Multi-dimensional Signal Analysis

Lecture 2H

Multi-resolution Analysis (II)

Discrete Wavelet Transform

Recap (CWT)

Continuous wavelet transform

- A mother wavelet $\psi(t)$
- Define

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right)$$

- Define the continuous wavelet transform (CWT) of f as

$$W_f(a,b) = \langle\, f \mid \psi_{a,b} \,\rangle$$

Recap (CWT II)

Continuing...

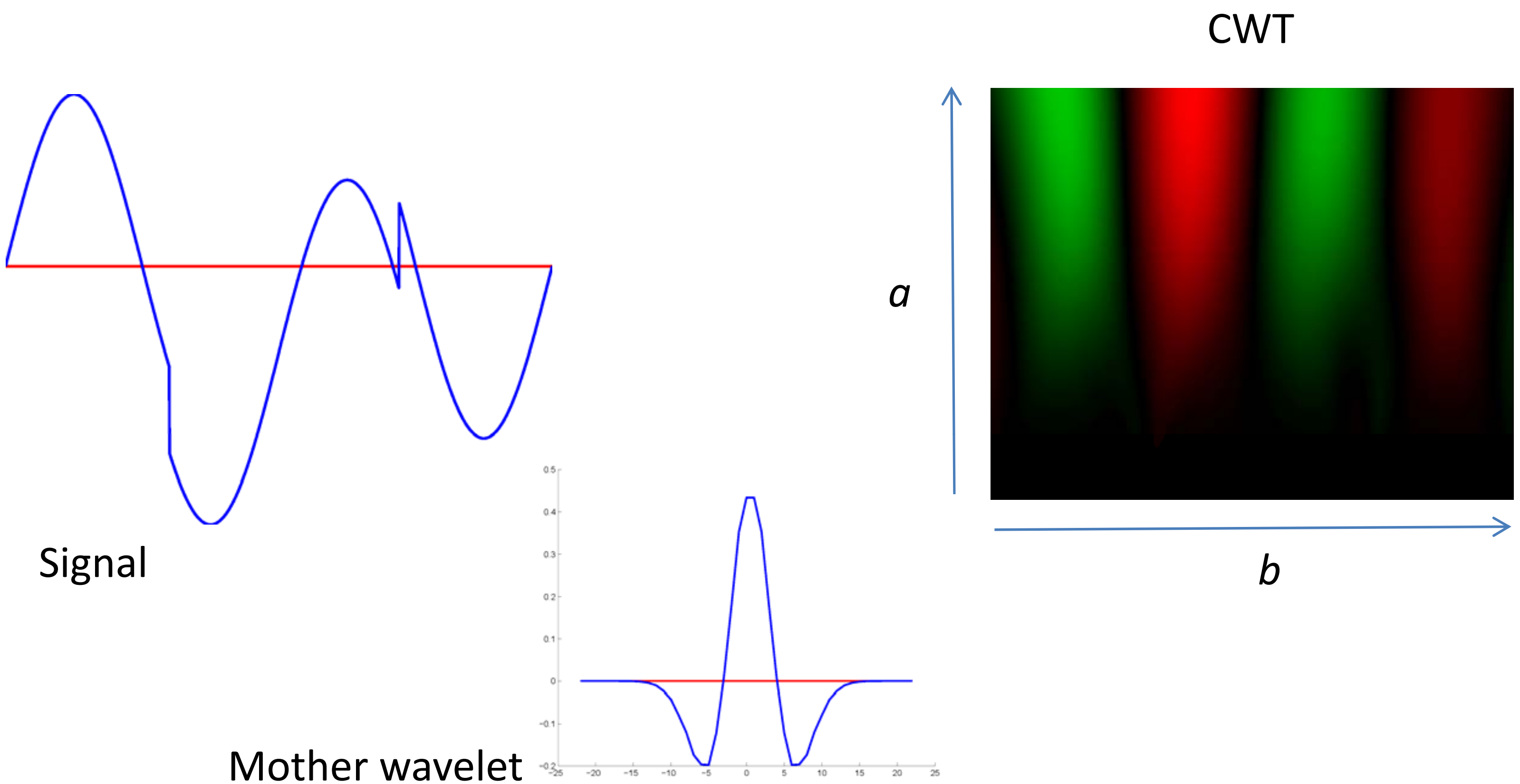
- f is a one-variable function
- W_f is a two-variable function
  - Scale and translation
- W_f has an inverse transform (ICWT) iff

$$0 < \int_{-\infty}^{\infty} \frac{|\Psi(v)|^2}{|v|} \, dv < \infty$$

Ψ is the FT of ψ

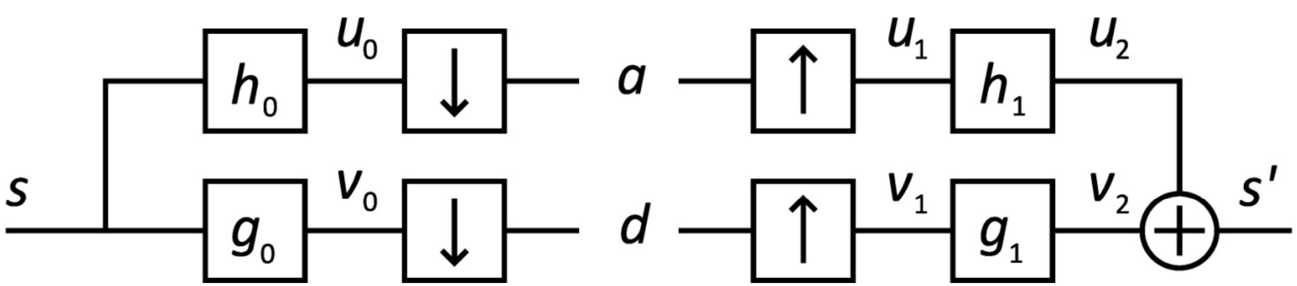
Recap (CWT III)

Example



Recap (2chFB)

- The 2-channel filter bank



- Perfect reconstruction: *s'* = *s*
  - If and only if

$$H_1(u)H_0(u) + G_1(u)G_0(u) = 2 \quad \text{(FB1)}$$

$$H_1(u)H_0(u+\pi) + G_1(u)G_0(u+\pi) = 0 \quad \text{(FB2)}$$

Recap (CMF)

The conjugate mirror filter bank (CMF)

- We choose

$$|H_1(u)|^2 + |H_1(u+\pi)|^2 = 2 \qquad (O)$$

This is the *alternating flip*

$$H_0(u) = \overline{H_1(u)}$$

$$G_1(u) = e^{-iu}\,\overline{H_1(u+\pi)} \quad \Rightarrow \quad g_1[k] = (-1)^{1-k}\,\overline{h_1[1-k]}$$

$$G_0(u) = e^{iu}\,H_1(u+\pi) = \overline{G_1}(u)$$

- Leads to FB1 and FB2 being satisfied!

Recap (SF I)

Scaling function $\phi(t)$: must satisfy

1. $\phi(t-k)$, $k \in \mathbb{Z}$, is an orthogonal set of functions of some function space V_0

2. $\phi(t)$ can be written as a linear combination of the functions $\phi(2t-k)$, $k \in \mathbb{Z}$

Recap (SF II)

1. and 2. *alone* lead to

- Also $2^{1/2}\phi(2\,t - k)$ is an ON basis for $V_1 \supset V_0$
- Define sequence h as $h[k] = \langle \phi(t) | 2^{1/2}\phi(2\,t - k) \rangle$
- Define $H(u)$ = DFT{h}
- Then $|H(u)|^2 + |H(u+\pi)|^2 = 2$
  - Same as condition (O) in the CMF
  - H here corresponds to H_1 in the CMF

Recap (SF III)

Cont...

- Define sequence *g* as $g[k] = (-1)^{1-k}\,\overline{h[1-k]}$
  - The alternating flip in CMF
  - *G* here corresponds to G_1 in the CMF
- Define *G*(*u*) = DFT{*g*}
- Define a function $\psi(t)$ with FT Ψ Given by

$$\Psi(u) = \frac{1}{\sqrt{2}}\, G\left(\frac{u}{2}\right)\, \Phi\left(\frac{u}{2}\right)$$

Recap (SF IV)

Cont...

- $\psi(t-k)$, $k \in \mathbb{Z}$, is also an ON-set
- Spans a subspace $W_0 \subset V_1$ and $W_0 \perp V_0$
- In fact, $V_1 = V_0 \oplus W_0$
  - $2^{1/2}\,\psi(2t-k)$ is an ON-basis of V_1
  - $\phi(t-k)$ is an ON-basis of V_0
  - $\psi(t-k)$ is an ON-basis of W_0

Recap (SF V)

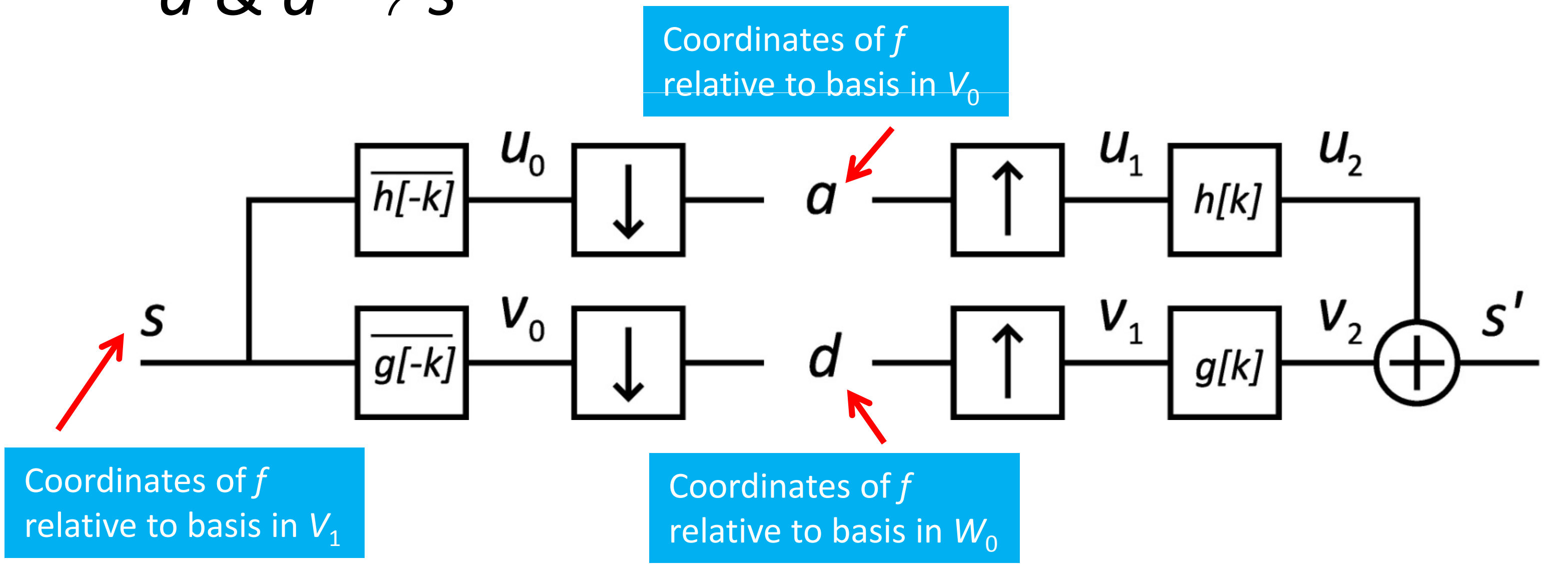
Cont...

- An $f \in V_1$ can be written as $f = f_0 + w_0$
  - $f_0 \in V_0$
  - $w_0 \in W_0$
- $f \in V_1$ has coordinates $s[k]$ relative the ON-basis in V_1
- $f \in V_1$ has coordinates $a[k]$ and $d[k]$ relative the ON-bases in V_0 and W_0, respectively
- Sequences $s[k]$ or $a[k]$ and $d[k]$ are alternative representations of $f \in V_1$

Recap (SF VI)

These representations can be mapped:

- $s \to a \,\&\, d$
- $a \,\&\, d \to s$



Recap (SF VII)

- The fact that V_1 is spanned by the ON-basis $2^{1/2}\,\phi(2\,t-k)$ when $V_0 \subset V_1$ is spanned by $\phi(t-k)$ means:

> The space V_1 contains functions that can have smaller "details" than V_0 does

Recap (SF VII)

The fact that $V_1 = V_0 \oplus W_0$ means:

W_0 contains the **details** that are missing in V_0 to make up V_1

Also:

V_0 contains an **approximation** of V_1 without the details that are in W_0

The story continues...

- There is an obvious relation between a CMF-bank and the results derived from 1. and 2. of the scaling function ϕ:
  - ϕ and ψ define sequences *h* and *g* which we can identify with the reconstructing filters h_1 and g_1 in the 2-channel CMF-bank
  - Any ϕ that satisfies 1. and 2. generates a 2-channel CMF-bank
    - And vice versa (!)

CMF

- In the CMF-bank, the signal space V is the space where the *discrete* input signal a lives
- The filter bank decomposes s into two components $u_2 \in V_0$ and v_2 in W_0
- With $V = V_0 \oplus W_0$
  - $a[n]$ are the coordinates of $u_2[k]$ relative $h_1[k - 2n]$
  - $d[n]$ are the coordinates of $v_2[k]$ relative $g_1[k - 2n]$

Scaling function

In the case of the scaling function:

- The signal space is V_1 which hosts functions of a *continuous* variable: $f(t)$
- The discrete signal $a[k]$ are the *coordinates* of $f_1 \in V_1$ relative to the ON-basis $2^{1/2}\,\phi(2\,t - k)$
- The discrete signals $a[k]$ and $d[k]$ are the coordinates of $f \in V_1$ relative to the ON-bases $\phi(t-k)$ and $\psi(t-k)$
  - Together they span $V_1 = V_0 \oplus W_0$

Looking ahead

In what follows, we will take the second view:

- Any discrete time function holds the coordinates of some continuous time function relative to some ON-basis of some space
  - $s[k]$ are the coordinates of $f_1 \in V_1$ relative $2^{1/2}\,\phi(2\,t-k)$
  - $a[k]$ are the coordinates of $f_0 \in V_0$ relative $\phi(t-k)$
  - $d[k]$ are the coordinates of $w_0 \in W_0$ relative $\psi(t-k)$
- $f_1 = f_0 + w_0 \in V_0 \oplus W_0$

Using 1. and 2. again

- Consider the set of functions given by

$$2\,\phi(4\,t-k), \qquad k \in \mathbb{Z}$$

- It relates to the set $2^{1/2}\,\phi(2\,t-k)$ in the same way as that set relates to $\phi(t-k)$ (why?)
  - All the results derived from 1. and 2. between V_1 and V_0 applies!!

The space V_2

Consequently:

- The set of functions given as

  $$2\,\phi(4\,t - k)$$

  forms an ON-basis of a space $V_2 \supset V_1 \supset V_0$
- The space V_2 contains functions of even finer details than V_1 does
  - And finer still than V_0 does

The space W_1

Furthermore:

- We can define a difference space W_1 such that

  $V_2 = V_1 \oplus W_1$

  and $W_1 \perp V_1$
- $2\,\phi(4\,t - k)$ is an ON-basis for V_2
- $2^{1/2}\,\phi(2\,t - k)$ is an ON-basis for V_1
- $2^{1/2}\,\psi(2\,t - k)$ is an ON-basis for W_1

  (why?)

Decomposition of V_2

Furthermore, $V_2 = V_1 \oplus W_1$ means that

- any $f_2 \in V_2$ can be decomposed into

$$f_2 = f_1 + w_1$$

where $f_1 \in V_1$ and $w_1 \in W_1$

Coordinates of $f_2 \in V_2$

The coordinates of f_2 are:

$s[k]$ relative the ON-basis $2\phi(4t - k)$ in V_2

- Alternatively:

$a[k]$ and $d[k]$ relative the ON-bases
$2^{1/2}\phi(2t - k)$ in V_1 and $2^{1/2}\psi(2t - k)$ in W_1

Coordinates of $f \in V_2$

- a and d are obtained from a as

$$a[k] = (s[\,\cdot\,] * \overline{h[-\,\cdot]})[2\,k]$$

$$d[k] = (s[\,\cdot\,] * \overline{g[-\,\cdot]})[2\,k]$$

Convolve and skip every second sample (the odd ones)

- s is obtained from a and d as

Insert zeros between every sample and convolve

$$s[k] = \sum_{n=-\infty}^{\infty} (a[n]\,h[k-2\,n] + d[n]\,g[k-2\,n])$$

Putting things together (I)

We have already seen that:

- $f_1 \in V_1$ can be decomposed as

$$f_1 = f_0 + w_0$$

where $f_0 \in V_0$ and $w_0 \in W_0$

Putting things together (II)

This means:

- any $f_2 \in V_2$ can be decomposed into

$$f_2 = f_0 + w_0 + w_1$$

where $f_0 \in V_0$ and $w_0 \in W_0$ and $w_1 \in W_1$

Putting things together (III)

This also means:

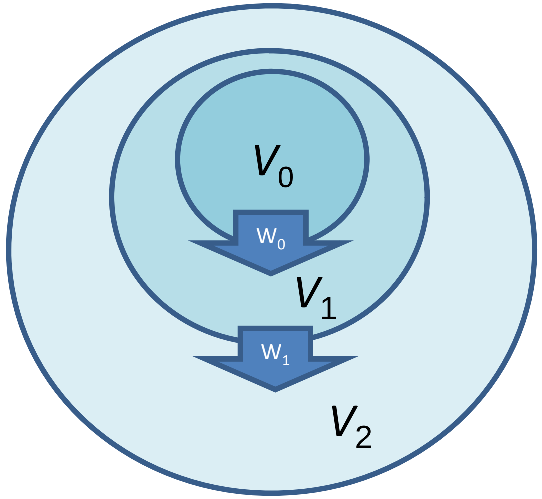
$$V_2 = V_0 \oplus W_0 \oplus W_1$$

$$V_0 \perp W_0 \perp W_1$$ (they are mutually orthogonal)

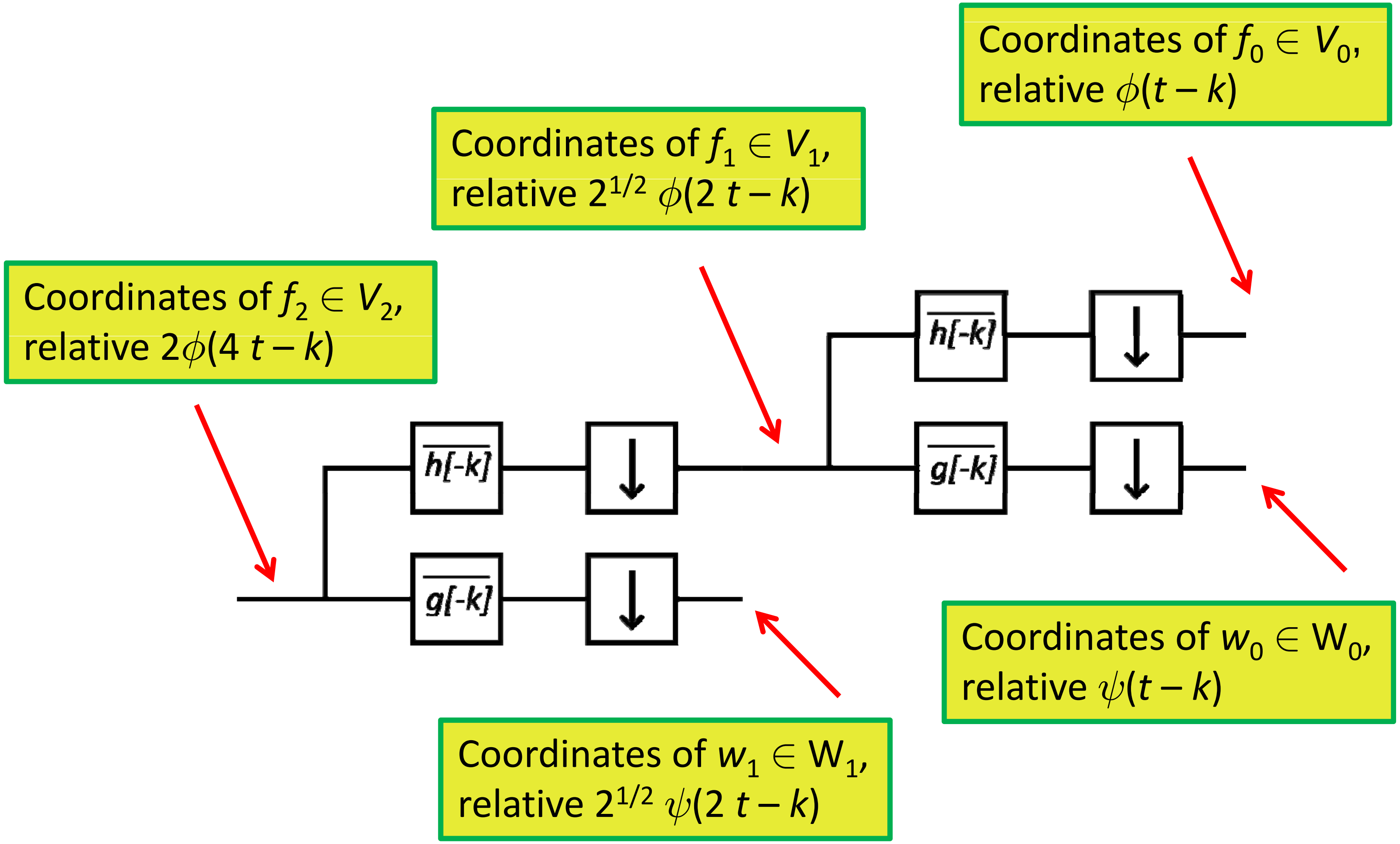
(why?)

Putting things together (IV)

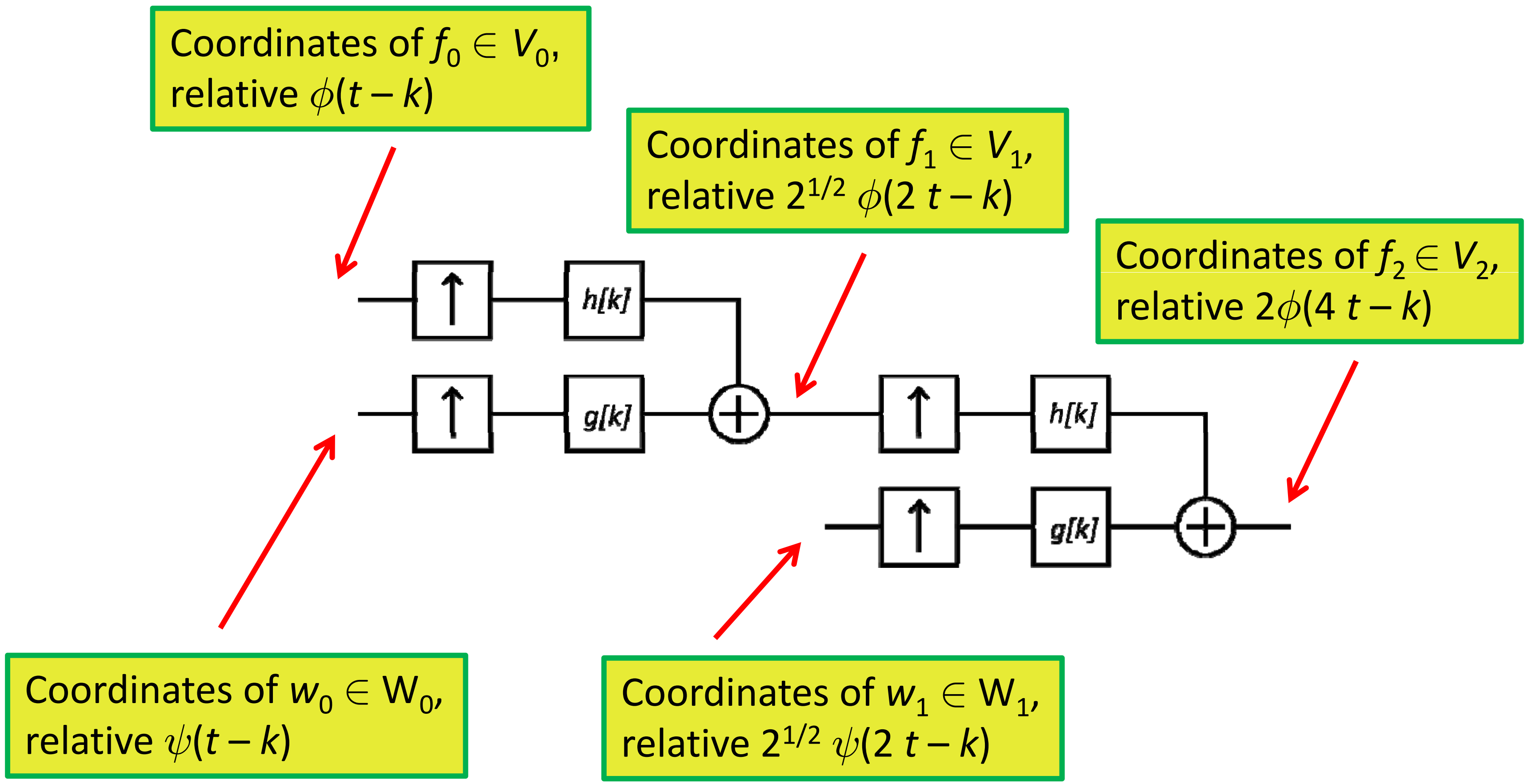
- We can see $f_1 \in V_1$ as an approximation of $f_2 \in V_2$, and $f_0 \in V_0$ as a coarser approximation of f_2
- The details that are missing in f_0 to get to f_2 are found in W_0 and W_1



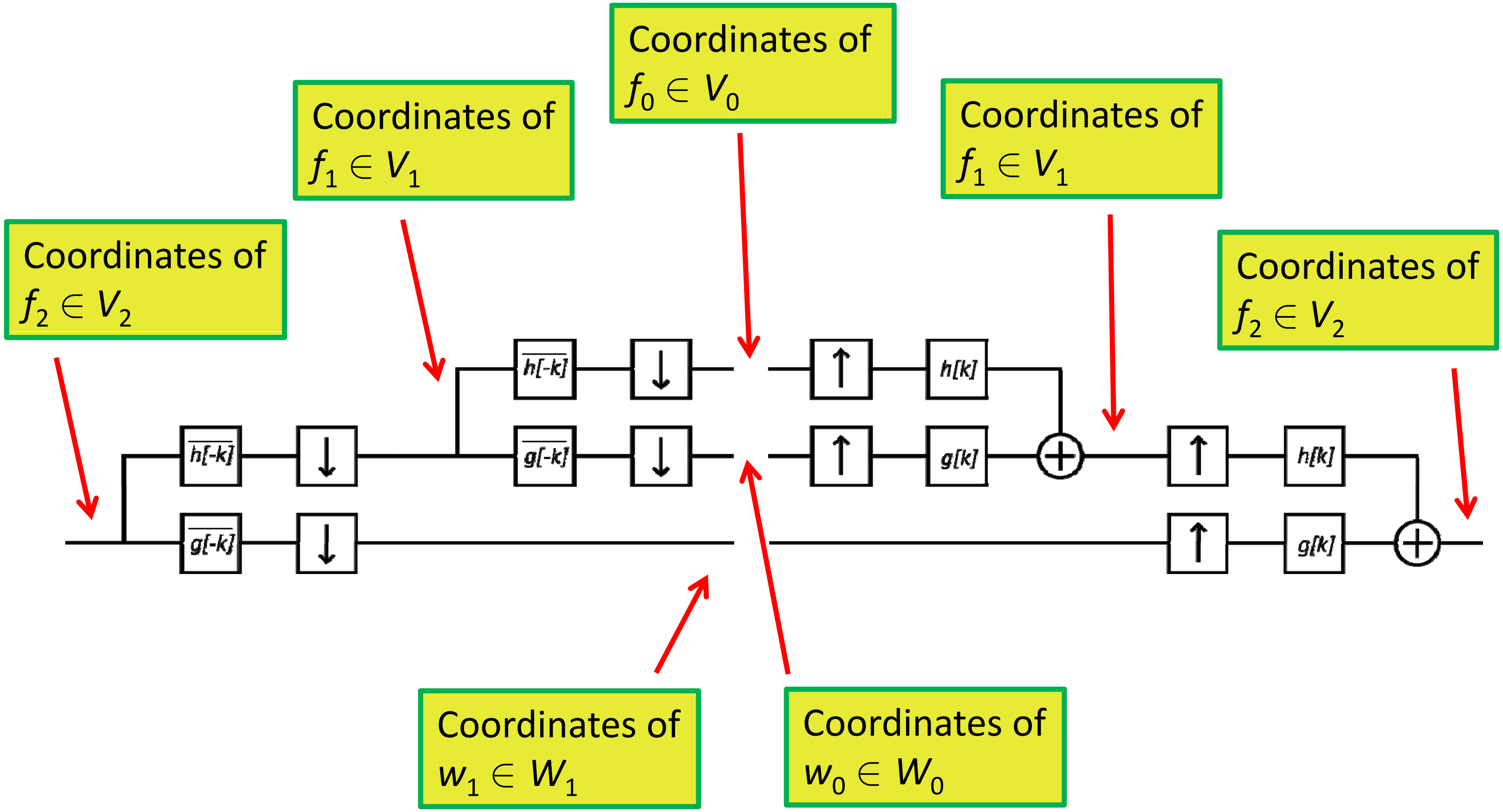
Analysis in two steps



Reconstruction on two steps



The full view



Generalisation

- Consider the set of functions given by

$$2^{p/2}\,\phi(2^p t - k), \qquad k \in \mathbb{Z},\ \ p \in \mathbb{Z},\ \ p \geq 1$$

- It relates to the set $2^{(p-1)/2}\,\phi(2^{(p-1)}t - k)$ in the same way as that set relates to $2^{(p-2)/2}\,\phi(2^{(p-2)}t - k)$
- And so on ...
- ... all the way to the set $\phi(t - k)$

The space V_p

- $2^{p/2}\phi(2^p t - k)$ is an ON-basis for a space V_p
- V_p can be decomposed as

$$V_p = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_{p\text{-}1}$$

The space V_p

- All spaces V_0, W_0, W_1, ..., W_{p-1} are mutually orthogonal

- V_0 is spanned by $\phi(t-k)$, $k \in \mathbb{Z}$
- W_j is spanned by $2^{j/2}\,\psi(2^j t - k)$, $k \in \mathbb{Z}$,
  - $j \in \mathbb{Z}, 0 \leq j \leq p-1$

The space V_p

- Any $f_p \in V_p$ can be decomposed as

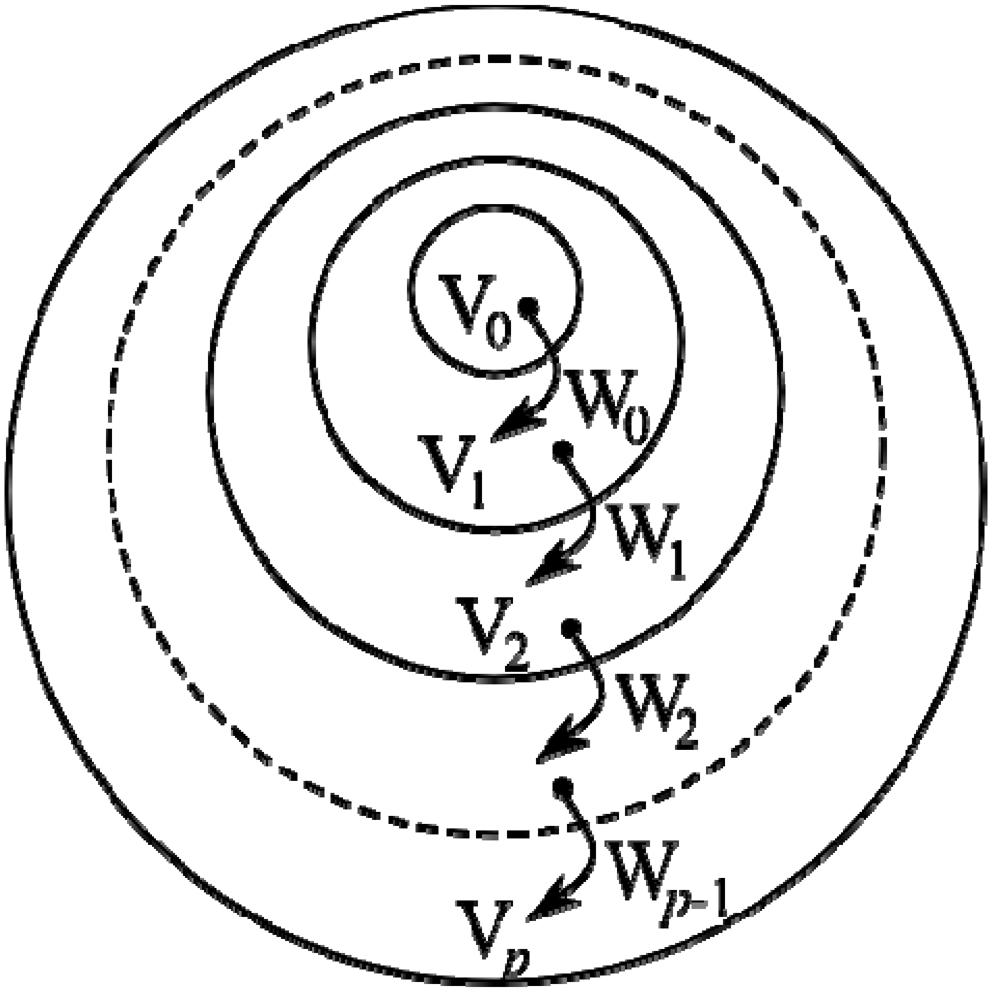
$$f_p = f_0 + w_0 + w_1 + \ldots + w_{p\text{-}1}$$

A very, very coarse approximation of f_p

Finer and finer details added to f_0 in order to construct f_p

- $f_0 \in V_0$ and $w_j \in W_j$

Decomposition of spaces



Decomposition of spaces

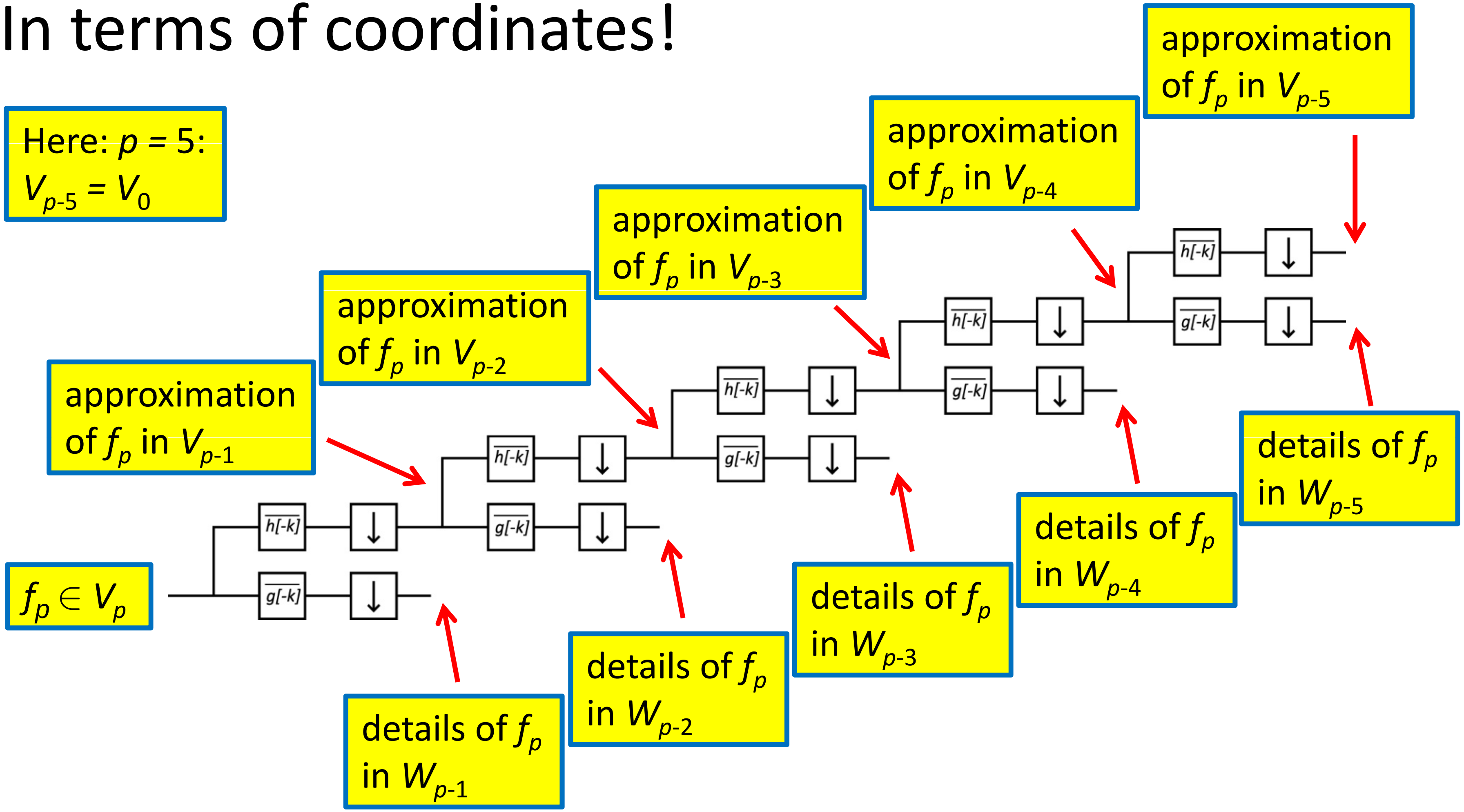
- V_p contains functions with details of some minimal size or scale
- Then $V_{p\text{-}1}$ contains function that have twice the size or scale at minimum compared to V_p
- And $V_{p\text{-}2}$ contains functions that have 4 times the size or scale at minimum compared to V_p
- And so on ...

The space V_0

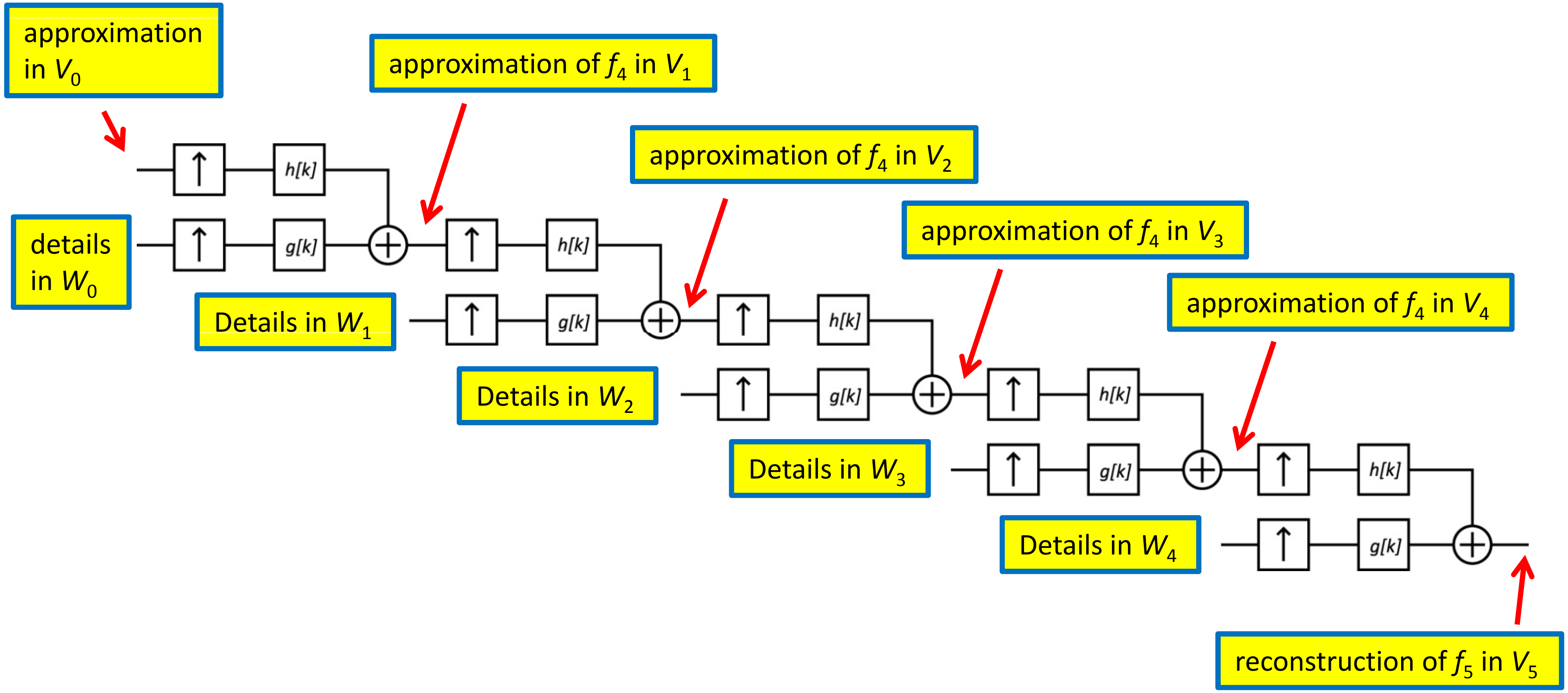
- We can choose *p* (a positive integer) as large as we want
  - Or is practically useful or necessary
- This specifies a certain “scale” of V_0 relative V_p
  - A factor 2^p larger
- V_0 contains the “coarsest” approximation of f_p that is reasonable for a particular application

Multi-level decomposition

In terms of coordinates!



Multi-level reconstruction



Coordinates in $W_{j\text{-}1}$

- With $f_j = f_{j\text{-}1} + w_{j\text{-}1} \in V_j$:

  coordinates of $w_{j\text{-}1}$ relative to the ON-basis in $W_{j\text{-}1}$ are given as

$$d_{j\text{-}1}[k] = \langle\, f_j \mid 2^{(j\text{-}1)/2}\psi(2^{(j\text{-}1)}\, t - k)\,\rangle$$

Coordinates in $W_{j\text{-}2}$

- With $f_{j\text{-}1} = f_{j\text{-}2} + w_{j\text{-}2} \in V_{j\text{-}1}$:

  coordinates of $w_{j\text{-}2}$ relative to the ON-basis in $W_{j\text{-}2}$ are given as

$$d_{j\text{-}2}[k] = \langle\, f_{j\text{-}1} \mid 2^{(j\text{-}2)/2}\psi(2^{(j\text{-}2)}\, t - k) \,\rangle$$

Coordinates in $W_{j\text{-}2}$

- But $f_{j\text{-}1} = f_j - w_{j\text{-}1}$ where $w_{j\text{-}1} \in W_{j\text{-}1}$
- Furthermore:
  - $W_{j\text{-}1} \perp W_{j\text{-}2}$
  - $w_{j\text{-}1} \perp W_{j\text{-}2}$
  - $2^{(j\text{-}2)/2}\psi(2^{(j\text{-}2)}\, t - k)$ is an ON-basis of $W_{j\text{-}2}$
  - $w_{j\text{-}1} \perp$ to all $2^{(j\text{-}2)/2}\psi(2^{(j\text{-}2)}\, t - k)$

$$\Rightarrow d_{j\text{-}2}[k] = \langle\, f_j \mid 2^{(j\text{-}2)/2}\psi(2^{(j\text{-}2)}\, t - k)\, \rangle$$

Coordinates in W_j

- In general, the coordinates of $w_j \in W_j$ is given by

$$\langle f_{j+1} \mid 2^{j/2}\psi(2^j t - k) \rangle = \langle f_p \mid 2^{j/2}\psi(2^j t - k) \rangle$$

(why?)

Wavelets

- ψ(x) is a wavelet
  - Satisfies the "wavelet condition"
  - Not thoroughly proven here but follows from the special properties of ϕ
- The details in terms of coordinates relative to the ON-basis in W_j are computed as

$$d_j[k] = \left\langle\, f_p(t) \mid 2^{j/2}\psi(2^j t - k) \,\right\rangle$$

Continuous Wavelet Transform (CWT)

- In the continuous wavelet transform we compute

$$W_{f_p}(a,b) = \langle\, f_p(t) \mid \psi_{a,b}(t)\,\rangle$$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\,\psi\left(\frac{t-b}{a}\right)$$

Scaling and translation of the mother wavelet

Mother wavelet

Discrete Wavelet Transform (DWT)

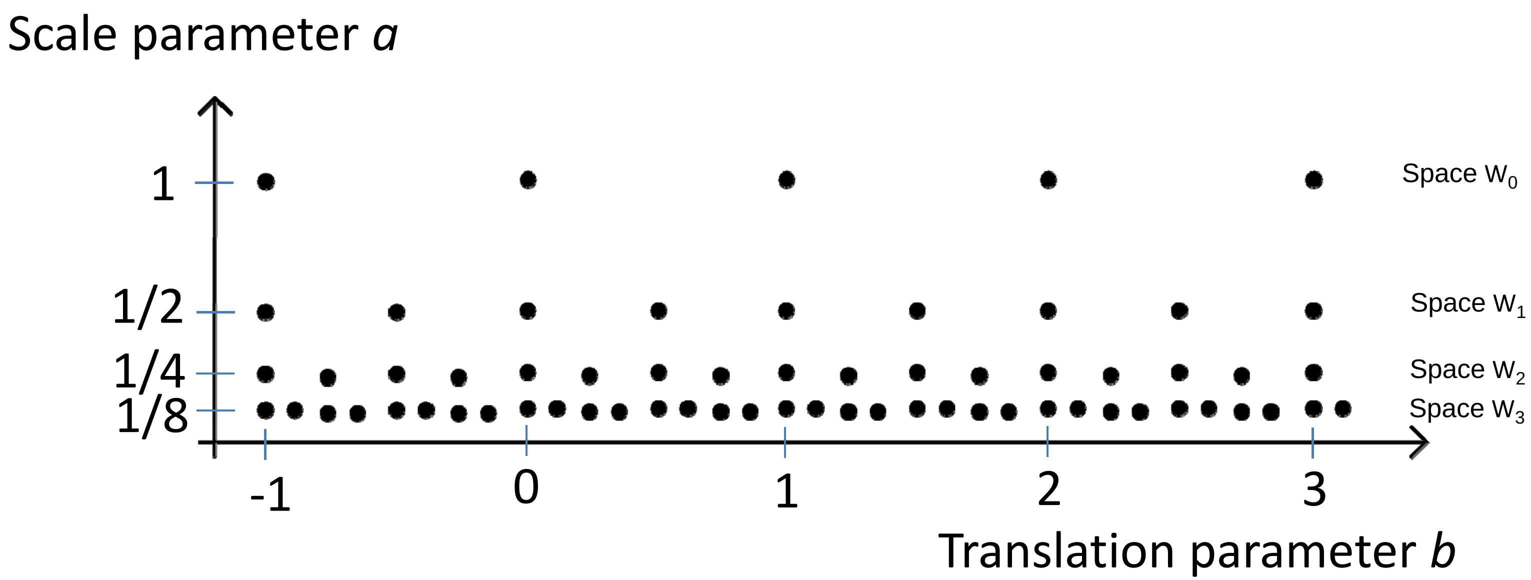
- Consequently, we can see the details of f_p in the different spaces W_j as a **sampling** of the continuous wavelet transform given by ψ:

$$d_j[k] = \left\langle f_p(t) \mid 2^{j/2}\, \psi(2^j t - k) \right\rangle =$$

$$= \left\langle f_p(t) \mid 2^{j/2}\, \psi\left(\frac{t - 2^{-j}k}{2^{-j}}\right) \right\rangle =$$

$$W_{f_p}(2^{-j}, 2^{-j}k), \quad j, k \in \mathbb{Z}, j \geq 0$$

DWT = Sampling of CWT



Sampling pattern of the CWT to get the DWT

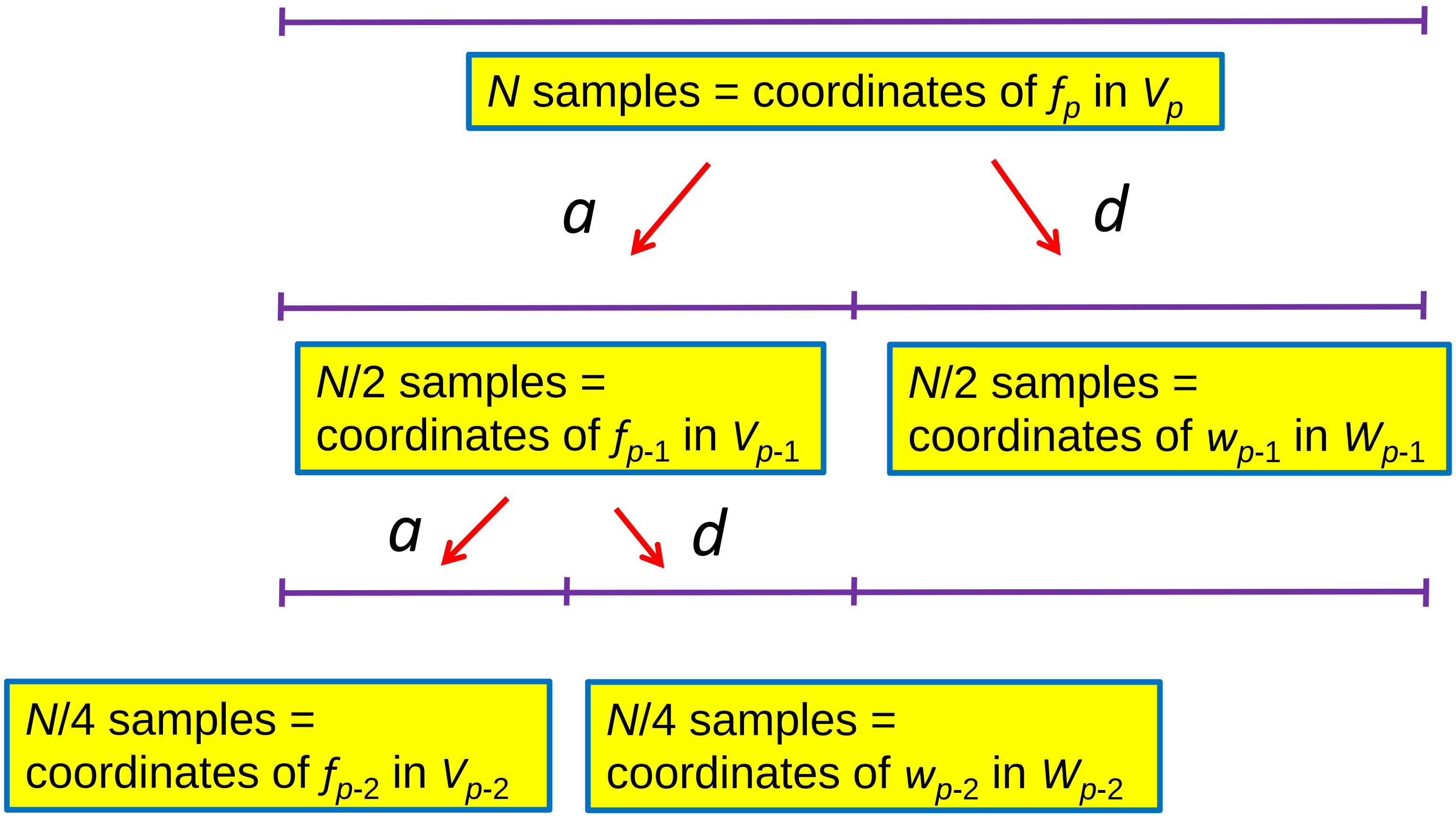
DWT

- In practice the DWT computes these samples of the CWT *and* the approximation at level V_0
- Together they can completely reconstruct the function f_p in terms of its coordinates relative to the ON-basis in V_p
- The reconstruction in V_p is made by a set of ON-basis functions
  - DWT is a transform based on an ON-basis
  - CWT is based on a frame

Multi-resolution analysis

- This approach of decomposing a function $f_p \in V_p$ into coarser and coarser approximations in together with the corresponding details is referred to as *multi-resolution analysis* (MRA)
- Formulated by **Stéphane Mallat**, 1989
  - **Ingrid Daubechies** described first ψ for MRA 1987
  - Filter banks have been around since the 1970'

Practical implementation of DWT



An observation

- Since $V_{j\text{-}1} \perp W_{j\text{-}1}$ and all bases are ON:

$$|f_j|^2 = |f_{j\text{-}1}|^2 + |\mathrm{w}_{\mathrm{j\text{-}1}}|^2$$

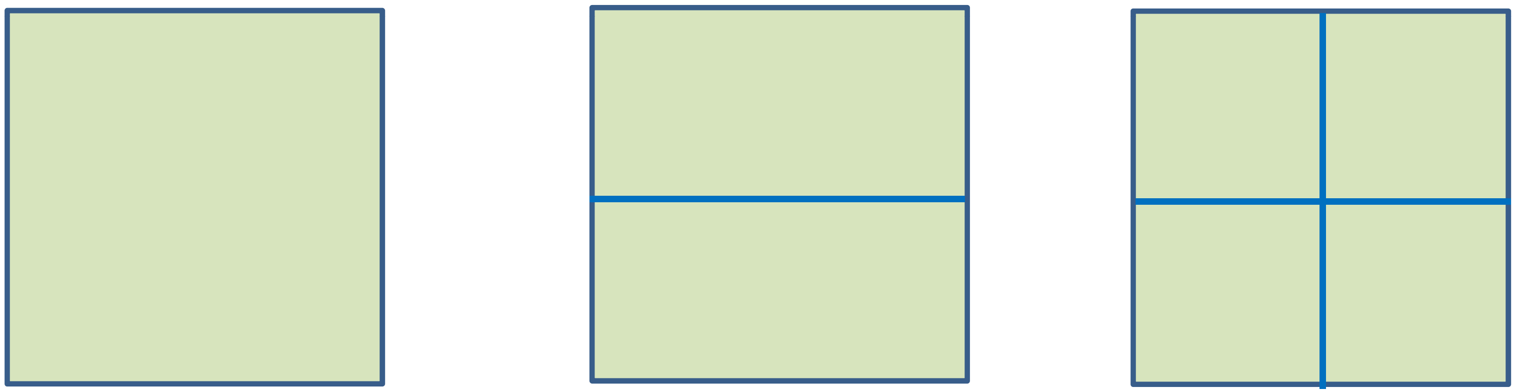
- Means: $|f_{j\text{-}1}| \leq |f_j|$

- Means:
  the more levels p we have, the smaller is f_0 for the same f_p

2D DWT

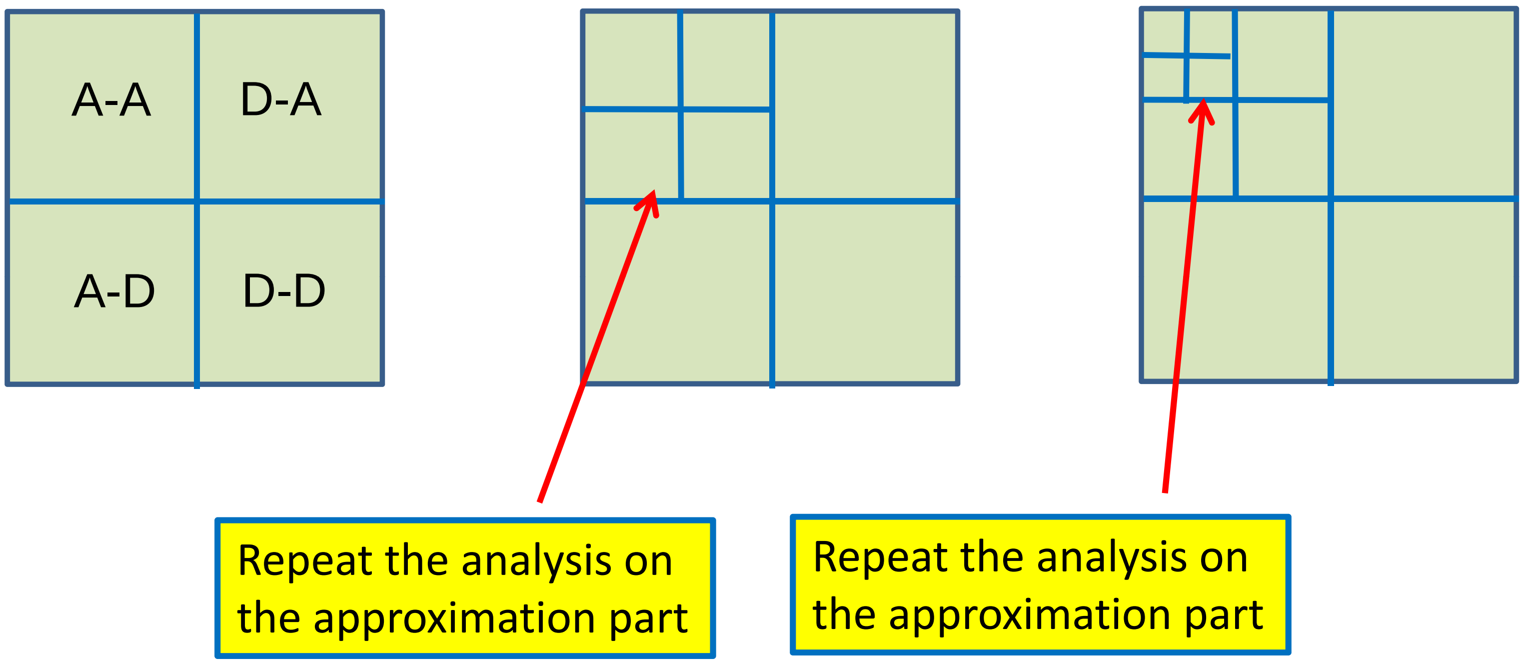
For 2D images

- Apply the 1D DWT first on each column, and then on each row
  - Or vice versa, gives the same result (why?)

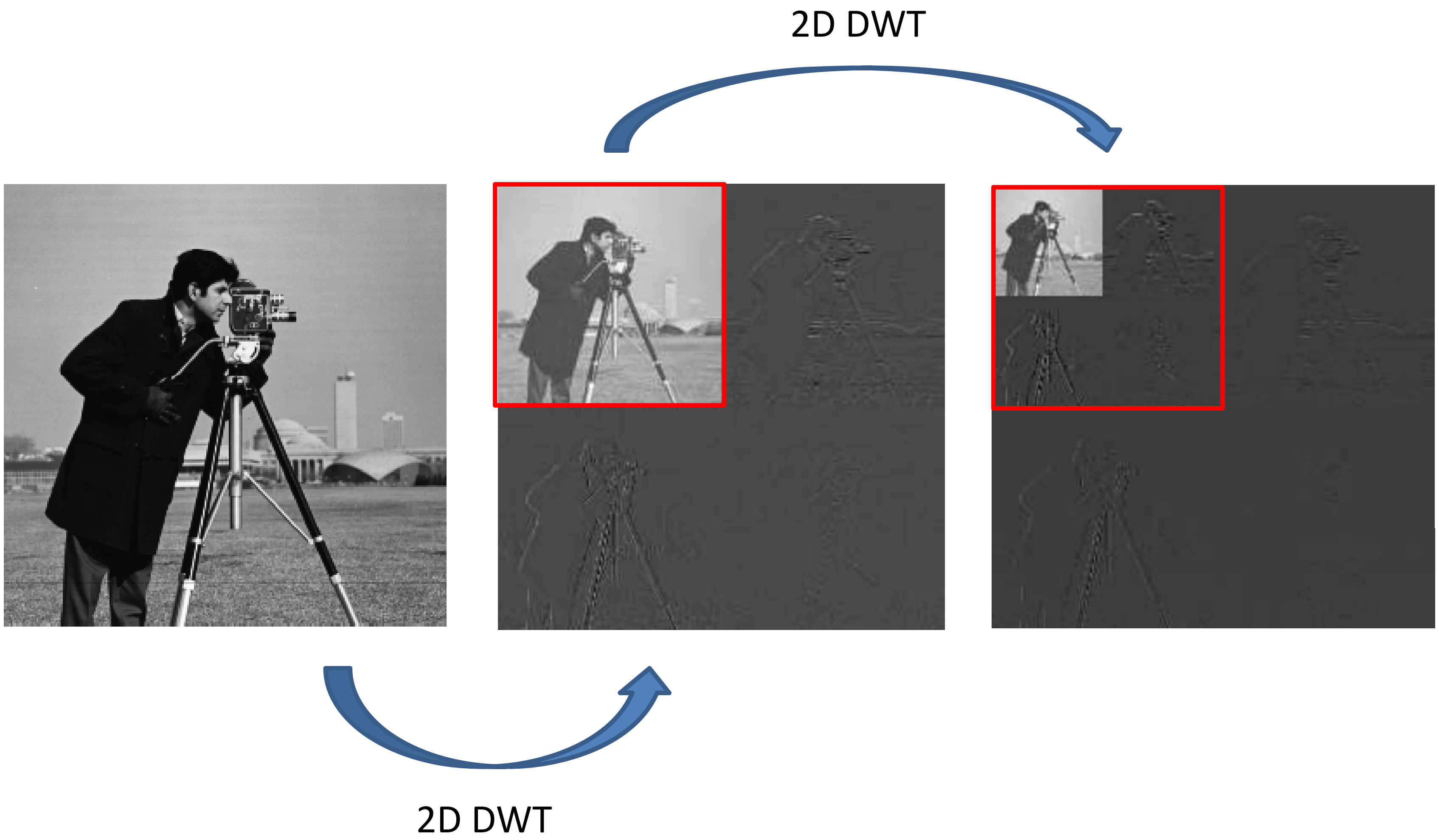


2D DWT

Horizontal - Vertical



2D DWT, Example



What you should know include

- The 2-channel filter bank
- Conditions FB1 and FB2 for perfect reconstruction
- The CMF-bank: condition on H_1 and the other three filters defined from H_1
- Alternating flip
- Orthogonality of the CMF-bank
- Definition of ϕ, as a function of continuous time t
- Definition of discrete sequence $h[k]$
- Construction of sequence g from h (alternating flip)
- Construction of ψ from ϕ (and g)
- Relation between MRA and CMF
- Construction of the sequence of spaces $V_p \supset V_{p-1} \supset \dots \supset V_1 \supset V_0$
- Construction of the difference spaces $W_{p-1}, \dots, W_0$, they are mutually orthogonal
- Decomposition of $f_p \in V_p$ as $f_p = f_0 + w_0 + \dots + w_{p-1}$, $f_0 \in V_0$ and $w_j \in W_j$
- DWT: coordinates in W_j = samples of CWT
- The corresponding sampling pattern
- Implementation of 2D DWT