Exercises for Linear representations in Signal Processing

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Exercises marked with (A) are on an advanced level and can be deferred until the others are solved. Exercises marked with (M) lead to numerical computations that are not straight-forward to do by hand, and are recommended to be carried out using Matlab or similar numerical calculation tools.

12 Dual bases

The vector space $\mathbb{R}^2$ has two basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

There is also a vector

$$\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

The scalar product between vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ is given as $\mathbf{a}^\top \mathbf{b}$.

12.1 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

12.2 What is the dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ corresponding to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$?

12.3 Verify that the resulting basis satisfies the dual relation relative to the original basis.

12.4 What are the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$?

12.5 What are the dual coordinates of $\mathbf{v}$ relative to the dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$?

12.6 Verify the last two answers.

Change the scalar product in $\mathbb{R}^2$ so that it is now given as $\mathbf{a}^\top \mathbf{G}_0 \mathbf{b}$ with

$$\mathbf{G}_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

12.7 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

12.8 What is the new dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$? Verify your result.

12.9 What are the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$? Verify your result.

12.10 What are the dual coordinates of $\mathbf{v}$? Verify.

12.11 (A) Verify that $\mathbf{G}_0$ represents a valid scalar product.

Consider the vector space $\mathbb{C}^2$. The scalar product between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ is here defined as

$$\langle\, \mathbf{a} \,|\, \mathbf{b} \,\rangle = \mathbf{b}^\star \mathbf{G}_0 \mathbf{a} \quad \text{where} \quad \mathbf{G}_0 = \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix}$$

In the following exercises, pay attention to how exactly the scalar product between two vectors in $\mathbb{C}$ is defined! In this vector space there is a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ and a vector $\mathbf{v}$:

$$\mathbf{b}_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

12.12 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

12.13 What is the new dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$? Verify your result.

12.14 What are the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$? Verify your result.

12.15 What are the dual coordinates of $\mathbf{v}$? Verify.

12.16 (A) Verify that $\mathbf{G}_0$ represents a valid scalar product.

12.17 (A) Show that any set of vectors $\{\tilde{\mathbf{e}}_k\}$ that satisfy the duality relation $\langle\, \mathbf{e}_k \,|\, \tilde{\mathbf{e}}_l \,\rangle = \delta_{kl}$ relative to a basis $\{\mathbf{e}_k\}$ must itself be a basis, the dual basis. *Hint:* Consider a finite dimensional vector space. It then suffices to show that the dual basis cannot be linearly dependent.

13 Subspaces

Consider the two-dimensional subspace of $\mathbb{R}^3$ that is spanned by the two basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The scalar product in $\mathbb{R}^3$ is here defined by $\mathbf{G}_0 = \mathbf{I}$. There is also a vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

13.1 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

13.2 What is the dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ corresponding to the subspace basis $\{\mathbf{b}_1, \mathbf{b}_2\}$?

13.3 Verify that the resulting dual basis satisfies the dual relation relative to the original basis.

13.4 What are the coordinates of $\mathbf{v}_1$, the orthogonal projection of $\mathbf{v}$ onto the subspace, relative to the subspace basis $\{\mathbf{b}_1, \mathbf{b}_2\}$?

13.5 What is $\mathbf{v}_1$?

13.6 What is $\mathbf{v}_0$, the component of $\mathbf{v}$ that lies in the orthogonal complement of the subspace? Verify that it lies in the orthogonal complement.

Consider the same subspace in $\mathbb{R}^3$, spanned by $\mathbf{b}_1, \mathbf{b}_2$, but now the scalar product in $\mathbb{R}^3$ is defined by

$$\mathbf{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

13.7 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

13.8 What is the dual basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ corresponding to the subspace basis $\{\mathbf{b}_1, \mathbf{b}_2\}$?

13.9 What is $\mathbf{v}_1$ in this case?

13.10 You are dealing with the same subspace as before. Why is $\mathbf{v}_1$ not the same as before?

Return to the general case of a real vector space, when $\mathbf{B}$ is a basis of some subspace and the scalar product is defined by some symmetric and positive definite matrix $\mathbf{G}_0$.

13.11 (A) Use the equation

$$\mathbf{v}_1 = \mathbf{B}\,(\mathbf{B}^\star \mathbf{G}_0 \mathbf{B})^{-1} \mathbf{B}^\star \mathbf{G}_0 \mathbf{v}$$

to show that the orthogonal projection $\mathbf{v}_1$ is independent of the choice of basis $\mathbf{B}$ in the subspace.

13.12 (A) Show that $\mathbf{P} = \mathbf{B}\,(\mathbf{B}^\star \mathbf{G}_0 \mathbf{B})^{-1} \mathbf{B}^\star \mathbf{G}_0$ is a projection operator, i.e., $\mathbf{P}^2 = \mathbf{P}\,\mathbf{P} = \mathbf{P}$. What subspace does it project onto?

13.13 (A) Let $\mathbf{v}_1$ be the orthogonal projection of $\mathbf{v} \in V$ onto the subspace spanned by $\mathbf{P}$. Show that $\|\mathbf{v}_1\|^2 = \langle\, \mathbf{v} \mid \mathbf{v}_1 \,\rangle = \langle\, \mathbf{v}_1 \mid \mathbf{v} \,\rangle$.

14 Normalized convolution, 1D signals

A time discrete signal f has the following values

$$f[k] = \{10, 10, 11, 13, 14, 20, 17, 13, 10\}$$

We want to analyze this signal with a basis consisting of the polynomials $\{1, x\}$ in a 5 sample window, with the origin $x = 0$ at the center element. The applicability function a is a triangular function:

$$a[k] = \{1, 2, 3, 2, 1\}$$

14.1 What is the basis matrix in this case?

14.2 What filters are convolved with the signal?

14.3 What is the scalar product matrix $\mathbf{G}_0$?

14.4 What is the metric $\mathbf{G}$ in this case? What is $\mathbf{G}^{-1}$?

14.5 Estimate the local mean and local first order derivative of the signal at the center point using normalized convolution. Verify that the values you obtain are reasonable.

14.6 (A) Extend the basis functions so that is also includes a "x^2" function. Then redo Exercises 14.4 and 14.5. Did the coordinates of the first two basis functions change? Motivate this result.

The previous calculations are based on the signal having full certainty for all samples. Assume instead that the signal certainty is given by

$$c[k] = \{1, 1, 1, 0, 1, 1, 1, 1, 1\}$$

14.7 How does this modify the scalar product $\mathbf{G}_0$? In particular, what is $\mathbf{G}_0$ at the center point of the signal?

14.8 What is the metric $\mathbf{G}$ and its inverse $\mathbf{G}^{-1}$ at the center point of the signal?

14.9 Re-estimate the local average and first order derivatives for this signal with missing data.

14.10 Compare this result to what happens if you do not take the certainty information into account, and just set missing data = 0.

In normalized convolution, we first apply the filters corresponding to the different basis functions, and then transform the result to proper coordinates relative to the basis functions. In the case of signal certainty = 1, however, the resulting coordinates can also be obtained directly by first transforming the filters to "dual" filters that correspond to the dual basis functions and then convolve with the dual filters.

14.11 Show that the inverse metric $\mathbf{G}^{-1}$ can be used to transform the filters so that the filter responses directly give proper coordinates relative to the basis functions. You can restrict yourself to the case of two basis functions.

14.12 What are the dual filters in the case of the full certainty signal above? Verify that they give the expected result.

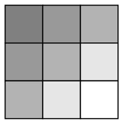
14.13 Normalized averaging of a signal f refers to the special case of normalized convolution where there is only a single basis function, which in addition is constant = 1, and g has variable certainty c. In this case the local signal's coordinate relative to the single basis function is given by

$$c_1 = \frac{(f \cdot c) * a}{c * a} \quad \text{(point-wise division)}$$

Show that this expression follows from applying normalized convolution to this particular basis. What is the practical interpretation of the signal c_1?

15 Normalized convolution, 2D signals

A 3×3 region in an image is illustrated to the left and has the pixel values given by f to the right.



$$f[k_1, k_2] = \begin{pmatrix} 10 & 11 & 12 \\ 11 & 12 & 14 \\ 12 & 14 & 17 \end{pmatrix}$$

The image is analyzed with the basis functions 1,x,y with the origin centered in a 3 x 3 region, where we assume that the x-axis points right and the y-axis points up. The applicability function is

$$a[k_1, k_2] = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

15.1 What is the basis matrix in this case?

15.2 What filters are applied/convolved onto the signal?

15.3 What is the scalar product matrix $\mathbf{G}_0$?

15.4 What is the metric $\mathbf{G}$ and its inverse $\mathbf{G}^{-1}$?

15.5 Estimate the local mean and first order derivatives in horizontal and vertical direction using normalized convolution. Verify the result.

A 2D signal is analyzed in terms of Cartesian separable basis functions (for example polynomials), using an applicability function a that is also Cartesian separable, based on normalized convolution.

15.6 (A) Show that also the convolution operations that correspond to the computations involved in normalized convolution are Cartesian separable. What is the advantage of this observation?

15.7 (A) A reasonable applicability function a should also be circular symmetric, i.e., a function only of the distance to the origin. Why is circular symmetry important? Give an example of a Cartesian separable applicability function that is also circular symmetric.

16 Filter optimization

The spectrum S_g of a signal g is defined as the expected squared magnitude of the signal's Fourier transform:

$$S_g(u) = \mathsf{E}\,|G(u)|^2 \quad \text{where} \quad G = \mathcal{F}g.$$

The concept of the signal spectrum is therefore connected to the idea that the observed signal is drawn from a statistical distribution of signals and that E is the expectation operator over this distribution. We assume in the following that the signal is of a single discrete variable. This signal is convolved by an FIR-filter f, that has a Fourier transform F. Ideally we want $F = F_{\text{ideal}}$, where F_{ideal} is some suitable frequency function. One way to determine the filter coefficients f is to minimize the expected difference between the resulting filter output, $h = g * f$, and the filter output produced by the ideal filter, $h_{\text{ideal}} = g * f_{\text{ideal}}$. This error is defined as

$$\epsilon = \mathsf{E} \sum_{k=-\infty}^{\infty} \big|h[k] - h_{\text{ideal}}[k]\big|^2.$$

16.1 Show that the minimization of ϵ is equivalent to filter optimization where the frequency weight function $W(u)$ is the signal spectrum S_g. This motivates why $W(u)$ in filter optimization should be related to the signal spectrum. In general, this result also motivates why filter optimization should be done relative to a weighting function in the frequency domain. *Hint:* Use Parseval's relation for discrete signals.

Filter optimization can be done with a spatial mask that, e.g., can be used to set coefficients close to the "corners" of the filter $= 0$. Assume that the filter is applied to a signal of outer dimension n. This signal is filtered with an FIR-filter that initially has its coefficients in an n-dimensional hyper-cube with side P. Using a spatial mask, the coefficients which are further away from the filter center than $P/2$ are set $= 0$. The remaining coefficients, which are optimized, represent an n-dimensional hyper-sphere of radius $P/2$.

16.2 How much fewer are the optimized coefficients relative to the initial hyper-cube for a given n? Use the approximation that the number of coefficients in both cases is proportional to the volume of the hyper-cube and the hyper-sphere, respectively. You may restrict your analysis to the cases $n = 1, 2, 3, 4$. *Hint:* Wikipedia has an entry on *n-sphere* that gives the volume of a hyper-sphere with a given radius and for different dimensions n.

The spatial mask can be used to reduce the computational complexity of the resulting filter, i.e., each filter output can be computed with fewer additions and multiplications the fewer filter coefficients that are $\neq 0$ (assuming the corresponding convolution operation can skip the zero valued coefficients in a filter kernel). The process of removing coefficients is typically incremental, and implies that a "full" filter is optimized first and then suitable filter coefficients are removed by means of the spatial mask, and the filter is re-optimized.

16.3 (A) Intuitively, it may seem like a good idea to remove the filter coefficients that have the smallest magnitudes. Show that it is not necessary correct to say that removing the filter coefficient of the smallest magnitude also always makes the smallest increase in the optimization error. *Hint:* use the result from exercise 13.13.

Filter optimization can also be made with a spatial term that punishes large coefficients at large distance from the origin.

16.4 Reducing filter coefficients far from the filter center affects the resulting frequency function of the filter in a particular way. How?

17 Principal Component Analysis

In Principal Component Analysis (PCA) we want to determine an ON-basis $\mathbf{B}$ of an M-dimensional subspace such that

$$\epsilon = \mathsf{E}\left[\|\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top\mathbf{v}\|^2\right]$$

is minimized over $\mathbf{B}$, where E is the expectation value operator over all observations of the signal $\mathbf{v}$. We assume a real and N-dimensional vector space V and a scalar product given by $\mathbf{G}_0 = \mathbf{I}$.

17.1 Show that *minimizing* ϵ is equivalent to *maximizing*

$$\epsilon_1 = \mathsf{E}\left[\mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v}\right].$$

17.2 For the case $M = 1$, i.e., when $\mathbf{B}$ has only a single column $\mathbf{b}_1$, show that the solution to the above minimization problem is given by

$\mathbf{b}_1$ is a normalised eigenvector corresponding to the largest eigenvalue of $\mathbf{C}$

where $\mathbf{C}$ is the signal's correlation matrix, defined as

$$\mathbf{C} = \mathsf{E}\left[\mathbf{v}\,\mathbf{v}^\top\right].$$

17.3 Show that in this case, $M = 1$, follows that

$$\epsilon = \sum_{k=2}^{N} \lambda_k,$$

where λ_k are the eigenvalues of $\mathbf{C}$, sorted in descending order, i.e., λ_1 is the largest.

17.4 For the general case, when $M > 1$, the subspace ON basis matrix $\mathbf{B}$ that minimizes ϵ is not unique. In this case, show that $\mathbf{B}' = \mathbf{B}\,\mathbf{Q}$ also is an ON-basis that solves the minimization problem, where $\mathbf{Q}$ is $M \times M$ orthogonal.

17.5 (A) For the case $M = 2$, apply Langrange's method on the PCA minimization problem with constraint $\mathbf{B}^\top\mathbf{B} = \mathbf{I}$. Show that the resulting equations do not provide a unique solution to the problem (consistent with the result of the previous exercise).

17.6 (A) Show that introducing the additional constraint of $\mathbf{B}^\top\mathbf{C}\,\mathbf{B}$ being diagonal we obtain the relations

$$\mathbf{C}\,\mathbf{b}_k = \lambda_k\mathbf{b}_k, \quad k = 1, \ldots, M,$$

where $\mathbf{b}_k$ is the k-th subspace basis vector, i.e., column k in $\mathbf{B}$. Also show that this allows us to generalize the statement about the solution to the PCA problem as

$\mathbf{B}$ is an ON-basis of the eigenvectors corresponding to the M largest eigenvalues of $\mathbf{C}$

You may simplify to the case $M = 2$.

17.7 Show that in the general case, $M > 1$, it follows that

$$\epsilon = \sum_{k=M+1}^{N} \lambda_k.$$

18 Frames

Given a set of M vectors $\{\mathbf{b}_k \in V, k = 1, \ldots, M\}$ the corresponding frame operator $\mathbf{F}$ is defined as a mapping $V \to V$ given by

$$\mathbf{F}\,\mathbf{v} = \sum_{k=1}^{M} \langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle\, \mathbf{b}_k$$

for every $\mathbf{v} \in V$.

18.1 Show that this leads to $\mathbf{F}$ being self-adjoint, i.e., for all $\mathbf{u}, \mathbf{v} \in V$ it is the case that

$$\langle\, \mathbf{F}\,\mathbf{u} \mid \mathbf{v} \,\rangle = \langle\, \mathbf{u} \mid \mathbf{F}\,\mathbf{v} \,\rangle$$

Investigate the case when V is a real vector space. *Hint:* use the linear property of the scalar product, in combination with fact that it is symmetric in its two arguments for this case.

18.2 (A) Show that the same property holds also for the case that V is a complex vector space.

18.3 Show that the frame condition can be expressed in terms of the frame operator as

$$A\|\mathbf{v}\|^2 \leq \langle\, \mathbf{v} \mid \mathbf{F}\,\mathbf{v} \,\rangle \leq B\|\mathbf{v}\|^2$$

18.4 (A) In the finite dimensional case, and when the scalar product is defined by $\mathbf{G}_0 = \mathbf{I}$, show that the frame bounds A and B correspond to the smallest and largest eigenvalues of the matrix $\mathbf{F}$.

18.5 (A) A set of vectors $\mathbf{b}_k$ constitute a basis of V. Show that they satisfy the frame condition, i.e., they constitute a frame.

18.6 With $V = \mathbb{R}^2$, define an infinite set of vectors as $\mathbf{b}_1 = (1, 0)$ and $\mathbf{b}_k = (0, 1/k)$, $k = 2, \ldots$. Show that this infinite set satisfy the frame condition and, therefore, constitute a frame. What are the frame bounds?

18.7 With $V = \mathbb{R}^2$, define an infinite set of vectors as $\mathbf{b}_1 = (1, 0)$ and $\mathbf{b}_k = (0, 1/\sqrt{k})$, $k = 2, \ldots$. Show that this infinite set does not satisfy the frame condition and, therefore, does not constitute a frame.

18.8 For $\mathbf{v} \in V$ and $\mathbf{B}$ a matrix that holds a frame of V in its columns, a set of reconstructing coefficients for $\mathbf{v}$ is given as

$$\mathbf{c} = \mathbf{B}^{\star}\mathbf{G}_0\mathbf{v}$$

An other set of reconstructing coefficients is given by $\mathbf{c} + \mathbf{c}_0$, where $\mathbf{c}_0$ is a null vector of $\mathbf{B}$. Show that $\mathbf{c}_0^{\star}\mathbf{c} = 0$, i.e., $\mathbf{c}_0$ is orthogonal to $\mathbf{c}$, Also show that this implies: $\mathbf{c}$ is the shortest vector of reconstructing coefficients.

Consider the following four vectors in $\mathbb{R}^2$:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The scalar product in $\mathbb{R}^2$ is given by $\mathbf{G}_0 = \mathbf{I}$.

18.9 Show that the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ form a frame for $\mathbb{R}^2$. What is the corresponding frame operator $\mathbf{F}$? What are the frame bounds A and B? Is it a tight frame?

18.10 What are the dual frame vectors?

18.11 Compute a set of reconstructing coefficients of $\mathbf{v}$ relative to the frame. It should have minimum norm. Verify that $\mathbf{v}$ can be reconstructed from the frame vectors with the reconstructing coefficients.

18.12 Describe the full set of reconstructing coefficients for $\mathbf{v}$ relative to the frame.

19 Sampling, over-sampling and reconstruction

A discrete noise signal $n[k]$ has the following properties

A. The samples have zero mean: $\mathsf{E}\{n[k]\} = 0$.

B. The samples are independent and have variance σ^2: $\mathsf{E}\{n[k]\, n[l]\} = \sigma^2\, \delta_{kl}$.

This discrete noise signal is reconstructed to a continuous time signal as

$$n_{\mathrm{rec}}(t) = \sum_{k=-\infty}^{\infty} n[k]\, \mathrm{sinc}(t-k)$$

19.1 Show that properties A+B lead to the following properties for the reconstructed noise signal:

A' $n(t)$ has zero mean for all t.

B' $\mathsf{E}\{n(t_1)\, n(t_2)\} = \sigma^2\, \mathrm{sinc}(t_1 - t_2) \quad \Rightarrow \quad \mathsf{E}\{n^2(t)\} = \sigma^2$.

19.2 Construct a signal $n(t)$ as the mean of $n_1(t)$ and $n_2(t)$, where both n_1 and n_2 have properties A'+B' and they are *independent*. Show that the variance of n is reduced to by a factor of 2 relative to n_1 a and n_2.

19.3 Show that the set of unit spaced sinc functions form an ON-basis for the space of 2π-band-limited functions.

19.4 Show that for a 2π-band-limited function f, a sample value at position $t = k$ is given by

$$f(k) = \langle\, f(t) \mid \mathrm{sinc}(t-k)\, \rangle = \int_{-\infty}^{\infty} f(t)\, \mathrm{sinc}(t-k)\, dt$$

19.5 Show that the set of half unit spaced sinc functions is not a basis of the same space, but instead form a frame. What are the frame bounds? Is it a tight frame?

19.6 Using the result from the previous exercise, how can a 2π-band-limited function f be reconstructed from samples from f at half integer positions?

20 Continuous Wavelet Transform

The continuous wavelet transform of a one-variable function f is defined as

$$W_f(a,b) = \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}}(t)\, dt \tag{1}$$

where $\psi_{a,b}$ is a family of wavelet functions generated from the mother wavelet ψ according to

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right) \tag{2}$$

20.1 Show that the factor $\frac{1}{\sqrt{a}}$, that is used in the definition of the wavelet family above, makes the norm of each wavelet function $\psi_{a,b}$ a constant independent of a, b.

Given the wavelet transform W_f of a function f, define an auxiliary function g as

$$g(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|a|^2}\, W_f(a,b)\, \psi_{a,b}(t)\, da\, db \tag{3}$$

20.2 By inserting the above expression for W_f into (3), show that g can be rewritten as

$$g(t) = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|a|^3}\, \overline{\psi}\left(\frac{y-b}{a}\right)\, \psi\left(\frac{t-b}{a}\right)\, da\, db\, dy \tag{4}$$

Hint: assume that changing order of integration is allowed here.

20.3 The innermost double integral in (4) is

$$I(t,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|a|^3}\, \overline{\psi}\left(\frac{y-b}{a}\right)\, \psi\left(\frac{t-b}{a}\right)\, da\, db$$

Show that $I(t,y)$ can be rewritten as

$$I(t,y) = \int_{-\infty}^{\infty} \frac{1}{|a|^3} \int_{-\infty}^{\infty} p(b)\, \overline{q}(b)\, db\, da \tag{5}$$

with

$$p(b) = \psi\left(\frac{t-b}{a}\right) \qquad q(b) = \psi\left(\frac{y-b}{a}\right) \tag{6}$$

Notice that the functions p and q still depend on a, b and t or y.

20.4 The innermost integral in (5) is

$$J = \int_{-\infty}^{\infty} p(b)\, \overline{q}(b)\, db$$

Show that J can be rewritten as

$$J = \frac{|a|^2}{2\pi} \int_{-\infty}^{\infty} e^{iu(t-y)}\, |\Psi(-au)|^2\, du$$

where Ψ is the Fourier transform of ψ. *Hint:* Parseval's formula is useful here.

20.5 Insert J back into (5) and show that $I(t, y)$ can be rewritten as

$$I(t, y) = \int_{-\infty}^{\infty} \frac{e^{iu(x-y)}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|a|} |\Psi(-au)|^2 \, da \, du \tag{7}$$

20.6 The innermost integral in (7) is

$$K = \int_{-\infty}^{\infty} \frac{1}{|a|} |\Psi(-au)|^2 \, da$$

Show that K can be rewritten as

$$K = \int_{-\infty}^{\infty} \frac{1}{|v|} |\Psi(v)|^2 \, dv$$

This means that K is a constant that only depends on the choice of the mother wavelet ψ.

20.7 Put the constant K back into (7) and show that

$$I(t, y) = K \, \delta(t - y)$$

20.8 Put this $I(t, y)$ back into (4) and show that this leads to

$$g(t) = K \, f(t)$$

Congratulation! You have now proven that f can be reconstructed from its continuous wavelet transform W_f:

$$f(t) = \frac{1}{K} \iint_{-\infty}^{\infty} \frac{1}{|a|^2} W_f(a, b) \, \psi_{a,b}(t) \, da \, db \tag{8}$$

20.9 Why do we need to make the requirement $0 < K < \infty$. *Hint:* consider (8).

20.10 Choose a mother wavelet as $\psi(t) = t \, e^{-\frac{1}{2}t^2}$. Show that this mother wavelet satisfies the feasibility condition with $K = 4\pi$.

20.11 Based on the mother wavelet ψ defined in the previous exercise, compute the wavelet transform W_f of some functions:

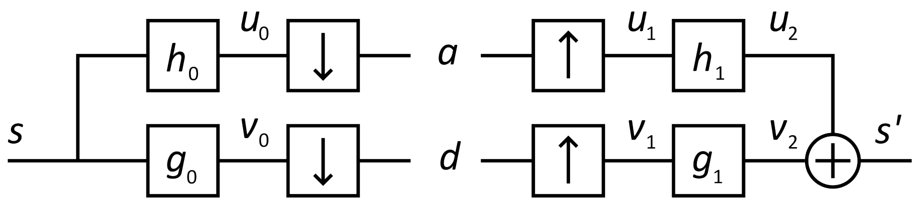
1. $f(t) = \delta(t)$
2. $f(t) = \text{step}(t)$
3. $f(t) = \text{rect}(t)$
4. $f(t) = \cos t$

20.12 Validate your result in the previous exercise, by in each case proving that f can be reconstructed from W_f in accordance to (8).

20.13 Translation and scaling of the variable of f have well-defined corresponding operations in the Fourier domain. What operations on W_f correspond to translation and scaling of f?

21 Filter banks

A two-channel filter bank looks like this



The down-sampling operation, denoted with ↓, implies that every second sample is removed, producing a sequence that has half as many samples per time unit. For example

$$a[k] = u_0[2\,k].$$

The up-sampling operation, denoted with ↑, implies that a zero is inserted between every sample in the input sequence. For example

$$u_1[k] = \begin{cases} a[k/2] & k \text{ even,} \\ 0 & k \text{ odd.} \end{cases}$$

The four discrete filters h_0, h_1, g_0, g_1 have Fourier transforms that are denoted as H_0, H_1, G_0, G_1. These transforms are all 2π-periodic functions.

21.1 Let $x[k]$ be a discrete signal, with corresponding Fourier transform $X(u)$. Show that the down-sampling operation followed by the up-sampling operation of $x[k]$ can be represented as the function $X'(u) = \frac{1}{2}[X(u) + X(\pi + u)]$ in the Fourier domain.

21.2 Use the result from the previous exercise to show that a necessary and sufficient condition for $s = s'$, for general s, is described by

$$H_0(u)H_1(u) + G_0(u)G_1(u) = 2, \tag{9}$$

$$H_0(u)H_1(u+\pi) + G_0(u)G_1(u+\pi) = 0. \tag{10}$$

21.3 (A) Given (9) and (10) it may seem reasonable to design a filter bank by choosing, e.g., filters h_0 and g_0 in an arbitrary way and then solve h_1 and g_1 from (9) and (10). Set $h_1 = (1, 1, 1)$ and $g_1 = (1, -1, 1)$. What filters h_0 and g_0 solve (9) and (10)? Why is this choice of filters not useful? *Hint:* start by determining H_1 and G_1 and insert them into (10) which allows you to express G_0 in terms of H_0. This can be inserted into (9) to give an expression for H_0 which, finally, can be transformed back to the filter h_0. *Hint:* the calculations may become simpler if you use the z-transform instead of the Fourier transform.

Instead of choosing two of the filters in an arbitrary way, there are better design methods. One is the *conjugate mirror filter bank*, where three of the filters depend on the fourth one, e.g., h_1, as follows

$$H_0(u) = \overline{H_1}(u), \qquad G_1(u) = e^{-iu}\,\overline{H_1}(u+\pi), \qquad G_0(u) = e^{iu}\,H_1(u+\pi) = \overline{G_1}(u). \tag{11}$$

21.4 Show that (11) leads to (10) being satisfied and (9) becomes the (O) condition

$$|H_1(u)|^2 + |H_1(u+\pi)|^2 = 2. \tag{12}$$

This means that (11) together with (12) lead to a perfect reconstruction filter bank.

21.5 Show that (11) can be formulated in the signal domain as

$$h_0[k] = \overline{h_1[-k]}, \qquad g_1[k] = (-1)^{1-k}\, \overline{h_1[1-k]}, \qquad g_0[k] = \overline{g_1[-k]}. \tag{13}$$

21.6 Assume that H_1 satisfies condition (O) in (12) and that the other three filters of the filter bank are given by (11). Show that in fact all four filters satisfy condition (O) in this case.

21.7 (A) Show that the first term in the left hand side of (12) can be written

$$|H_1(u)|^2 = \sum_{m,n\in\mathbb{Z}} h_1[m+n]\, \overline{h_1[n-m]}\, e^{-2ium} + \sum_{m,n\in\mathbb{Z}} h_1[m+n+1]\, \overline{h_1[n-m]}\, e^{-iu(2m+1)}.$$

21.8 (A) Use the result from the previous exercise to show that the entire left hand side of (12) can be written

$$|H_1(u)|^2 + |H_1(u+\pi)|^2 = 2\sum_m \langle\, h_1[\cdot] \mid h_1[\cdot - 2\,m]\,\rangle\, e^{-2ium}. \tag{14}$$

This is a general result for arbitrary h_1 and does not depend on the conjugate mirror filter property.

21.9 (A) Show that (12) and (14) together imply that

$$\langle\, h_1[\cdot] \mid h_1[\cdot - 2\,m]\,\rangle = \delta[m].$$

This means that the filter h_1 generates an orthonormal set of discrete functions by shifting it multiples of 2.

21.10 (A) Use the result from the previous exercise to show that the filter choices in (11) imply that each of the four filters g_0, g_1, h_0, h_1 generate an orthonormal set of discrete functions by shifting them multiples of 2.

21.11 (A) Show that the filter choice in (11) leads to

$$G_1(u)\, \overline{H_1}(u) + G_1(u+\pi)\, \overline{H_1}(u+\pi) = 0. \tag{15}$$

21.12 (A) Use the same type of approaches as previously to show that (15) implies that

$$\langle\, g_1[\cdot] \mid h_1[\cdot - 2m]\,\rangle = 0.$$

This means that all multiple-of-2-shifts of g_1 are orthogonal to all multiple-of-2-shifts of h_1.

21.13 (A) Show that the conjugate mirror filter bank implies analyzing a input signal with a set of orthogonal functions, corresponding to the convolution with h_0 and g_0 and subsequent down-sampling, and reconstructing the input signal again by a linear combination with the same set of orthogonal functions, corresponding to the up-sampling and convolution with h_1 and g_1. This implies that the conjugate mirror filter bank constitutes an *orthogonal filter bank.*

The simplest type of orthogonal filter bank is generated from $h_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

21.14 Show that this h_1 has a Fourier transform H_1 that satisfies (12).

21.15 What are the corresponding filters g_1, h_0, g_0?

21.16 With this choice of filters, what is the delay in the reconstructed signal s' relative to the input signal s if we shift all filters the minimum amount to make them causal?

Another conjugate mirror filter bank is generated by $h_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.

21.17 Show that this h_1 has a Fourier transform H_1 that satisfies (12).

21.18 What are the corresponding filters g_1, h_0, g_0?

21.19 If we shift all filters the minimum amount to make them causal, what is the delay in the reconstructed signal s' relative to the input signal s?

21.20 How would you characterize the four filters of the filter bank, e.g., as low-pass, band-pass, high-pass, etc?

21.21 (A) Give a general expression for the coefficients of an arbitrary FIR filter of length 4, $h_1 = (a, b, c, d)$, that produces a conjugate mirror filter bank. Assume real filter coefficients. *Hint:* use the constraints that emerge in exercise 21.9.

21.22 (A) Prove that the filters in a conjugate mirror filter bank must be of even length. *Hint:* use the constraints that emerge in exercise 21.9.

22 Discrete Wavelet transform

A scaling function $\phi(t)$ satisfies the following properties

(a) $\phi(t-k), k \in \mathbb{Z}$ is an ON-basis for some function space V_0.

(b) $\phi(t)$ can be written as a linear combination of the scaled functions $\phi(2\,t-k), k \in \mathbb{Z}$.

Here orthogonality is defined in terms of the standard scalar product between functions:

$$\langle\, \phi_1 \mid \phi_2 \,\rangle = \int_{-\infty}^{\infty} \phi_1(t)\, \overline{\phi}_2(t)\, dt.$$

22.1 Show that from (a) follows also that the function set $\{2^{1/2}\phi(2\,t-k), k \in \mathbb{Z}\}$ is an orthonormal set of functions.

22.2 Show that they span some space V_1 that includes V_0.

From property (b) of the scaling function follows that

$$\phi(t) = \sum_{k=-\infty}^{\infty} h[k]\, 2^{1/2}\, \phi(2\,t-k), \tag{16}$$

for some sequence $h[k], k \in \mathbb{Z}$.

22.3 Show that (16) implies that any integer shifted version of ϕ can be written as a linear combination of the ON-basis $\{2^{1/2}\psi(2\,t-k), k \in \mathbb{Z}\}$ according to

$$\phi(t-l) = \sum_{m=-\infty}^{\infty} h[m-2l]\, 2^{1/2}\, \phi(2\,t-m), \quad l \in \mathbb{Z}.$$

22.4 (A) Show that property (a) is equivalent to choosing ϕ such that

$$\sum_{k=\infty}^{\infty} |\Phi(v+2\pi k)|^2 = 1, \tag{17}$$

where $\Phi(u)$ is the Fourier transform of $\phi(t)$. *Hint:* use Parseval's theorem.

22.5 (A) Take the Fourier transform of (16) to show that

$$\Phi(u) = 2^{-1/2}\, H\left(\frac{u}{2}\right) \Phi\left(\frac{u}{2}\right) \tag{18}$$

where H is the discrete Fourier transform of the sequence $h[k]$, defined in (16).

22.6 Show that (18) inserted into (17) leads to

$$|H(u)|^2 + |H(u+\pi)|^2 = 2$$

Notice that this is the same condition as condition (O) in (12). *Hint:* split the sum into separate sums for even k and for odd k.

Given the Fourier transform $H(u)$, define a new function as

$$G(u) = e^{-iu}\,\overline{H(u+\pi)}. \tag{19}$$

$H(u)$ is the transform of a time-discrete function $h[k]$, therefore it is 2π-periodic. This means that also the new function, $G(u)$, is 2π-periodic and can be associated with a time-discrete function $g[k]$ as its inverse time-discrete Fourier transform. Finally, define a new function $\psi(t)$ in such a way that its Fourier transform $\Psi(u)$ is given by

$$\Psi(u) = \frac{1}{\sqrt{2}}\, G\left(\frac{u}{2}\right)\, \Phi\left(\frac{u}{2}\right),$$

22.7 Use Parseval's theorem, together with (17) and the fact that G is 2π-periodic to show that

$$g[n] = \langle \psi(x) | 2^{1/2}\phi(2x-n)\rangle.$$

22.8 (A) Show that

$$\psi(t-l) = \sum_{k=-\infty}^{\infty} (-1)^k\, \overline{h[k]}\, 2^{1/2}\, \phi(2t-k+2l+1), \quad l \in \mathbb{Z},$$

which means that all functions $\psi(t-l)$ lie in V_1.

22.9 (A) Show that

$$\langle\, \psi(t-m) \mid \phi(t-l)\,\rangle = 0 \quad \text{for } l, m \in \mathbb{Z},$$

which means that all functions $\psi(t-m)$ are orthogonal to V_0.

22.10 (A) Show that

$$\langle\, \psi(t-n) \mid \psi(t)\,\rangle = \delta[n] \quad \text{for } n \in \mathbb{Z},$$

which means that all functions $\psi(t-n)$ form an orthonormal set.

Let $f \in V_1$, i.e., it can be written as a linear combination of the ON-basis $2^{1/2}\,\phi(2\,t-k)$:

$$f(t) = \sum_n s[n]\, 2^{1/2}\, \phi(2t-n),$$

for some sequence s of coordinates of f relative the basis.

22.11 (A) Show that a function $f \in V_1$ also can be written as a linear combination of the two orthonormal sets $\{\phi(t-k)\}$ and $\{\psi(t-k)\}$ for $k \in \mathbb{Z}$:

$$f(t) = \sum_n a[n]\, \phi(t-n) + \sum_n d[n]\, \psi(t-n).$$

This means that these two ON sets together span V_1. Hint: show that sequences $a[n]$ and $d[n]$ have discrete Fourier transforms given by

$$A(u) = \frac{1}{2} S\left(\frac{u}{2}\right) \overline{H}\left(\frac{u}{2}\right) + \frac{1}{2} S\left(\frac{u}{2}+\pi\right) \overline{H}\left(\frac{u}{2}+\pi\right), \tag{20}$$

$$D(u) = \frac{1}{2} S\left(\frac{u}{2}\right) \overline{G}\left(\frac{u}{2}\right) + \frac{1}{2} S\left(\frac{u}{2}+\pi\right) \overline{G}\left(\frac{u}{2}+\pi\right). \tag{21}$$

22.12 (A) Show that (20) and (21) imply that sequences a and d are computed from s as

$$a = (s[\,\cdot\,] * \overline{h[-\,\cdot]})[2\,k] = \text{Convolve } s[k] \text{ with } \overline{h}[-k] \text{ and down-sample with a factor 2,}$$

$$d = (s[\,\cdot\,] * \overline{g[-\,\cdot]})[2\,k] = \text{Convolve } s[k] \text{ with } \overline{g}[-k] \text{ and down-sample with a factor 2.}$$

22.13 (A) Show that the sequence s can be reconstructed from sequences a and d as:

$$s[k] = \sum_{n=-\infty}^{\infty} a[n]\, h[k-2n] + \sum_{n=-\infty}^{\infty} d[n]\, g[n-2n]. \tag{22}$$

22.14 Show that (22) implies that sequence s is computed from sequences a and d as:

$$\begin{gathered} s = \text{up-sample } a \text{ with a factor 2 and convolve with } h\, + \\ \text{up-sample } d \text{ with a factor 2 and convolve with } g \end{gathered}$$

where "up-sample with a factor 2" means insert a zero between every original sample.

22.15 Show that the simple scaling function

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1, \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

leads to a ψ function in terms of the so-called Haar wavelet:

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}, \\ -1 & \frac{1}{2} \le t < 1, \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Hint: First find the sequence g and construct ϕ from g.

22.16 Draw both ϕ and ψ and use the figures to motivate why the two sets of functions given by $\phi(t-k)$, and $\psi(t-k), k \in \mathbb{Z}$, form two ON-bases which are mutually orthogonal.

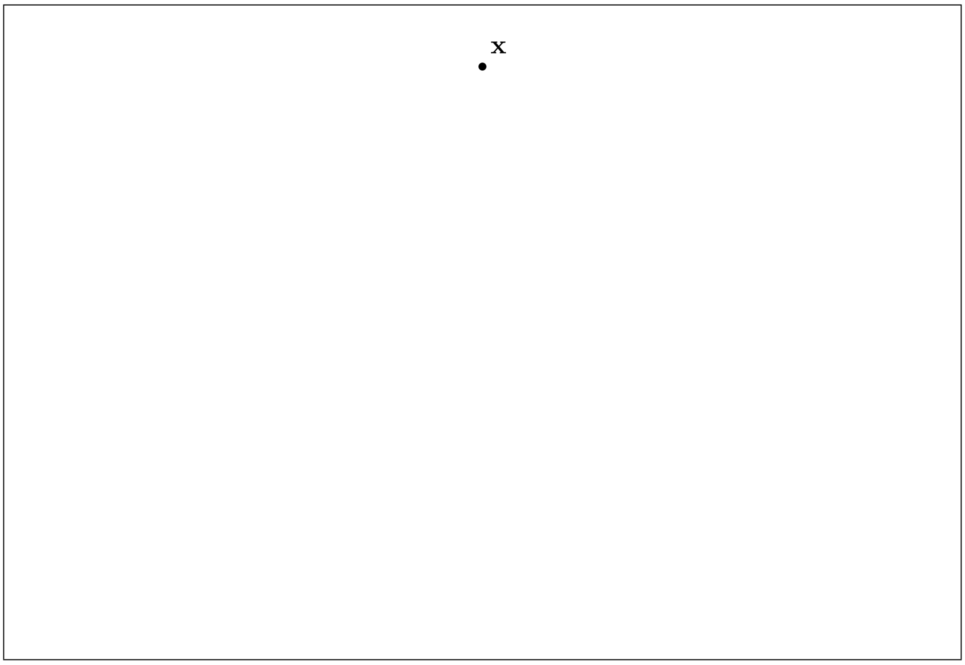
23 Stereo Geometry

23.1 In a pair of stereo images, a point in one image corresponds to a line in the other image. Motive why this is so.

23.2 The figure below shows a 3D point $\mathbf{x}$. This point is observed by two pinhole cameras; *camera1* and *camera2*. In the figure, draw points, lines, and planes, corresponding to the following geometric objects:

- The *camera centers*, $\mathbf{n}_1$ and $\mathbf{n}_2$, and the *virtual image planes*, $\mathbf{p}_1$ and $\mathbf{p}_2$, of each of the two cameras.
- The *projection lines* from $\mathbf{x}$ to each of the two cameras, $\mathbf{L}_1$ and $\mathbf{L}_2$, and the projections of $\mathbf{x}$ in the two images, as the *image points* $\mathbf{y}_1$ and $\mathbf{y}_2$.
- The *epipoles* (epipolar points) $\mathbf{e}_{12}$ and $\mathbf{e}_{21}$.
- The *epipolar lines*, $\mathbf{l}_1$ and $\mathbf{l}_2$, corresponding to the image points $\mathbf{y}_1$ and $\mathbf{y}_2$.

You need to draw each object in such a way that its defining properties are clearly illustrated, and also correctly label each of the requested objects in the figure.



23.3 The *fundamental matrix* $\mathbf{F}$ defines a matching constraint between points in stereo images, called the *epipolar constraint*. How is this constraint formulated? Describe the additional variables that occur in the constraint.

23.4 Two cameras, with camera centers at $\mathbf{n}_1$ and $\mathbf{n}_2$, observe a scene. The fundamental matrix $\mathbf{F}$, which defines the epipolar geometry between images from the two cameras, is known. How can you determine the projection of $\mathbf{n}_2$ into the first image from $\mathbf{F}$?

23.5 Two points in stereo images, $\mathbf{y}_1$ in image 1 and $\mathbf{y}_2$ in image 2, have been classified as being *corresponding points*.

a) What does this mean?
b) How can you test if they really are corresponding points?

23.6 The epipolar constraint $\mathbf{y}_1^T \mathbf{F} \mathbf{y}_2 = 0$ must be satisfied if $\mathbf{y}_1$ and $\mathbf{y}_2$ are the homogeneous coordinates of corresponding image points. The opposite may not be true, however. Explain how two image points can satisfy the epipolar constraint without corresponding to the same 3D point.

23.7 How are the epipoles $\mathbf{e}_{12}$ and $\mathbf{e}_{21}$ in a pair of stereo images related to the fundamental matrix?

23.8 How many constraints on the fundamental matrix, $\mathbf{F}$, are provided by one pair of corresponding image points?

23.9 Epipolar geometry in terms of the fundamental matrix $\mathbf{F}$ can be estimated from a set of $n \geq 8$ corresponding points in the images of two pinhole cameras, e.g., using the 8-point algorithm. The estimated $\mathbf{F}$ is well-defined only under certain basic assumptions. Describe at least one situation that leads to a degenerate $\mathbf{F}$.

23.10 The 8-point algorithm for estimating the fundamental matrix from stereo correspondences has a step which sets $\det(\mathbf{F}) = 0$ on the estimated fundamental matrix $\mathbf{F}$. What are the practical consequences for the epipolar lines and epipolar points if this step is not made?

23.11 Why is it correct to say that the fundamental matrix $\mathbf{F}$, which is a 3×3 matrix, has 7 degrees of freedom (rather than $3 \times 3 = 9$)?

23.12 Given two camera matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, the corresponding fundamental matrix is computed as $\mathbf{F} = [\mathbf{e}_{12}]_\times \mathbf{C}_2 \mathbf{C}_1^T (\mathbf{C}_1 \mathbf{C}_1^T)^{-1}$. Explain how $\mathbf{e}_{21}$ is determined from $\mathbf{C}_1$ and $\mathbf{C}_2$, and explain why it follows that $\det \mathbf{F} = 0$ from this expression.

23.13 Given two camera matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, the corresponding fundamental matrix $\mathbf{F}$ can be computed as $\mathbf{F} = [\mathbf{e}_{21}]_\times \mathbf{C}_2 \mathbf{C}_1^+$ where $\mathbf{e}_{21}$ is the epipole in image 2 and $\mathbf{C}_1^+$ is the pseudo-inverse of $\mathbf{C}_1$. Why is it sufficient to know only $\mathbf{C}_1$ and $\mathbf{C}_2$ in order to compute $\mathbf{F}$?

24 Triangulation

24.1 Reconstruction of a 3D point based on the positions of its projections in a pair of stereo images can be done using a linear method. How can you derive *linear equations* in the unknown 3D point $\mathbf{x}$ based on the usual camera projection equations:

$$\mathbf{y}_1 \sim \mathbf{C}_1 \, \mathbf{x} \qquad \text{and} \qquad \mathbf{y}_2 \sim \mathbf{C}_2 \, \mathbf{x}$$

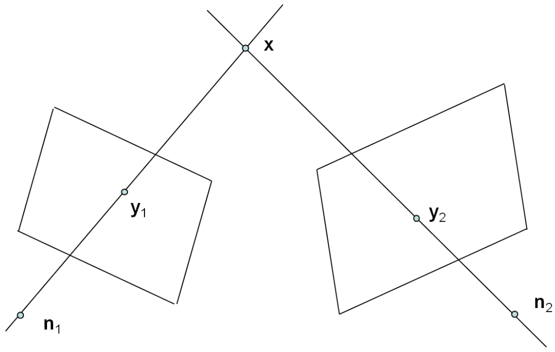
24.2 The mid-point method for triangulation of 3D points cannot give a reliable result for a particular case that can occur in practice. Which case?

24.3 Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two points in a pair of stereo images, one in each image. Why is not meaningful to triangulate a 3D point from $\mathbf{y}_1$ and $\mathbf{y}_2$ unless the two image points satisfy (approximately) the epipolar constraint?

24.4 We want to triangulate a 3D point from its projections onto a stereo image pair. This is done by determining the intersection of the two projection lines. Describe how we can determine if the two lines really intersect.

The figure below shows two cameras, with their image planes and camera centers $\mathbf{n}_1$ and $\mathbf{n}_2$. It also shows two image points, one in each image, with homogeneous coordinates given by $\mathbf{y}_1$ and $\mathbf{y}_2$.

Finally, the figure shows the *reprojection lines* of the two image points. How can you determine if the two lines intersect at some point $\mathbf{x}$ without explicitly computing the two reprojection lines? What information do you use instead?



25 Rectification

25.1 Rectification of two stereo images corresponds to a virtual adjustment of the principal axes of the two cameras in a specific way. In what way?

25.2 An alternative to rectification of images taken by a stereo rig is to make a mechanical adjustment of the cameras in the rig. What type of adjustment is that?

25.3 Where are the epipolar points located for a pair of *rectified* stereo images?

25.4 The two images produced by a stereo rig should be rectified. This is done by applying a rectifying homography transformation $\mathbf{H}_1$ and $\mathbf{H}_2$ onto the respective image. What is the defining algebraic relation for $\mathbf{H}_1$ and $\mathbf{H}_2$ to make them rectifying homographies? Which additional matrix appears in this relation?

25.5 A pair of stereo images are related by a fundamental matrix $\mathbf{F}$. We apply homographies, $\mathbf{H}_1$ and $\mathbf{H}_2$, to the two images in order to rectify the images. How are $\mathbf{H}_1, \mathbf{H}_2$, and $\mathbf{F}$ related in this case?

25.6 Why do we want to rectify stereo images?

25.7 What is the geometric consequence of applying rectifying transformations on a pair of stereo images?