TSBB06 Multi-dimensional Signal Processing

Lecture 2B
Dual Bases

Bases and coordinates

- Let V be an N-dimensional vector space with basis $\mathbf{b}_k$, $k = 1, \ldots, N$
- Any $\mathbf{v} \in V$ can then be written as

$$\mathbf{v} = \sum_{k=1}^{N} c_k \mathbf{b}_k$$

for some set of coordinates c_k

- How do we determine these coordinates?

Dual basis

- Assume that we can find a set of *N* vectors $\tilde{\mathbf{b}}_k$ such that

$$\left\langle \mathbf{b}_i | \tilde{\mathbf{b}}_j \right\rangle = \left\langle \tilde{\mathbf{b}}_i | \mathbf{b}_j \right\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- This set is unique (why?)
- This set is a basis of *V* (why?)
- This set depends on the scalar product (why?)
- Called a **dual basis**

Dual basis

- Compute the scalar product between $\mathbf{v} \in V$ and a dual basis vector $\tilde{\mathbf{b}}_k$

$$\left\langle \mathbf{v} | \tilde{\mathbf{b}}_k \right\rangle = \left\langle c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \ldots + c_N \mathbf{b}_N | \tilde{\mathbf{b}}_k \right\rangle =$$

$$c_1 \left\langle \mathbf{b}_1 | \tilde{\mathbf{b}}_k \right\rangle + c_2 \left\langle \mathbf{b}_2 | \tilde{\mathbf{b}}_k \right\rangle + \ldots + c_N \left\langle \mathbf{b}_N | \tilde{\mathbf{b}}_k \right\rangle =$$

$$c_1 \cdot 0 + \ldots + c_k \cdot 1 + \ldots + c_N \cdot 0 = c_k$$

Coordinates and dual bases

Main result (in this part of the course!):

- If we have the dual basis, the coordinates of a vector are given as the scalar product between the vector and the dual basis
- We summarise this as

$$\mathbf{v} = \sum_{k=1}^{N} \left\langle \mathbf{v} \middle| \tilde{\mathbf{b}}_k \right\rangle \mathbf{b}_k$$

This works also for the infinite countable case

Orthonormal bases

- By definition, an orthonormal (or unitary) basis $\mathbf{b}_k$, $k = 1, \ldots, N$, satisfies

$$\langle \mathbf{b}_i | \mathbf{b}_j \rangle = \delta_{ij}$$

- Consequently: an ON-basis is its own dual basis
  - In this case <u>only</u>: coordinates are given as the scalar product between vector and basis

Change of basis

- A dual basis can also be useful if we want to change from one basis to another
- Let $\mathbf{b}_k$ and $\mathbf{b}'_k$, $k = 1, \ldots, N$ be two bases

$$\mathbf{v} = c_1\,\mathbf{b}_1 + c_2\,\mathbf{b}_2 + \ldots + c_N\,\mathbf{b}_N$$
$$\mathbf{v} = c'_1\,\mathbf{b}'_1 + c'_2\,\mathbf{b}'_2 + \ldots + c'_N\,\mathbf{b}'_N$$

where c_k and c'_k are the corresponding coordinates

$\mathbf{b}_k$ is the old basis
$\mathbf{b}'_k$ is the new basis

Change of basis

- Taking the scalar product between **v** and the new dual basis, and expanding **v** in the old basis gives us

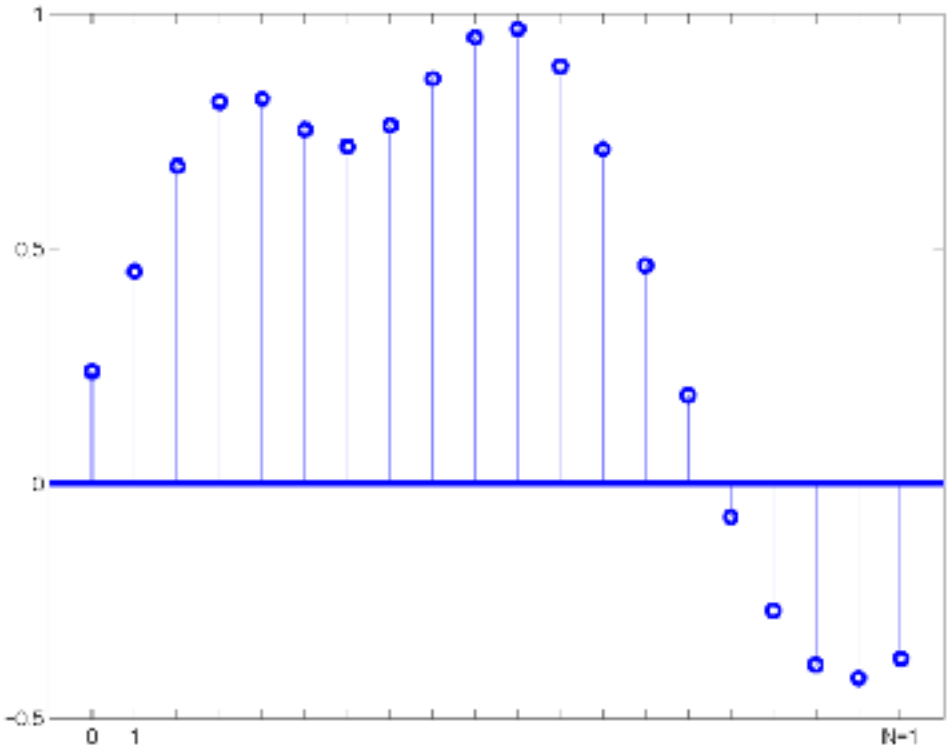
$$c'_k = \left\langle \mathbf{v} | \tilde{\mathbf{b}}'_k \right\rangle =$$

$$= \left\langle c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \ldots + c_N \mathbf{b}_N | \tilde{\mathbf{b}}'_k \right\rangle =$$

$$= c_1 \left\langle \mathbf{b}_1 | \tilde{\mathbf{b}}'_k \right\rangle + c_2 \left\langle \mathbf{b}_2 | \tilde{\mathbf{b}}'_k \right\rangle + \ldots + c_N \left\langle \mathbf{b}_N | \tilde{\mathbf{b}}'_k \right\rangle$$

An example (DFT)

- Consider a discrete signal f[k] of N samples, enumerated from 0 to N - 1



An example (DFT)

- An element of a vector space $V = \mathbb{C}^N$
  - Initially in $\mathbb{R}^N$ but we need to use $\mathbb{C}^N$
- We use the scalar product

$$\langle \mathbf{f} | \mathbf{g} \rangle = \sum_{k=0}^{N-1} f[k] g^*[k]$$

An example (DFT)

- This signal can be seen as the linear combination of *N* impulse functions together with the sample values

$$\mathbf{f} = \sum_{k=0}^{N-1} f[k]\boldsymbol{\delta}_k$$

We use bold face to denote elements of vector space *V*

where $\boldsymbol{\delta}_k = \delta[x-k]$

- The impulse functions $\boldsymbol{\delta}_k$, $k = 0, \ldots, N-1$ form an orthonormal (ON) basis for *V* (why?)

An example (DFT)

- The coordinates of ***f*** relative to this basis are very easy to find:
  - They are the function values
- We call this basis the *canonical basis* of this function space
  - **The canonical basis is its own dual basis**

- Let us now look at another basis for the same space

An example (DFT)

- Consider the set of functions defined as

$$\mathbf{b}_k = e^{2\pi i k x / N} \;, \quad k = 0, \ldots, N-1$$

- We note that

$$\langle \mathbf{b}_k | \mathbf{b}_l \rangle = \sum_{x=0}^{N-1} e^{-2\pi i l x / N} e^{2\pi i k x / N} = N \delta_{kl}$$

- Which means that

$$\tilde{\mathbf{b}}_k = \frac{1}{N} \mathbf{b}_k$$

An example (DFT)

- We have two bases for *V*,
  - the canonical basis
  - the complex exponential basis

  and we know the coordinates of $\mathbf{f} \in V$ in the first basis (the canonical basis)
- We also have the dual bases
  - We can determine the coordinates in both bases

An example (DFT)

- The coordinates of **f** relative to the second basis are given as

$$c'_k = \left\langle \mathbf{f} | \tilde{\mathbf{b}}_k \right\rangle = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi i k x / N}$$

- We recognise this as the discrete Fourier transform (DFT) of the function *f*

Interpretation of DFT (I)

- $c_k = F(2\pi k/N)$

- $F(u_k)$ at $u_k = 2\pi k/N$ is the *scalar product* of the signal *f* and the dual basis function

$$\frac{1}{N} e^{2\pi i k x/N}$$

Note: no minus sign!!

An example (DFT)

- Alternatively, we can compute c'_k by transforming from the coordinates in the canonical basis to coordinates in the new basis:

  new coordinates =

  linear combination of old coordinates and

  < old basis | new dual basis >

An example (DFT)

- This gives us

$$c'_k = \sum_{l=0}^{N-1} f[l] \left\langle \boldsymbol{\delta}_l | \tilde{\mathbf{b}}_k \right\rangle$$

- Since $\left\langle \boldsymbol{\delta}_l | \tilde{\mathbf{b}}_k \right\rangle = \frac{1}{N} e^{-2\pi i k l / N}$

we get

$$c'_k = \frac{1}{N} \sum_{l=0}^{N-1} f[l] e^{-2\pi i k l / N}$$

Same expression as before!

Interpretation of DFT (II)

- $F(u_k)$ at $u_k = 2\pi k/N$ is the result of transforming coordinates from the canonical basis to the basis

$$e^{2\pi i k x/N}$$

An example (DFT)

- A third interpretation of the DFT is given directly from the expression

$$F(u) = \frac{1}{N} \sum_{l=0}^{N-1} f[l] e^{-iul/N}$$

- The function *F*(*u*) is a linear combination of the functions

$$\frac{1}{N} e^{-iul/N}$$

with the coefficients *f*[*l*]

Interpretation of DFT (III)

- The resulting transform function *F*, either of a discrete variable $u_k = 2\pi k/N$, or a continuous variable *u*, is a linear combination of exponential functions

An example (DFT)

Conclusions:

- The DFT can be seen as either
  - a direct coordinate computation in terms of scalar product between **f** and the dual basis of the complex exponentials
  - A coordinate transformation when we change from the canonical basis $\boldsymbol{\delta}_k$ to the basis $\mathbf{b}_k$ of complex exponentials
  - A linear combination of complex exponentials

An example (DFT)

- This description generalises to the continuous Fourier transform but is more elaborate to prove formally since we have to deal with infinite and uncountable bases

Back to dual bases

- A very useful result:
  - Given a basis $\mathbf{b}_k$ with a dual basis $\tilde{\mathbf{b}}_k$
  - The dual basis of $\tilde{\mathbf{b}}_k$ is the original basis $\mathbf{b}_k$

- This is straightforward to show (how?), at least in the finite dimensional case

- Thus: $\tilde{\tilde{\mathbf{b}}}_{\mathbf{k}} = \mathbf{b}_{\mathbf{k}}$

Dual coordinates

This result implies for $\mathbf{v} \in V$:

$$\mathbf{v} = \sum_{k=1}^{N} c_k \mathbf{b}_k \qquad c_k = \left\langle \mathbf{v} | \tilde{\mathbf{b}}_k \right\rangle$$

$$\mathbf{v} = \sum_{k=1}^{N} \tilde{c}_k \tilde{\mathbf{b}}_k \qquad \tilde{c}_k = \langle \mathbf{v} | \mathbf{b}_k \rangle$$

These are the dual coordinates of **v**

Transforming coordinates

- The coordinates and the dual coordinates must be related according to the coordinate transformation rule:

  new coordinates =

  linear combination of old coordinates and < old basis | new dual basis >

Transforming coordinates

- Transforming from coordinates to dual coordinates means
  - old basis $\mathbf{b}_k$
  - new basis $\tilde{\mathbf{b}}_k$
  - new dual basis $\mathbf{b}_k$
- and gives us

$$\tilde{c}_k = \sum_{l=1}^{N} c_l \left\langle \mathbf{b}_l \middle| \mathbf{b}_k \right\rangle$$

Transforming coordinates

- Alternatively, transforming from dual coordinates to ”standard” coordinates means
  - old basis $\tilde{\mathbf{b}}_k$
  - new basis $\mathbf{b}_k$
  - new dual basis $\tilde{\mathbf{b}}_k$
- and gives us

$$c_k = \sum_{l=1}^{N} \tilde{c}_l \left\langle \tilde{\mathbf{b}}_l | \tilde{\mathbf{b}}_k \right\rangle$$

Expanding the basis in the dual basis

- We now go one step further and express one of the basis vectors as a linear combination of the dual basis:

$$\mathbf{b}_k = \sum_{l=1}^{N} \tilde{\mathbf{b}}_l \langle \mathbf{b}_k | \mathbf{b}_l \rangle$$

These are the dual coordinates of $\mathbf{b}_k$

Expanding the basis in the dual basis

- Let us define the matrix **G** with elements

$$G_{lk} = \langle \mathbf{b}_k | \mathbf{b}_l \rangle$$

Notice the order of the indices!

it allows us to write the previous expression as

$$\mathbf{b}_k = \sum_{l=1}^{N} \tilde{\mathbf{b}}_l G_{lk}$$

Expresses a basis vector as a linear combination of the dual basis

Expanding the basis in the dual basis

- By inverting **G** we can expand each dual basis vector in the original basis

$$\tilde{\mathbf{b}}_k = \sum_{l=1}^{N} \mathbf{b}_l [\mathbf{G}^{-1}]_{lk}$$

- This suggests a recipe for computing the dual basis of a general (non-orthogonal) basis:
  - Compute $\mathbf{G}$, with $G_{lk} = \langle \mathbf{b}_k | \mathbf{b}_l \rangle$
  - Invert **G**
  - Form linear combinations of the basis with **G**

The matrix **G**

- From the previous relation follows directly

$$\left\langle \tilde{\mathbf{b}}_k | \tilde{\mathbf{b}}_m \right\rangle = \sum_{l=1}^{N} \left\langle \mathbf{b}_l | \tilde{\mathbf{b}}_m \right\rangle \left[\mathbf{G}^{-1}\right]_{lk} = \sum_{l=1}^{N} \delta_{lm} \left[\mathbf{G}^{-1}\right]_{lk} = \left[\mathbf{G}^{-1}\right]_{mk}$$

- Furthermore, from $\langle \mathbf{a} | \mathbf{b} \rangle = \langle \mathbf{b} | \mathbf{a} \rangle^*$ follows

$$G_{kl} = G_{lk}^*$$

i.e., **G** is *Hermitian*
(or symmetric in the case *V* is real)

G and scalar products

- Let $\mathbf{u}$, $\mathbf{v} \in V$, and let $\mathbf{b}_k$ be a basis of V:

$$\mathbf{u} = \sum_{k=1}^{N} u_k \mathbf{b}_k \qquad \mathbf{v} = \sum_{k=1}^{N} v_k \mathbf{b}_k$$

Coordinates of **u** and **v**, respectively, in the basis $\mathbf{b}_k$

$$\mathbf{c}_u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \qquad \mathbf{c}_v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

G and scalar products

- The scalar product between **u** and **v**:

$$\langle \mathbf{u} | \mathbf{v} \rangle = \left\langle \sum_{k=1}^{N} u_k \mathbf{b}_k \middle| \sum_{l=1}^{N} v_l \mathbf{b}_l \right\rangle$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} u_k v_l^* \langle \mathbf{b}_k | \mathbf{b}_l \rangle$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} u_k v_l^* G_{lk} = \mathbf{c}_v^* \mathbf{G} \mathbf{c}_u$$

G and scalar products

- This means that **G** defines the scalar product in terms of coordinates relative to the basis $\mathbf{b}_k$
- Sometimes it is easier to describe a vector in terms of its coordinates relative to some basis, rather than as an abstract element of some vector space
- If **G** is at hand, it is then straightforward to compute scalar products by multiplying the coordinate vectors of **u** and **v** from left and right onto **G**

$$\langle \mathbf{u} | \mathbf{v} \rangle = \mathbf{c}_v^* \mathbf{G} \mathbf{c}_u$$

Notice the order of the vectors!

G and dual coordinates

- Previously, we derived how ”standard” coordinates are transformed into dual coordinates and vice versa:

$$\tilde{c}_k = \sum_{l=1}^{N} c_l \langle \mathbf{b}_l | \mathbf{b}_k \rangle = \sum_{l=1}^{N} c_l G_{kl}$$

$$c_k = \sum_{l=1}^{N} \tilde{c}_l \left\langle \tilde{\mathbf{b}}_l | \tilde{\mathbf{b}}_k \right\rangle = \sum_{l=1}^{N} \tilde{c}_l \left[\mathbf{G}^{-1} \right]_{kl}$$

G, summary

- **G** depends on the scalar product and on the basis
- **G** is called (the) *metric (tensor)*
  - *A.k.a.* the *Gram matrix or Gramian*
- **G** is Hermitian (symmetric if *V* is real)
- Gives the dual basis, for a specific basis
- Gives scalar products given coordinates
- Transforms ”standard” coordinates to dual ones

  $\mathbf{G}^{-1}$ transforms in the opposite way

Two operations

- We have made extensive use of two operations:
  - scalar products
  - linear combinations
- Given a basis $\mathbf{b}_k$ for V, the scalar product between $\mathbf{v} \in V$ and the dual basis gives the coordinates of **v** relative the basis
- Given the coordinates of **v**, **v** can be reconstructed as a linear combination of the coordinates and the basis vectors

Two operations

In the following:

- Scalar products between a signal and a set of vectors (a basis of V or of a subspace of V) are referred to as an *analyzing operation*
  – Produces some type of coordinates
- Linear combinations between coordinates and a basis is a *reconstructing operation*
  – Or *synthesising operation*
  – Produces vectors or signals
- Analyzing and reconstructing operations are in a *dual* relation to each other (how?)

Finite dimensional signals

- In this course, many signals that are described are finite dimensional
  - E.g. a segment of some infinite discrete signal
- Some are one-variable (e.g., time) but they are often multi-variable (e.g., spatial)
  - We can rearrange multi-dimensional signals as column vectors (how?)

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}$$

f and **g** contain coordinates relative the canonical basis

Finite dimensional signals

- We apply a scalar product that is defined in the canonical basis in terms of a matrix $\mathbf{G}_0$
  - $\mathbf{G}_0$ is the metric in the canonical basis!

$$\langle \mathbf{f} | \mathbf{g} \rangle = \mathbf{g}^* \mathbf{G}_0 \mathbf{f}$$

Note the order of the vectors!

The basis matrix

- Let $\mathbf{b}_k$, k = 1, ..., N be a basis of V
- Let $\mathbf{B}$ denote a matrix that contains the vectors $\mathbf{b}_k$ in its columns, the *basis matrix*

$$\mathbf{B} = \begin{pmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \\ | & | & & | \end{pmatrix}$$

The dual coordinates

- The dual coordinates of **v** is given by

$$\tilde{\mathbf{c}} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_N \end{pmatrix} = \begin{pmatrix} \langle \mathbf{v} | \mathbf{b}_1 \rangle \\ \langle \mathbf{v} | \mathbf{b}_2 \rangle \\ \vdots \\ \langle \mathbf{v} | \mathbf{b}_N \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^* \mathbf{G}_0 \mathbf{v} \\ \mathbf{b}_2^* \mathbf{G}_0 \mathbf{v} \\ \vdots \\ \mathbf{b}_N^* \mathbf{G}_0 \mathbf{v} \end{pmatrix}$$

which can be written more compactly as

$$\tilde{\mathbf{c}} = \mathbf{B}^* \mathbf{G}_0 \mathbf{v}$$

Analyzing operator

- An analyzing operator computes the scalar product with the basis for a vector

- $\mathbf{B}^*\mathbf{G}_0$ is an analyzing operator
  - It gives **v**'s dual coordinates when applied to **v**

- We define $\mathbf{A} = \mathbf{B}^*\mathbf{G}_0$

Reconstructing operator

- Given the dual coordinates of **v**, we can reconstruct **v** by means of a linear combination with the dual basis

$$\mathbf{v} = \tilde{\mathbf{B}}\tilde{\mathbf{c}}$$

where the *dual basis matrix* is

$$\tilde{\mathbf{B}} = \begin{pmatrix} | & | & & | \\ \tilde{\mathbf{b}}_1 & \tilde{\mathbf{b}}_2 & \cdots & \tilde{\mathbf{b}}_N \\ | & | & & | \end{pmatrix}$$

Reconstructing operator

- In this context, $\tilde{\mathrm{B}}$ is a reconstructing operator

- We define $\mathbf{R} = \tilde{\mathbf{B}}$

- In summary: **v = R A v** for all $\mathbf{v} \in V$

$\Rightarrow$ **R A = I** (The $N \times N$ identity matrix)

Analyzing and Reconstructing operators

- If, instead, the dual basis has been used for the analysis, we obtain directly the coordinates
- $\tilde{\mathbf{A}} = \tilde{\mathbf{B}}^* \mathbf{G}_0$ is an analyzing operator
  - It gives **v**'s coordinates when applied to **v**
- In this context, the reconstructing operator is $\tilde{\mathbf{R}} = \mathbf{B}$, forming linear combinations with the basis together with the coordinates

The dual basis revisited

- These relations imply

$$\mathbf{R}\mathbf{A} = \tilde{\mathbf{B}}\mathbf{B}^*\mathbf{G}_0 = \mathbf{I} \;\Rightarrow$$

$$\tilde{\mathbf{B}}\mathbf{B}^*\mathbf{G}_0\mathbf{B} = \mathbf{B}$$

- We note that $\mathbf{B}^*\mathbf{G}_0\mathbf{B}$ contains all possible scalar products between basis vectors, i.e.,

$$\mathbf{B}^*\mathbf{G}_0\mathbf{B} = \mathbf{G}$$

G is the scalar product in basis **B**
$\mathbf{G}_0$ is the scalar product in the canonical basis

The dual basis revisited

- This gives us

$$\tilde{\mathbf{B}}\mathbf{G} = \mathbf{B} \qquad \tilde{\mathbf{B}} = \mathbf{B}\mathbf{G}^{-1}$$

which is exactly the same relation between the basis and the dual basis as before (now in matrix form)

What you should know includes

- Definition of a dual basis
- Computation of coordinates based on dual basis
- What are dual coordinates
- Definition of a metric **G**
- Computation of the dual basis using **G**
- Computation of coordinates from dual coordinates using **G**
- The concepts analysing and reconstructing (or synthesising) operations or operators