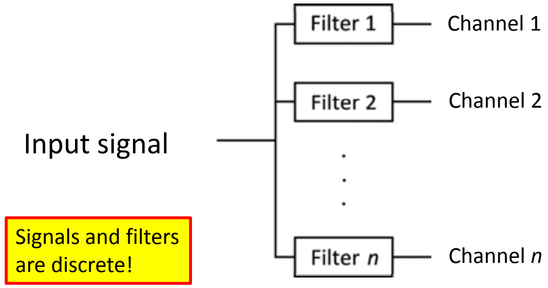
Multi-dimensional Signal Analysis

Lecture 2G
Filter Banks and
Multi-resolution Analysis (I)

Filter banks

- A common type of processing unit for discrete signals is a filter bank, where some input signal is filtered by *n* filters, producing *n* channels

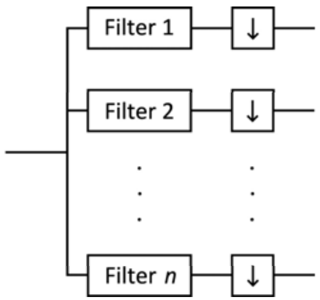


This an example of how to create *n* channels

Signals and filters are discrete!

Filter banks

- In order to have the same amount of data (on average per time unit) before and after the filtering: down-sample the filter outputs by a factor *n*
- Keep every *n*-th sample



Filter banks, applications

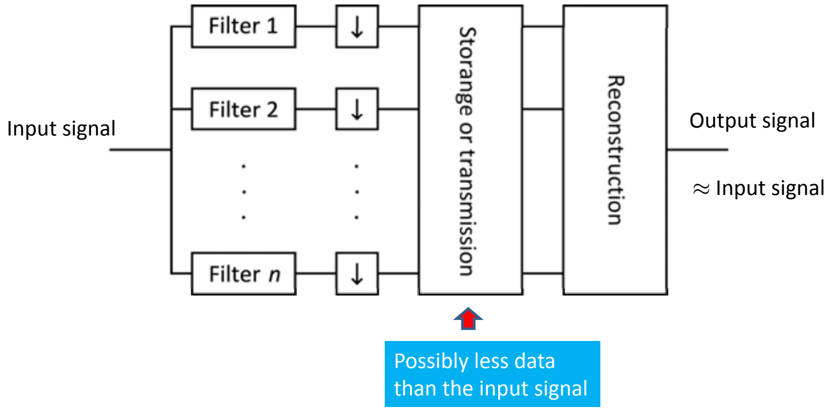
- One application for filter banks is to decompose the input signal into different *bands* or *channels*
- Here *band* is rather imprecise and can mean frequency band but also more general subspaces

Filter banks, applications

- Each band is processed independently relative to the others
  - Filtering, e.g., LP-filter to reduce noise
  - Data compression, to reduce data
  - Numerical resolution, to reduce data
  - Even: removal, to reduce data
- Each band is then stored or transmitted
- From each band we want to be able to reconstruct the input signal again
  - Possibly only to some degree of approximation
  - Or *perfect reconstruction:* output = input (+ shift!)

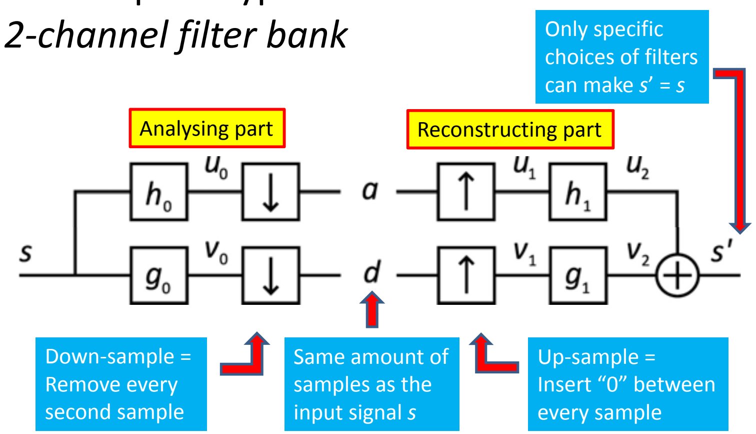
Filter banks, applications

- In general, something like this:



The 2-channel filter bank

- The simplest type of filter bank is the *2-channel filter bank*



The 2-channel filter bank

- Let the DFTs of h_0, h_1, g_0, g_1 be denoted as H_0, H_1, G_0, G_1 (they are 2π-periodic!)
- We can show that necessary and sufficient conditions for $s' = s$ are given by

$$H_1(u)H_0(u) + G_1(u)G_0(u) = 2 \quad \text{(FB1)}$$

$$H_1(u)H_0(u+\pi) + G_1(u)G_0(u+\pi) = 0 \quad \text{(FB2)}$$

- FB1: No distortion
- FB2: No aliasing from the sub-sampling

(why?)

Ex 21.2

A practical note

Assuming that all four filters are causal:

- The filters introduce delays in the output signal *s'*
- This means that, at best, we can accomplish *s'*[*k*] = *s*[*k* - *d*] for some positive integer *d*
- We will ignore this practical matter for now and consider the requirement *s'*[*k*] = *s*[*k*]

As a consequence, some filters may become *non-causal*

- Can be fixed by time-shift if they also are FIR
- Output will then be delayed accordingly
- We want the four filters to be FIR

Finding the four filters

- Bad idea: choose two of the filters *arbitrarily* and solve the other two from FB1 and FB2 (why?)
- Better idea: Apply some design principles for the four filters that produce useful filters
  - Many possibilities exists!

A particular choice of filters

- FB2 is automatically satisfied if

$$H_0(u) = \overline{H_1(u)}$$
$$G_1(u) = e^{-iu}\,\overline{H_1(u+\pi)}$$
$$G_0(u) = e^{iu}\,H_1(u+\pi) = \overline{G_1(u)}$$

Called: *Conjugate Mirror filters*

- FB1 then becomes

$$|H_1(u)|^2 + |H_1(u+\pi)|^2 = 2 \qquad \text{(O)}$$

Ex 21.4

Alternating flip

- The conjugate mirror filter condition can also be formulated in the time domain as

$$h_0[k] = \overline{h_1[-k]}$$
$$g_1[k] = (-1)^{1-k}\,\overline{h_1[1-k]}$$
$$g_0[k] = (-1)^{k+1}\,h_1[k+1] = \overline{g_1[-k]}$$

Called: *Alternating flip*

Ex 21.5

An orthogonal set

- We can show that the condition (O) implies that

$$\langle h_1[\,\cdot\,] \mid h_1[\,\cdot - 2\,m] \rangle = \delta[m]$$

- h_1 is always orthogonal to itself when shifted by multiples of 2 (unless m = 0)

Ex 21.8

Convolution and linear combinations

- We have already seen that the convolution operation can be interpreted in terms of producing linear combinations
- Here:

$$u_2[k] = \sum_{n=-\infty}^{\infty} a[n]\, h_1[k - 2\,n]$$

- The signal u_2 is a linear combination of shifted versions of h_1
  - Shift by multiples of 2
  - We just saw that they form an ON-set!

More orthogonality

- In the same way we can show that the conjugate mirror filter bank leads to

$$\langle g_1[\,\cdot\,] \mid g_1[\,\cdot - 2\,m] \rangle = \delta[m]$$

$$\langle g_1[\,\cdot\,] \mid h_1[\,\cdot - 2\,m] \rangle = 0$$

Ex 21.11 (why?)

More orthogonality

- This means that also $v_2[k]$ is formed as a linear combination of orthonormal discrete functions $g_1[k-2\,m]$ ⟵ These are functions of k, one for each m
- Finally, it also means that the two orthogonal sets $h_1[k-2\,m]$ and $g_1[k-2\,m]$ are mutually orthogonal

These are functions of k, one for each m

Reconstruction

- The output signal from the filter bank, $s' = s$, is formed as the linear combination of two orthogonal sets of functions (an ON-basis)
- Means: the coefficients in this linear combination are the *coordinates* relative the reconstructing basis functions. These are
  - $a[m]$ for the basis functions $h_1[k - 2m]$
  - $d[m]$ for the basis functions $g_1[k - 2m]$

Analysis

- In the filter bank, $a[m]$ is formed as

$$\begin{aligned} a[m] &= (s * h_0)[2\,m] \\ &= \sum_{k=-\infty}^{\infty} s[k]\, h_0[2\,m - k] \\ &= \sum_{k=-\infty}^{\infty} s[k]\, \overline{h_1[k - 2\,m]} = \langle s[\,\cdot\,] | h_1[\,\cdot - 2\,m] \rangle \end{aligned}$$

Analysis

- Similarly: $d[m] = \langle\, s[\cdot] \mid g_1 [\cdot - 2\, m] \,\rangle$
- This result is expected!
- We are analysing $s[k]$ using an ON-basis
- The corresponding coordinates are then given as the scalar products between $s[k]$ and the basis functions
  - They are their own duals!

Orthogonal filter bank

- In summary: we choose H_1 such that

$$|H_1(u)|^2 + |H_1(u + \pi)|^2 = 2$$

and then the other three filters such that

$$\begin{aligned} H_0(u) &= \overline{H_1(u)} \\ G_1(u) &= e^{-iu}\, \overline{H_1(u + \pi)} \\ G_0(u) &= e^{iu}\, H_1(u + \pi) = \overline{G_1}(u) \end{aligned}$$

- This produces an *orthogonal filter bank*

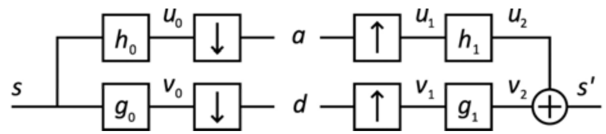
Orthogonal filter bank

- If it is possible to choose h_1 as a FIR filter, then h_0, g_1, g_0 also become FIR
- Is this possible?
- Yes: $h_1 = 2^{-1/2}$ (1, 1) is the simplest possible choice
  - $g_1 = 2^{-1/2}$ (-1, 1)
  - $h_0 = 2^{-1/2}$ (1, 1)
  - $g_0 = 2^{-1/2}$ (1, -1)

Called: *Haar* filter bank

Other choices also exist!

Splitting of the signal space



- The signal u_2 is constructed as a linear combination of 2-shifts of h_1
  - Span some subspace V_0 of the signal space V
- The signal v_2 is constructed as a linear combination of 2-shifts of g_1
  - Span some subspace W_0 of the signal space V
- The output signal $s' \in V$ is $u_2 + v_2$
  - $V = V_0 \oplus W_0$, $V_0 \perp W_0$

Filter banks

- The conjugate mirror filter bank is only one special solution to FB1 and FB2!
- There are other ways of obtaining $s' = s$
  – Still leads to perfect reconstruction
- Some approaches do not lead to an orthogonal filter bank
  – Can be useful anyway!

Scaling function

Consider a function $\phi(t)$ that satisfies

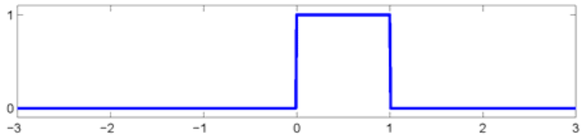
1. $\phi(t-k)$, $k \in \mathbb{Z}$, is an orthogonal basis of some function space V_0
2. $\phi(t)$ can be written as a linear combination of the functions $\phi(2t-k)$, $k \in \mathbb{Z}$

- Such a function ϕ is called a *scaling function*
- These relatively simple and innocent assumptions lead to a framework for defining a discrete wavelet transform in terms of orthogonal filter banks

Example

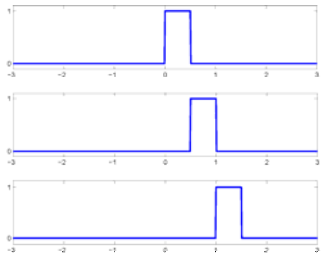
- The simplest example of a scaling function is

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Example

- In this case: V_0 is the space of **piece-wise constant functions** on integer intervals
- The functions $\phi(2t - k)$ look like



$k = 0$

$k = 1$

$k = 2$

$\phi(t) = \phi(2\,t) + \phi(2\,t - 1)$

Consequences

- The fact that $\phi(t- k)$ is an orthogonal basis for $k \in \mathbb{Z}$ means

$$\langle \phi(t) | \phi(t-k) \rangle = \int_{-\infty}^{\infty} \phi(t)\, \overline{\phi}(t-k)\, dt = \delta[k], \quad k \in \mathbb{Z}$$

- From this follows that also $2^{1/2}\phi(2t - k)$, $k \in \mathbb{Z}$, form an orthogonal basis
  - Spans a function space $V_1 \neq V_0$

Ex 22.1

Ex 22.2

Consequences (II)

- Property 2. of the scaling functions leads to

$$\phi(t) = \sum_{k=-\infty}^{\infty} h_k\, 2^{1/2}\, \phi(2\,t - k)$$

for some sequence of scalars h_k

Coordinates of ϕ relative to the ON-basis $2^{1/2}\phi(2x - k)$, $k \in \mathbb{Z}$

- Since we have an ON-basis:

$$h_k = \langle \phi(t) | 2^{1/2}\phi(2\,t-k) \rangle = 2^{1/2} \int_{-\infty}^{\infty} \phi(t)\, \overline{\phi}(2\,t-k)\, dt$$

Consequences (III)

- In fact, any $\phi(x - l)$, $l \in \mathbb{Z}$, can be written as a linear combination of the basis $2^{1/2}\phi(2\,t - k)$:

$$\phi(t-l) = \sum_{m=-\infty}^{\infty} h_{m-2l}\, 2^{1/2}\, \phi(2\,t - m)$$

Ex 22.3

Consequences (IV)

Notice that:

- The functions $\phi(t - k)$ are shifted by multiples of "1" relative to t
  - Spans a space denoted V_0
- The functions $2^{1/2}\phi(2t - k)$ are shifted by multiples of "1/2" relative to t
  - Spans a space denoted V_1 ⇐ Functions in V_1 can contain finer details than V_0
- Both bases are orthonormal within each space
- $V_0 \subset V_1$ ⇐ Since V_1 has a basis that spans every basis vector in V_0

Consequences (V)

- Returning to property 1. we have seen that

$$\langle \phi(t) | \phi(t-k) \rangle = \delta[k]$$

- This can also be formulated as

$$\sum_k |\Phi(v + 2\pi k)|^2 = 1$$

Ex 22.4

where Φ is the Fourier transform of ϕ

The sequence *h*

- The sequence of scalars $h[k]$, $k \in \mathbb{Z}$, depends *only* on the choice of scaling function ϕ
- Remember that ϕ itself has particular properties, described by 1. and 2.
  - Also $h[k]$ has particular properties!
- $h[k]$ can be interpreted as the coefficients of a discrete filter with Fourier transform

$$H(u) = \sum_{k=-\infty}^{\infty} h_k\, e^{-iuk}$$

This is a 2π-periodic function

The function *H*

- From 1. and 2. follows that *H* must be related to Φ as

$$\Phi(u) = 2^{-1/2}\, H\left(\frac{u}{2}\right) \Phi\left(\frac{u}{2}\right)$$

Ex 22.5

or

$$H(u) = 2^{1/2} \frac{\Phi(u)}{\Phi(2\,u)}$$

This must be a 2π-periodic function!

The function *H*

- The fact that ϕ has special properties and that *H* is related to ϕ leads to

$$|H(u)|^2 + |H(u+\pi)|^2 = 2$$

Ex 22.6

- We recognise this as condition (O) on the filter h_1 in a conjugate mirror filter bank!

The function ψ(x)

- Define a new function $\psi(t)$, with a Fourier transform Ψ given by

$$\Psi(u) = \frac{1}{\sqrt{2}}\, G\left(\frac{u}{2}\right)\, \Phi\left(\frac{u}{2}\right)$$

Where

$$G(u) = e^{-iu}\, \overline{H(u+\pi)}$$

or

$$g[n] = (-1)^{1-n}\, h[1-n]$$

This is the *alternating flip* condition in a conjugate mirror filter bank

The sequence *g*

- The sequence *g* is derived from *h*
- But, *h* depends only on ϕ:
  - *g* depends only on ϕ
- Also ψ depends only on the choice of ϕ
- Finally, we can derive

$$\langle \psi(t) | 2^{1/2} \phi(2t-n) \rangle = g[n]$$

Ex 22.7

The sequences *h* and *g*

We summarise:

$$h[k] = \langle \phi(t) | 2^{1/2} \phi(2\,t-k) \rangle$$
$$g[k] = \langle \psi(t) | 2^{1/2} \phi(2\,t-k) \rangle$$

Coordinates of ϕ relative basis in V_1

Coordinates of ψ relative basis in V_1

At least if we can prove that $\psi \in V_1$

or, more generally:

$$\langle \phi(t-l) | 2^{1/2} \phi(2\,t-k) \rangle = h[k-2\,l]$$
$$\langle \psi(t-l) | 2^{1/2} \phi(2\,t-k) \rangle = g[k-2\,l]$$

Properties of ψ, (I)

- We can show that:

Ex 22.8

$$\psi(t-l) = \sum_k (-1)^k\, \overline{h_k}\, 2^{1/2}\, \phi(2t-k+2l+1)$$

which means that all $\psi(t - l) \in V_1$, $l \in \mathbb{Z}$

$\Rightarrow$ g_n are the coordinates of ψ in the ON-basis of V_1

Example

- For the simple example of a scaling function

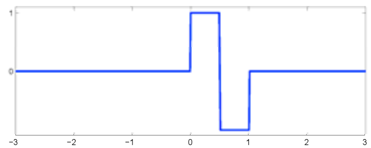
$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

we get

$g[k] = \{2^{-1/2}, -2^{-1/2}\}$

Called Haar wavelet

and the corresponding function $\psi(t)$:



Properties of ψ, (II)

- Using the previous relations, we can also prove that

Ex 22.9

$$\langle \psi(t-m) | \phi(t-l) \rangle = 0 \quad \text{for } l, m \in \mathbb{Z}$$

which means that all $\psi(t - l)$ are orthogonal to the space V_0 since $\phi(t-m)$ is a basis for this space

- $\psi(t-m) \in V_1$ and $\psi(t-m) \perp V_0$

Properties of ψ, (III)

- We can also show hat

$$\langle \psi(t-n) | \psi(t) \rangle = \delta[n]$$

Ex 22.10

(why?)

which means that the set $\psi(t-n)$ is orthonormal for $n \in \mathbb{Z}$

- In summary:
  - All $\psi(t - n)$ are $\perp$ to V_0
  - They form an orthonormal set

Properties of V_1

- Let f be an arbitrary function in V_1. It can then be written as

$$f(t) = \sum_n s[n]\, 2^{1/2}\, \phi(2t-n)$$

for some sequence $s[n]$ since $2^{1/2}\phi(2t-n)$ is an ON-basis for V_1

Properties of V_1

- Using the previous results, we can show that it is also possible to write this $f \in V_1$ as

$$f(t) = \sum_n a[n]\, \phi(t-n) + \sum_n d[n]\, \psi(t-n)$$

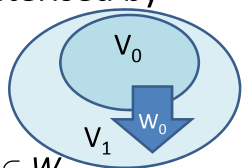
Ex 22.11

for some sequences $a[k]$ and $d[k]$

- This means that $\phi(t-n)$ and $\psi(t-n)$ together span V_1

The difference space W_0

- V_0 is a subspace of V_1
- Any $f \in V_1$ can be written as a unique linear combination of a component in V_0 and a component in V_1, and this component is orthogonal to V_0
- There is a *difference space* W_0 characterised by
  - o $W_0 \subset V_1$
  - o $W_0 \perp V_0$
  - o $V_1 = V_0 \oplus W_0$
  - o For any $f \in V_1 : f = f_0 + w_0, \quad f_0 \in V_0, \quad w_0 \in W_0$



The difference space W_0

- V_1 has an ON-basis $2^{1/2}\,\phi(2t-k)$, $k \in \mathbb{Z}$
- V_0 has an ON-basis $\phi(t-k)$, $k \in \mathbb{Z}$
- W_0 has an ON-basis $\psi(t-k)$, $k \in \mathbb{Z}$

- $W_0 \perp V_0$
- $V_1 = V_0 \oplus W_0$

- Coordinates of $f \in V_1$ are given by $s[k]$
- Coordinates of $f \in V_0 \oplus W_0$ given by $a[k]$ and $d[k]$

Sequences a, b and c

- Since $f \in V_1$ is completely represented either by sequence $s[k]$, or by sequences $a[k]$ and $d[k]$:
  - There must be a (linear) mapping $s \rightarrow (a, d)$
  - There must be a (linear) mapping $(a, d) \rightarrow s$

A linear mapping $a \rightarrow b$ & c

From previous results follows:

Ex 22.12

$$a[k] = (s[\,\cdot\,] * \overline{h[-\,\cdot]})[2\,k]$$

$$d[k] = (s[\,\cdot\,] * \overline{g[-\,\cdot]})[2\,k]$$

Convole and skip every second sample (the odd ones)

A linear mapping b & $c \rightarrow a$

From previous results also follows:

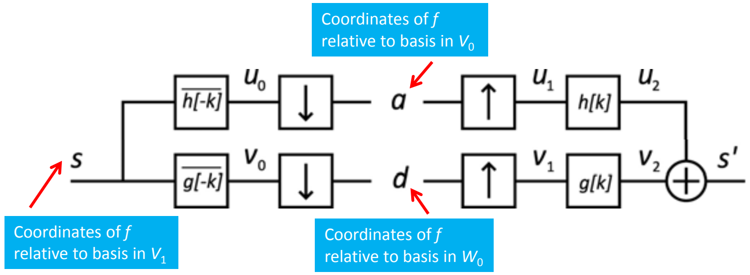
Ex 22.13

$$s[k] = \sum_{n=-\infty}^{\infty} (a[n]\,h[k-2\,n] + d[n]\,g[k-2\,n])$$

Insert zeros between every sample and convolve

A 2-channel filter bank

- We can summarise these results as



An orthogonal 2-channel filter bank with

- $h_0 = h$
- $g_0 = g$ (as alternating flip of h_0)

What you should know

- The anatomy of the 2-channel filter bank
- The 2 constraints on the 4 filters to produce $s'=s$
- The conjugate mirror filter (CMF) bank
- Orthogonal 2-channel filter bank
- How a CMF decomposes the signal space into V_0 and W_0, and how they are related
- Definition of a scaling function ϕ
- Leads to a unique function ψ
- How ϕ and ψ decompose the signal space into V_0 and W_0, and how they are related