & 1 \end{pmatrix}$$

Affine transformations include all of the previous transformations, and more, e.g.:

- Non-uniform scaling
- Reflections
- ...

DoF = 6 (4 from A, 2 from $\bar{t}$)

Geometric Transformations in 2D

Table: Geometric transformations in 2D. Each type includes, as subgroups, the types listed below it.

Type	Matrix	Constraints	DoF	Invariants
Affine	$\begin{pmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \mathbf{0}^\top & 1 \end{pmatrix}$	$\det \mathbf{A} \neq 0$	6	parallel lines
Similarity	$\begin{pmatrix} s\mathbf{Q} & \bar{\mathbf{t}} \\ \mathbf{0}^\top & 1 \end{pmatrix}$	$\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$ and $s \neq 0$	4	$+$ angles
Rigid	$\begin{pmatrix} \mathbf{R} & \bar{\mathbf{t}} \\ \mathbf{0}^\top & 1 \end{pmatrix}$	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$	3	$+$ distance

Transformation Groups

(1) Assume $T_1 = \begin{pmatrix} A_1 & \bar{t}_1 \\ 0^T & 1 \end{pmatrix}$ $T_2 = \begin{pmatrix} A_2 & \bar{t}_2 \\ 0^T & 1 \end{pmatrix}$, then

$$T_1 T_2 = \begin{pmatrix} A_1 A_2 & A_1 \bar{t}_2 + \bar{t}_1 \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} A_3 & \bar{t}_3 \\ 0^T & 1 \end{pmatrix}$$

(closure)

(2) Neutral element: $T_N = \begin{pmatrix} I & 0 \\ 0^T & 1 \end{pmatrix} = I_{3\times 3}$

(3) Inverse: see slide 8

(4) Associativity: Follows from matrix product $(T_1 T_2) T_3 = T_1 (T_2 T_3)$