Exercises for Introduction to Representations and Estimation in Geometry

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Numbered chapters in this collection of exercises do not necessary correspond to the same chapter in the IREG compendium. References to the IREG compendium from this document are based on the same version of the two documents.

Exercises marked with (A) are on an advanced level and can be deferred until the others are solved. Exercises marked with (M) lead to numerical computations that are not straight-forward to do by hand, and are recommended to be carried out using Matlab or similar numerical calculation tools.

1 Basic geometric objects

Solve the exercises in this section by means of Cartesian coordinates, using standard techniques from geometry and linear algebra.

A 2D line is represented by the equation

$$u\, l_1 + v\, l_2 = \Delta,$$

where l_1, l_2, Δ are parameters that determine what line it is, and (u, v) are the Cartesian coordinates of a point that lies on the line. Assume that the parameters are normalized such that $l_1^2 + l_2^2 = 1$ and $\Delta \geq 0$.

1.1 Show that the point on the line that is closest to the origin has Cartesian coordinates $\bar{\mathbf{y}}_0 = \Delta\,(l_1, l_2)$, i.e., Δ is the distance to the line from the origin. *Hint:* Alternative 1: verify first that $\bar{\mathbf{y}}_0$ really lies on the line, and that moving from $\bar{\mathbf{y}}_0$ in a direction along the line makes the distance larger. Alternative 2: Start at the origin and move along the perpendicular direction $\bar{\mathbf{l}}$ until you reach the line, i.e., find s such that $\bar{\mathbf{y}}(s) = (0, 0) + s\,\bar{\mathbf{l}}$ is on the line, where $\bar{\mathbf{l}}$ is perpendicular to the line . This happens for some s_0 that can be determined from your equations, and gives the point $\bar{\mathbf{y}}(s_0)$ that is on the line and is closest to the origin, i.e., $\bar{\mathbf{y}}(s_0) = \bar{\mathbf{y}}_0$.

1.2 Derive an expression for the Cartesian coordinates of the point on the line that has shortest distance to an arbitrary point with Cartesian coordinates (u_0, y_0). *Hint:* You can uses similar techniques as in the previous exercises, but replace the origin with (u_0, y_0). A third option in this case is to first change coordinate system (translate) so that (u_0, y_0) becomes the new origin, find the closest point on the line and, finally, return to the origin coordinate system.

1.3 (A) Do the results in exercises 1.1 and 1.2 depend on the type of coordinate system you are using (right-handed or left-handed), as long as it is Cartesian?

A 3D plane, *plane1*, is represented by the equation $2x_1 + 4x_2 - x_3 = 3$, that is satisfied for all points with coordinates (x_1, x_2, x_3). A 3D point has coordinates $\bar{\mathbf{x}}_1 = (-1, 2, 5)$.

1.4 What is the distance between the point $\bar{\mathbf{x}}_1$ and *plane1*? *Hint:* Use the techniques that you learned in your linear algebra course.

The point $\bar{\mathbf{x}}_1$ together with a second point $\bar{\mathbf{x}}_2 = (2, -1, 5)$ defines a 3D line.

1.5 The line intersects with the plane mentioned above. What are the coordinates of the intersecting point? **Verify** that your result is correct by checking that the point lies both in the plane and on the line. *Hint:* Use the techniques that you learned in your linear algebra course.

A third point is given as $\bar{\mathbf{x}}_3 = (4, 3, 1)$.

1.6 The three points $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3$ define a second 3D plane, *plane2*. What is the equation of this plane? **Verify** your result by checking that $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3$ all lie in the resulting plane. *Hint:* Use the techniques that you learned in your linear algebra course.

1.7 The intersection of *plane1* and *plane2* forms a 3D line. Describe this line in parametric form. **Verify** your result by checking that any point on the line lies in both planes. *Hint:* Use the techniques that you learned in your linear algebra course.

A 3D line is represented in parametric form as $\bar{\mathbf{x}}(s) = (1, 5, -7) + s\,(-3, 2, 1)$.

1.8 What is the distance between the point $\bar{\mathbf{x}}_1$ and the 3D line? *Hint:* Use the techniques that you learned in your linear algebra course.

2 Homogeneous representations in 2D

Here are the homogeneous coordinates of some 2D points:

$$\mathbf{y}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \mathbf{y}_2 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \quad \mathbf{y}_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \quad \mathbf{y}_4 = \begin{pmatrix} 2 \\ -1 \\ -0.5 \end{pmatrix} \quad \mathbf{y}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Here are the dual homogeneous coordinates of some 2D lines:

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{l}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{l}_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \quad \mathbf{l}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

2.1 Normalize the homogeneous coordinates of the 2D points such that you can see which 2D points they represent. Plot the points in a figure. *Hint:* use P-normalization.

2.2 Normalize the homogeneous coordinates of the 2D lines such that you can determine the parameters of the lines. *Hint:* use D-normalization. Alternatively: determine two points that lie on each line. With that information at hand, plot the lines in the same figure as the points.

2.3 Determine the point of intersection between two of the lines, **verify** that the result is correct by plotting the point in the figure. *Hint:* use the cross product formula.

2.4 Determine the line that intersects two of the points above, **verify** that the result is correct by determining the parameters of the line and plotting it in the figure. *Hint:* use the cross product formula.

2.5 Select a line and a point from those presented above, and determine the signed distance between the two. **Verify** the result by determining the distance from the figure. Try different combinations of points and lines that should give both positive, negative, and zero distances. *Hint:* The scalar product formula requires proper normalization.

2.6 (A) Let $\bar{\mathbf{y}}_1$ and $\bar{\mathbf{y}}_2$ be the Cartesian coordinates of two distinct points, lying on a line $\mathbf{l}$. Let $\mathbf{y}_1 \in \mathbb{R}^3$ and $\mathbf{y}_2 \in \mathbb{R}^3$ be the homogeneous coordinates of the same two points. Show that any linear combination of the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ represents the homogeneous coordinates of a point that lies on the line $\mathbf{l}$. Show that any point lying on $\mathbf{l}$ can be written as a linear combination of the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$.

2.7 Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be the dual homogeneous coordinates of two lines. How can you determine the angle between the two lines from the vectors $\mathbf{l}_1$ and $\mathbf{l}_2$? *Hint:* use the normal vector corresponding to each line.

2.8 (A) Let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ be the homogeneous coordinates of thee points, and let $\mathbf{Y} = (\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,)$ be a matrix that holds the homogeneous coordinates in its columns. Consider the matrix inverse $\mathbf{Y}^{-1}$. Exactly when does this matrix inverse exist? Characterize the rows of $\mathbf{Y}^{-1}$. *Hint:* consider the geometric interpretation of the algebraic relation $\mathbf{Y}^{-1}\mathbf{Y} = \mathbf{I}$ in terms of how rows of $\mathbf{Y}^{-1}$ and the columns of $\mathbf{Y}$ are related.

2.9 (A) Let $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ be the dual homogeneous coordinates of three lines, and let $\tilde{\mathbf{Y}} = (\,\mathbf{l}_1\,\mathbf{l}_2\,\mathbf{l}_3\,)$ be a matrix that holds the dual homogeneous coordinates in its columns. Consider the matrix inverse $\tilde{\mathbf{Y}}^{-1}$. Exactly when does this matrix inverse exist? Characterize the rows of $\tilde{\mathbf{Y}}^{-1}$. *Hint:* See previous exercise.

3 Transformations in 2D

Here is a matrix that represents a similarity transformation:

$$\mathbf{M}_1 = \begin{pmatrix} 3 & 4 & 2 \\ -4 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

3.1 Decompose $\mathbf{M}_1$ into a sequence starting with a scaling, then a rotation, and finally a translation. *Hint:* $\mathbf{M}$ can be decomposed directly into a linear affine transformations $\mathbf{M}_a$ followed by a translation $\mathbf{M}_t$: $\mathbf{M} = \mathbf{M}_t\mathbf{M}_a$. Then find the scaling of the linear transformation in $\mathbf{M}_a$ that reduces it into a rotation.

3.2 Decompose $\mathbf{M}_1$ into a sequence starting with a translation, then a rotation, and finally a scaling. *Hint:* Only the translation part differs from the previous exercise.

Here is a matrix that represents an affine transformation:

$$\mathbf{M}_2 = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

3.3 (A) This transformation is a combination of a translation, rotation, scaling, and shearing transformations. Try to work out how $\mathbf{M}_2$ can be decomposed into such a combination of transformations. *Hint:* There are multiple correct answers. Choose four points as the corners of a square and determine where these point end up after the transformation. This can give information about what the transformation does.

3.4 (A) Some of the transformations mentioned in the previous exercise can be applied in arbitrary order and give the same result (i.e., they *commute*), and some do not. Give some examples of both cases.

3.5 Pick a 2D point and a 2D line that you know intersect the point, e.g., among the ones mentioned in the early exercises. Transform the point with the affine transformation $\mathbf{M}_2$, apply the dual transformation of $\mathbf{M}_2$ to the homogeneous coordinates of the line, and demonstrate that the transformed line still intersects the transformed point. *Hint:* the point and the line should intersect both before and after the transformation.

3.6 (A) Since an affine transformation is rather general, it may not be obvious that it transforms a straight line to a straight line. Prove this fact using homogeneous coordinates.

3.7 (A) The homogeneous coordinates of 2D points are transformed by some matrix $\mathbf{M}$: $\mathbf{y}' = \mathbf{M}\,\mathbf{y}$. According to theory, the dual homogeneous coordinates of a line, $\mathbf{l}$, is then transformed as $\mathbf{l}' = \mathbf{M}^{-T}\mathbf{l}$. Furthermore, the Plücker coordinates of the same line, $\mathbf{L}$, is transformed as $\mathbf{L}' = \mathbf{M}\,\mathbf{L}\,\mathbf{M}^\top$. Given the result of the previous exercise, show that these two transformations are compatible.

3.8 $\mathbf{T}$ is the 3×3 transformation matrix corresponding to a rigid transformation:

$$\mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \text{where} \quad \mathbf{R} \in SO(2), \quad \mathbf{t} \in \mathbb{R}^2.$$

Determine the corresponding expression for $\mathbf{T}^{-1}$. *Hint 1:* Use the inverse of a block matrix described in Toolbox, Section 8.1.4. *Hint 2:* Verify that the resulting $\mathbf{T}^{-1}$ satisfy $\mathbf{T}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$.

3.9 $\mathbf{T}_1$ and $\mathbf{T}_2$ are two transformation matrices corresponding to rigid transformations:

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{pmatrix}, \mathbf{T}_2 = \begin{pmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix}, \quad \text{where} \quad \mathbf{R}_1, \mathbf{R}_2 \in SO(2), \quad \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^2.$$

Determine the corresponding expressions for the concatenations $\mathbf{T}_1\mathbf{T}_2$ and $\mathbf{T}_2\mathbf{T}_1$, and show that this resulting transformation again is a rigid transformation.

3.10 Use the results from exercises 3.8 and 3.9 to show that rigid transformations form a group under concatenation. *Hint:* there are four properties of the group transformation that must be verified.

3.11 Determine the expression for the rigid transformation that rotates by the rotation matrix $\mathbf{R}$ about the point $\mathbf{t}$. *Hint 1:* You can specify it as a combination of a translation, a rotation, and then the inverse of the first translation. *Hint 2:* Verify that the resulting transformation transforms $\bar{\mathbf{t}}$ to itself.

3.12 Exercise 3.11 suggests that an arbitrary rigid transformation

$$\mathbf{T} = \begin{pmatrix} \mathbf{R} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix}$$

can be seen as a rotation by $\mathbf{R}$ about some point $\bar{\mathbf{t}}_0$. How is $\bar{\mathbf{t}}_0$ determined from $\mathbf{R}$ and $\bar{\mathbf{t}}$? *Hint:* use the result from exercise 3.11.

3.13 (A) Determine necessary conditions on the parameters of the two rigid transformations in exercise 3.9 that makes the two transformations commute.

3.14 (A) Two points have Cartesian coordinates

$$\bar{\mathbf{y}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{\mathbf{y}}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

They are rigidly transformed to the corresponding points:

$$\bar{\mathbf{y}}'_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \bar{\mathbf{y}}'_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Determine the transformation matrix $\mathbf{T}$ of the rigid transformation. *Hint 1:* The mean of the two points, and their difference, must be transformed in the same way, by $\mathbf{T}$. *Hint 2:* Verify that the resulting transformation really is a rigid transformation ($\mathbf{R} \in SO(2)$), and that it transforms $\bar{\mathbf{y}}_k$ to $\bar{\mathbf{y}}'_k$.

3.15 Can we choose the four points $\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}'_1, \bar{\mathbf{y}}'_2$ in an arbitrary way in exercise 3.14?

3.16 $\mathbf{T}$ is the transformation matrix that represents a uniform scaling, with scale factor s, relative to the point $\bar{\mathbf{y}}_0$. Determine an expression for $\mathbf{T}$. *Hint 1:* Use a similar approach as in exercise 3.11. *Hint 2:* Check that the transformation makes $\bar{\mathbf{y}}_0$ a fixed point, i.e., $\bar{\mathbf{y}}_0$ is transformed to $\bar{\mathbf{y}}_0$.

4 Homogeneous representations in 3D

Here are the homogeneous coordinates of two 3D points:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

4.1 What 3D points do these homogeneous coordinates represent? *Hint:* P-normalization.

Here are the dual homogeneous coordinates of a 3D plane:

$$\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

4.2 What is the normal vector and distance to the origin for the plane $\mathbf{p}$? *Hint:* D-normalization.

4.3 What is the distance from each of the two points $\mathbf{x}_1$ and $\mathbf{x}_2$ to the plane $\mathbf{p}$. *Hint:* Don't forget proper normalization before using the scalar product formula.

4.4 How can you determine if the two points lie on the same side of the plane? Do they? *Hint:* How does the sign of the distance from a point to the line depend on the position of the point and the position of the origin?

4.5 What are the Plücker coordinates $\mathbf{L}_1$ of the 3D line that intersects the points $\mathbf{x}_1$ and $\mathbf{x}_2$? *Hint:* Just plug into suitable expression.

4.6 What are the dual Plücker coordinates of this line, $\tilde{\mathbf{L}}_1$, based on the duality mapping? *Hint:* Check IREG Section 5.3.5.

4.7 Pick two distinct 3D points, not on the line $\mathbf{L}$, and determine two planes $\mathbf{p}_1$ and $\mathbf{p}_2$ that both intersect the line $\mathbf{L}_1$. *Hint:* Use $\mathbf{p} \sim \tilde{\mathbf{L}}\,\mathbf{x}$ to determine a plane $\mathbf{p}$ that intersects with a line $\mathbf{L}$ and a point $\mathbf{x}$.

4.8 Form the dual Plücker coordinates as $\mathbf{p}_1\mathbf{p}_2^\top - \mathbf{p}_2\mathbf{p}_1^\top$ and verify that this is the same as in exercise 4.6.

4.9 In general, what algebraic relation can you use to check if a point $\mathbf{x}$ lies on the line $\mathbf{L}$?

4.10 What is the intersecting point $\mathbf{x}_0$ of the 3D line $\mathbf{L}_1$ and the plane $\mathbf{p}$? Verify that your result is a point in the plane and on the line.

4.11 The Plucker coordinates of a second line are given as

$$\mathbf{L}_2 = \begin{pmatrix} 0 & 1 & 3 & 3 \\ -1 & 0 & 4 & 6 \\ -3 & -4 & 0 & 6 \\ -3 & -6 & -6 & 0 \end{pmatrix}$$

How can you check that $\mathbf{l}_2$ holds the Plücker coordinates of a 3D line? Does it? *Hint:* Use the internal constraint defined for Plücker coordinates.

4.12 What is the direction of this line? Determine a point $\mathbf{x}'$ that lies on the line, for example the one that lies closest to the origin. *Hint:* Use L-normalization, described in IREG Section 5.3.3. You can verify you result by finding a second point $\mathbf{x}''$ on the line and forming the Plücker coordinates of the line through $\mathbf{x}'$ and $\mathbf{x}''$. It should be equivalent to $\mathbf{L}_2$.

4.13 How can you determine if the two lines $\mathbf{L}_1$ and $\mathbf{L}_2$ intersect? Do they intersect in this case? *Hint:* Check out IREG Section 5.4.7.

4.14 (A) Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be the Plücker coordinates of two arbitrary lines. Show that $\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{0}$ if and only if the two lines are identical. *Hint:* Express $\mathbf{L}_1$ as the Plücker coordinates of a line that passes through the points $\mathbf{x}_1$ and $\mathbf{x}_2$, and $\tilde{\mathbf{L}}_2$ as the dual Plücker coordinates of a line that intersects both planes $\mathbf{p}_1$ and $\mathbf{p}_2$. Plug all that into the equation $\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{0}$ and see what happens.

4.15 (A) Use the relation in exercise 4.14 to derive the duality mapping, IREG Equation (4.14). *Hint:* Express $\mathbf{L}$ and $\tilde{\mathbf{L}}$ as two general 4×4 anti-symmetric matrices. Use the result from exercise **??**: $\mathbf{L}\,\tilde{\mathbf{L}} = \mathbf{0}$, and solve the elements of $\tilde{\mathbf{L}}$ as expressions in the elements of $\mathbf{L}$. For example, start by solving $[\tilde{\mathbf{L}}]_{ij}$, for $ij = \{12, 13, 14, 23, 24\}$ from $[\mathbf{L}\,\tilde{\mathbf{L}}]_{kl} = 0$ where $kl = \{11, 12, 13, 22, 23\}$. Then use the internal constraint for the Plücker coordinates in IREG Equation (5.29) to get the final expressions for how the elements of $\tilde{\mathbf{L}}$ depend on the elements of $\mathbf{L}$.

4.16 (A) If $\mathbf{L}$ are the Plücker coordinates of a 3D line, what type of algebraic operation does $\mathbf{L}^2 = \mathbf{L}\mathbf{L}$ represent in $\mathbb{R}^4$? *Hint:* For a line, you can always choose two points on the line such that their homogeneous coordinates are orthogonal.

4.17 (A) Use the result from exercise 4.16 to show that $\mathbf{L}^2\mathbf{x} \sim \mathbf{x}$ is a necessary and sufficient condition for point $\mathbf{x}$ to lie on the line $\mathbf{L}$.

5 Transformations in 3D

5.1 Show that a 3D affine transformation preserves parallel planes as parallel. *Hint:* Assume two parallel planes, and investigate how the normal vectors of the two planes are modified by the transformation.

5.2 Show that a 3D affine transformation preserves proper points, lines, and planes as proper. *Hint:* Start with the canonical form of the homogeneous coordinates of a proper point and investigate how these homogeneous coordinates are modified by the transformation. A proper line or plane must include at least one proper point.

5.3 Show that a 3D affine transformation make points, lines, and the plane at infinite stay at infinity. *Hint:* Start with the homogeneous coordinates of a point at infinity and investigate how these homogeneous coordinates are modified by the transformation. A line at infinity and the plane at infinity include include only points at infinity.

5.4 Describe how a unique affine transformation can be determined from 4 3D points in general positions, before and after the transformation. *Hint:* Assume that the Cartesian coordinates of a pair of points are known, but the parameters of the affine transformation are not. Try to establish an equation system $\mathbf{A}\,\mathbf{z} = \mathbf{b}$ in the unknown transformation parameters $\mathbf{z}$.

5.5 (A) Describe some configuration of the three pairs of points for which the affine transformation in exercise 5.4 cannot be uniquely determined. *Hint:* Look at $\det(\mathbf{A})$ and try to find some configuration that makes $\det(\mathbf{A}) = 0$.

5.6 A special case of affine transformations consists of reflections in a plane $\mathbf{p}$, defined in IREG Equation (6.5). Show that the corresponding transformation matrix on the homogeneous coordinates of point is given by IREG Equation (6.6). *Hint:* Describe the transformation relative to a new ON-coordinate system with its origin in the plane.

5.7 (A) Describe how a unique similarity transformation can be determined from 3 points in 3D.

6 Introduction to estimation

6.1 A set of 10 2D points are defined by their Cartesian coordinates (u, v) where $u = -1, 1$ and $v = -2, -1, 0, 1, 2$. Plot the points in a figure, and guess a good estimate of a line that fits these points. Estimate a line that fits the points by minimizing a geometric error measured along the vertical and horizontal axis, respectively. Are the two estimated lines very similar or very different to the one you guessed? *Hint:* Use the expressions for the estimated line parameters derived in IREG, Section 12.1.

A set of N 3D points have Cartesian coordinates $\{\bar{\mathbf{x}}_k\}$ and from this set we want to estimate a 3D line. This can be done in different ways depending on how the line is represented, and which cost function is minimized.

6.2 From the given set of points, estimate the line using a parametric representation: $\bar{\mathbf{x}} = \bar{\mathbf{x}}_0 + t\,\hat{\mathbf{n}}$. Use a geometric error defined in terms of the sum of the squared distances from the points to the line. In other words: find the optimal choice of $\bar{\mathbf{x}}_0$ and unit vector $\hat{\mathbf{n}}$. *Hint:* Show first that $\bar{\mathbf{x}}_0$ can be chosen as the mean (center of gravity) position of the points, and generalize the derivation in IREG Section 12.2 to the 3D case for finding $\hat{\mathbf{n}}$. The distance between a point and the line is treated in exercise 1.8.

6.3 Use the same geometric error as in exercise 6.2, but now use the Plücker coordinates $\mathbf{L}$ to represent the line. *Hint:* you can formulate $\mathbf{L}$ directly from the solution of exercise 6.2.

6.4 (A) If you had not first solved exercise 6.2, how would you determine the Plücker coordinates that minimize the geometric error in the previous exercise? In other words: how would you minimize the error over the elements of $\mathbf{L}$ (or $\tilde{\mathbf{L}}$)? How do you guarantee that the internal constraint of $\mathbf{L}$ (or $\tilde{\mathbf{L}}$) is satisfied?

6.5 (A) Formulate an algebraic error for the estimation of the line. *Hint:* use the relation $\tilde{\mathbf{L}}\,\mathbf{x} = \mathbf{0}$ if the point $\mathbf{x}$ lies on the line with dual Plücker coordinates $\tilde{\mathbf{L}}$.

6.6 (A) How would you determine the Plücker coordinates that minimize the algebraic error in the previous exercise?

6.7 (A) Compare the three approaches: (1) using a parametric representation, (2) Plücker coordinates and geometric error, and (3) Plücker coordinates and algebraic error. Are they equivalent?

6.8 A 3D point $\mathbf{x}$ is estimated as the intersection of N 3D lines, with Plücker coordinates $\mathbf{L}_k$, where $k = 1, \ldots, N$. Formulate a data matrix $\mathbf{A}$ such that the estimation problem can be based on the homogeneous linear equation $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. *Hint:* use the relation $\tilde{\mathbf{L}}\,\mathbf{x} = \mathbf{0}$ if the point $\mathbf{x}$ lies on the line with dual Plücker coordinates $\tilde{\mathbf{L}}$.

6.9 In the previous exercise, we can find $\mathbf{x}$ as an eigenvector corresponding to eigenvalue zero of the matrix $\mathbf{A}^\top\mathbf{A}$. Simplify the matrix $\mathbf{A}^\top\mathbf{A}$ in this case.

7 Homographies

7.1 Show that a homography maps a line in 3D to a line. *Hint:* either investigate how the Cartesian coordinates of a point on the line are mapped by the homography to some curve, and show that the curve is a line, or look at how the subset of $\mathbb{R}^4$ that holds the homogeneous coordinates of point on the line are mapped by the homography to some other subset.

7.2 How are the Plücker coordinates of the transformed line expressed in the Plücker coordinates of the original line?

7.3 A square in a 2D space has vertices with Cartesian coordinates $(\pm 1, \pm 1)$ relative to some coordinate system. The square is transformed by a homography represented by the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

What does the transformed square look like? What shape does the interior of the square have after the transformation? *Hint:* Select a suitable set of points and determine how they are transformed by $\mathbf{H}$. Plot the transformed points, and lines between these points, in a figure and draw your conclusions.

7.4 Find two distinct proper points that are mapped to infinity by the homography $\mathbf{H}$ in the previous exercise.

7.5 Determine the dual homogeneous coordinates $\mathbf{l}$ of the line that passes through the two points from the previous exercise. Determine the dual transformation of $\mathbf{H}$, apply it to $\mathbf{l}$, and show that the result is the line at infinity.

8 The pinhole camera

8.1 Show that the image of a general 3D line in accordance to the pinhole camera model is a 2D line. *Hint:* Can be solved more or less in a similar way as exercise 7.1.

8.2 What lines are not mapped to lines by the camera? What is the image of such a line?

8.3 Given a 3×4 camera matrix $\mathbf{C}$, how can you determine three distinct planes that pass through the camera center? *Hint:* the homogeneous coordinates of the camera center, $\mathbf{n}$, satisfy $\mathbf{C}\,\mathbf{n} = \mathbf{0}$.

8.4 (A) Give a geometric interpretation of the third row of the camera matrix. Assume a normalized camera. *Hint:* what do the normalized image coordinates look like for points lying on the plane represented by the third row of the camera matrix?

Here is a camera matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

that maps 3D points defined in some world coordinate system to image coordinates.

8.5 What is the 3D position of the camera center in the world coordinate system? *Hint:* the homogeneous coordinates of the camera center, $\mathbf{n}$, satisfy $\mathbf{C}\,\mathbf{n} = \mathbf{0}$.

8.6 What are the image coordinates of the two 3D points $\mathbf{x}_1$ and $\mathbf{x}_2$ defined in exercise 4.1? Assume that their Cartesian 3D coordinates are defined relative to the same world coordinate system as the camera. *Hint:* Don't forget to do P-normalization of the result.

8.7 What is the image of the 3D line $\mathbf{L}_1$, defined in exercise 4.5, when using this camera? *Hint:* you can determine this line either by projecting two points on the line to the image and then determine the intersecting 2D line, or by applying the camera matrix on the Plücker coordinates of the 3D line to get the Plücker coordinates of the 2D line. Both approaches give the same result.

8.8 The world coordinate system is translated by the vector $\bar{\mathbf{t}}' = (1, 1, 1)$. What is the camera matrix relative to this new world coordinate system? **Verify** the the camera center has translated accordingly.

Here is a camera matrix that has been determined for a camera for an experiment at the Computer Vision Laboratory:

$$\mathbf{C} = \begin{pmatrix} 0.9230 & -2.1272 & 1.7078 & 821.8 \\ -2.7109 & -0.8041 & 0.8523 & 1626.3 \\ -0.00019 & -0.00061 & -0.00035 & 1.0 \end{pmatrix}$$

It maps 3D points measured in millimeters in some world coordinate system to image coordinates measured in pixels. The camera image is 2592×1944 pixels in size. This particular camera matrix can be decomposed as

$$\mathbf{C} \sim \mathbf{K}\,(\,\mathbf{R}\,|\,\bar{\mathbf{t}}\,) = \underbrace{\begin{pmatrix} 3795.3 & -1.5 & 948.3 \\ 0.0 & 3795.8 & 1298.2 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}}_{\mathbf{K}} \left(\underbrace{\begin{matrix} 0.3947 & -0.5532 & 0.7336 \\ -0.8815 & -0.0029 & 0.4722 \\ -0.2591 & -0.8330 & -0.4888 \end{matrix}}_{\mathbf{R}} \quad \underbrace{\begin{matrix} -45.2 \\ 117.4 \\ 1358.4 \end{matrix}}_{\bar{\mathbf{t}}} \right)$$

Notice that $\mathbf{R}$ is a 3×3 rotation matrix: $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$.

8.9 (M) Where is the camera center in the 3D world coordinate system?

8.10 (M) What is the direction of the optical axis of the camera?

8.11 Where is the principal point in pixel coordinates? Is it at the center of the image?

8.12 The two elements at positions (1,1) and (2,2) in the matrix $\mathbf{K}$ are not exactly equal, and there is a non-zero value in position (1,2). What is the effect of that? How will that camera map a square that is positioned right in front of the camera and parallel to the image plane?

There reason for the effects described in Exercise 8.12 can be a combination of: (1) the camera matrix has not been determined (or rather estimated) with sufficiently high accuracy, and (2) the camera is in practice not a perfect pin-hole camera.

8.13 Let $\mathbf{C}$ be a camera matrix with pseudo-inverse $\mathbf{C}^+$, i.e., $\mathbf{C}\,\mathbf{C}^+ = \mathbf{I}$. Let $\mathbf{y}$ be the homogeneous coordinates of some point in the image. How can we interpret $\mathbf{C}^+\mathbf{y}$? *Hint:* it represents a 3D point, but what type of point?

8.14 Let a camera matrix $\mathbf{C}$ that is defined relative to some world coordinate system be decomposed into internal and external parameters as $\mathbf{C} = \mathbf{K}\,(\mathbf{R}\,|\,\mathbf{t})$. What is the rigid transformation from the world coordinate system to the camera centered coordinate system? What is the rigid transformation from the camera centered coordinate system to the world coordinate system?

8.15 (A) Let $\mathbf{L}$ be the Plücker coordinates of some 3D line. This line is mapped by a pin-hole camera $\mathbf{C}$ to a 2D line $\mathbf{l}$ in the image defined by. What is the algebraic relation between $\mathbf{l}$, $\mathbf{C}$, and $\mathbf{L}$?

8.16 (A) Let $\mathbf{l}$ be the dual homogeneous coordinates of a 2D line in the camera image that is produced by camera matrix $\mathbf{C}$. All 3D points that are projected onto the line $\mathbf{l}$ must then lie in a plane $\mathbf{p}$. How is $\mathbf{p}$ related algebraically to $\mathbf{C}$ and $\mathbf{l}$?

8.17 (A) In the literature it is stated that the camera matrix $\mathbf{C}$ must have full rank (=3). Let $\mathbf{C}$ be a camera matrix of rank 2. Describe how $\mathbf{C}$ projects 3D points to 2D points in its image plane. In what way is this camera degenerated compared to the rank 3 case?

9 Estimation of transformations

Three 2D points approximately lie on a line. They have a homogeneous representation given by

$$\mathbf{y}_1 = \begin{pmatrix} -4.89 \\ 105.4 \\ 1 \end{pmatrix} \quad \mathbf{y}_2 = \begin{pmatrix} 0.88 \\ 100.2 \\ 1 \end{pmatrix} \quad \mathbf{y}_3 = \begin{pmatrix} 5.55 \\ 95.68 \\ 1 \end{pmatrix}$$

9.1 (M) What is the corresponding data matrix $\mathbf{A}$? What is its SVD profile? Is it ambiguous or not?

9.2 (M) What is the estimate of the intersecting line using the homogeneous method on this data?

9.3 (M) Define a geometric error for this estimate. What is the numerical value of the geometric error for this estimate?

9.4 (M) What is the Hartley-transformation corresponding to this dataset? Give the transformation matrix and the transformed coordinates.

9.5 (M) What is the data matrix from the transformed points? What is its SVD profile? Is it ambiguous or not?

9.6 (M) What is the estimate of the intersecting line using the homogeneous method on the transformed data? How does it differ from the first estimate?

9.7 (M) What is the geometric error of this second estimate? Did it change relative to the error in exercise 9.3? *Hint:* In order to compare the errors they should be calculated in the same coordinate system.

A homography is given by the matrix

$$\mathbf{H} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},$$

and the coordinates of some point before and after the transformation are given by

$$\text{before:} \quad \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad \text{after:} \quad \begin{pmatrix} u'_k \\ v'_k \end{pmatrix}.$$

9.8 Formulate a geometric error for the estimation of $\mathbf{H}$ given a set of corresponding points.

9.9 How would you determine a minimum of the geometric error over all choices of $\mathbf{H}$?

9.10 Let $\mathbf{l}_k$ and $\mathbf{l}'_k$ be a set of 2D lines before and after a homography transformation $\mathbf{H}$ of the Cartesian coordinates. How do you estimate $\mathbf{H}$ in this case, using an algebraic minimization? How many lines do you need at minimum?

9.11 You are estimating a 2D homography from two sets of corresponding points based on the minimizing an algebraic error by means of the homogeneous method. The points in each set turn out to be co-linear, i.e., they lie on a line that is specific for each set. What is the dimension of the solution space in this case?

9.12 A 2D line is estimated from only two points, but these points have a measurement error with a standard deviation of σ. Show that the expected error in the orientation of the estimated line increases if the points are closer than if they are farther apart. *Hint:* choose a coordinate system such that the unperturbed positions of the points are on the horizontal axis.

9.13 (A) If you have a method for estimating homographies, how can you use it to determine correspondences between two point sets that you know is related by a homography? *Hint:* it is allowed to guess correspondences!

9.14 (A) You are estimating a camera matrix $\mathbf{C}$ from a set of corresponding 2D and 3D points, using an algebraic error derived from DLT. What happens with the solution space if the 3D points happen to be chosen such that they lie approximately on a plane? What happens if they lie approximately on a 3D line?

10 Representations of 3D rotations

10.1 $\mathbf{R} \in SO(3)$ is a rotation matrix. Define a minimal parameterization of $\mathbf{R}$ based on its elements. Choose three of its elements as parameters, and show that the entire $\mathbf{R}$ can be determined from these three parameters. What is the range of the parameters? Are there any singularities or ambiguities? Is this a useful representation?

10.2 The special case of small rotations is discussed in IREG, Section 11.2.1, where IREG Equation (11.16) describes an approximate representation of a rotation matrix $\mathbf{R} \in SO(3)$ corresponding to a small rotation angle α. Assume that $\mathbf{R}$ is given for this special case, how can you then determine the rotation axis $\hat{\mathbf{n}}$ and the rotation angle α? Try to find a simpler way than what is described in IREG, Section 11.2.2?

10.3 Use the Cayley transformation to represent two rotations, $\mathbf{R}_1$ and $\mathbf{R}_2$, in terms of two matrices $\mathbf{M}_1, \mathbf{M}_2 \in so(3)$. Determine the matrix $\mathbf{M} \in so(3)$ representing the product of $\mathbf{R}_1\mathbf{R}_2$. *Hint:* multiply $\mathbf{R}_1$ and $\mathbf{R}_2$, expressed as Cayley transformations of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Then apply the inverse Cayley transformation on the result to obtain $\mathbf{M}$.

10.4 Twisted rotations are defined in IREG, Section 11.7.2. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be a pair of twisted rotations. Show that all four combinations of $\mathbf{R}_1$ and $\mathbf{R}_2$ in IREG Equation (11.77) correspond to rotations by 180°.

10.5 Show that the two rotation axes $\hat{\mathbf{n}}$, corresponding to the first two combinations, and $\hat{\mathbf{n}}'$, corresponding to the last two combinations, are related as in IREG Equation (11.78).

10.6 Show that two quaternions $\mathbf{q}_1, \mathbf{q}_2 \in S^3$ are related as in IREG Equation (11.85) if they form a twisted pair.

11 Estimation involving rotations

The orthogonal Procrustes problem is described in Section 2.3.6 of the compendium, and an approach for finding an optimal rotation that is guaranteed to lie in $SO(n)$ is discussed in Section 12.1. Both these are based on SVD.

11.1 (A) Show that the trick of changing the sign of the smallest singular value, when the determinant of one but not both of $\mathbf{U}$ and $\mathbf{V}$ is $= -1$, is correct for $n = 2$ in order to get a rotation in $SO(2)$.