TSBB09 Image Sensors
Camera calibration 2, Lecture E

- Camera calibration 2
  - Zhang's method for 3D camera calibration
  - Radial distortion
  - OpenCV: s extended version of Zhang’s method
  - Where is the camera center in a real lens?

- Literature
  - "A flexible new technique for camera calibration" by Zhengyou Zhang, Microsoft Research. Available as short article or long report.
  - Literature
    - "Introduction to Representations and Estimation in Geometry" (IREG) by Klas Nordberg
    - "Mathematical Toolbox for Studies in Visual Computation at Linköping University" by Klas Nordberg

Camera calibration, general

- Photogrammetry
  - A 3D calibration object is manufactured with good precision.
  - Disadvantage: expensive and complicated.
  - A 2D calibration object is manufactured with good precision. It can be a plane with squares. It is shown for the camera in different orientations. Zhang’s approach.
  - Advantage: cheap and simple. Lab task!

- Self-calibration
  - The camera is moving in a static scene.
  - Advantage: Flexible.
  - Disadvantage: The results are not always reliable.

See also Zhang, section 1: Motivations

3D Camera calibration according to Zhang

- A, R, and t in C=A[Rt] can be determined individually

 Calibration procedure, see Zhang: Section 3.3

 1) Print a pattern and attach it to a planar surface.
 2) Take a few images of the model plane under different orientations by moving the plane. Fig. 1.
 3) Detect feature points in the images and relate them to points in the world.
 4) Determine n C-matrices by calibrating n homographies. Determine A and [Rt] from the n C-matrices.
 5) Estimate the coefficients of the lens radial distortion from the linear least square solution of an equation system.
 6) Refine all parameters, including the lens radial distortion parameters in a non-linear minimization algorithm.

5) and 6) are not included in the lab “Camera Calibration”, but in the lab “Camera Calibration 2”.

1,2) Hold the pattern in some different orientations and take images
3) Detect interesting points in the images and relate them to points in the world

$$(u_i, v_i) \text{ corresponds to } (X_i, Y_i)$$

From $n$ calibration planes we can determine $n$ C-matrices by calibrating $n$ homographies using the previously described technique.

4) Determine $A$ and $[Rt]$ from the $n$ C-matrices

$s(u, v, 1)^T = A[Rt] \cdot (X, Y, 1)^T = C \cdot (X, Y, 1)^T$ \hspace{1cm} Eq. (24)

Depending on the variable $s$, $C$ can only be determined up to a scale factor. Zhang set $C_{33} = 1$ and introduces $\lambda$ as scale factor.

$$\lambda \cdot A \cdot \begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & 1 \end{pmatrix}$$

$r_1$ and $r_2$ are derived from rotation and are orthonormal!

4) Determine $A$ and $[Rt]$, cont.

Proof on next slide!

5) Proof of the constraints (3) and (4)

From the previous slide:

$$\lambda \cdot A \cdot [r_1 \ r_2 \ t] = [h_1 \ h_2 \ h_3]$$ \hspace{1cm} Before Eq. (3)

Two important constraints:

$$h_1^T A^{-T} A^{-1} h_2 = 0$$ \hspace{1cm} Eq. (3)

$$h_i^T A^{-T} A^{-1} h_1 = h_i^T A^{-T} A^{-1} h_2$$ \hspace{1cm} Eq. (4)

Proof on next slide!
4) Determine A and \([\text{Rt}]\) from the \(n\) C-matrices, cont.

**Form a \(B\)-matrix and a \(b\)-vector:**

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{12} & B_{22} & B_{23} \\
B_{13} & B_{23} & B_{33}
\end{bmatrix} = \{\text{insert and calculate}\} = \begin{bmatrix}
\frac{1}{\alpha^2} & -\frac{\gamma}{\alpha^2 \beta} & \frac{v_0 \gamma - u_0 \beta}{\alpha^2 \\
-\frac{\gamma}{\alpha^2 \beta} & \frac{1}{\beta^2} & -\frac{v_0 \beta}{\alpha^2} \\
\frac{v_0 \gamma - u_0 \beta}{\alpha^2 \beta^2 + 1} & -\frac{v_0 \beta}{\alpha^2 \beta^2} & \frac{v_0^2 + \beta^2}{\beta^2 + 1}
\end{bmatrix}
\]

**Note:**

- That the \(B\)-matrix is symmetric
- And that we can solve \(\alpha, \beta, \ldots\) from it.

**Eq. (5)**

\[
b = [B_{11}, B_{12}, B_{13}, B_{22}, B_{23}, B_{33}]^T
\]

**Eq. (6)**

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This is valid:

\[
\begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
h_{12} & h_{22} & h_{23} \\
h_{13} & h_{23} & h_{33}
\end{bmatrix} = \begin{bmatrix}
h_1 \\
h_2 \\
h_3
\end{bmatrix}
\]

Then:

\[
h_i^T B h_j = v_{ij}^T b
\]

**Eq. (7)**

**Check of:**

\[
\begin{bmatrix}
v_{12} \\
(v_{11} - v_{22})^T
\end{bmatrix} b = 0
\]

**Eq. (8)**

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This is a homogenous equation system, which can be solved by using SVD-technique, see next lecture, “Short about camera geometry...” from previous lecture or “Mathematical Toolbox …”
4) Determine A and [Rt] from the n C-matrices, cont.

When \( b \) is known, \( B \) is simply obtained. Then the parameters \( \alpha, \beta, \gamma, u_0, v_0 \) are obtained from Eq. (5). \( A \) is now determined!

\[ B = A^{-T} A^{-1} \]

Note: It is slightly better to solve \( A \) from \( B \) above by using Cholesky decomposition (see Mathematical Toolbox). Then the parameters \( \alpha, \beta, \gamma, u_0, v_0 \) can be directly obtained from \( A \) and they will probably be more accurate.

How many calibration planes are needed?

- One calibration plane gives one calibration matrix \( C \).
- One calibration matrix \( C \) gives one Eq. (8) with 2 equations.
- There are 5 unknowns in \( A \).
- If the skew \( \gamma = 0 \), there are 4 unknowns in \( A \).
- How many calibration planes, at least, are needed to determine \( A \)?
  - 3 planes are needed.
  - 2 planes are needed if \( \gamma = 0 \).

\[ C = A[Rt] \]

is determined up to 8 parameters by 1 calibration plane. There are 6 degrees of freedom in \( [Rt] \), 3 rotation angles and 3 translation directions. Consequently 8-6=2 equations are obtained for solving \( A \) from one calibration plane.

5) Radial distortion

- Radial distortion is the most common

- Other types of distortion: the human eye of an astigmatic person, fisheye-lenses, telescope

Radial distortion can be included in the calibration procedure.
5) Radial distortion, examples

Example from Aftonbladet: Image inside the "frimurar" room. (Anders Björck, Hasse Aro and the Swedish king are members.)

5) Radial distortion, equations

\[ \begin{align*}
(u, v, 1)^T &= A \cdot (u_0, v_0, 1)^T = A \cdot (x, y, 1)^T \\
\end{align*} \]

inner parameters: \( a, \beta, \gamma, u_0, v_0 \)

undistorted image coordinates: \((u, v)\)
distorted image coordinates: \((\bar{u}, \bar{v})\)
distorted normalized image coordinates: \((\bar{x}, \bar{y})\)

Model:
\[ \begin{align*}
\bar{x} &= x + x \cdot (k_1 r^2 + k_2 r^4) \\
\bar{y} &= y + y \cdot (k_1 r^2 + k_2 r^4)
\end{align*} \]

\( r^2 = x^2 + y^2 \)

k1 and k2 are the coefficients of radial distortion

\( \bar{u} = u + (u - u_0) \cdot (k_1 r^2 + k_2 r^4) \)  
\( \bar{v} = v + (v - v_0) \cdot (k_1 r^2 + k_2 r^4) \)  
\[ \text{Eq. (11)} \]

\( \bar{u} = \alpha x + \gamma y + u_0 \)  
\( \bar{v} = \beta y + v_0 \)  
\[ \text{Eq. (12)} \]

The center of the radial distortion is the same as the principal point.
5) Radial distortion, equations

\[
\begin{align*}
\hat{u} &= u + (u - u_0) \cdot \left( k_1(x^2 + y^2) + k_2(x^2 + y^2)^2 \right) \\
\hat{v} &= v + (v - v_0) \cdot \left( k_1(x^2 + y^2) + k_2(x^2 + y^2)^2 \right)
\end{align*}
\]

Eq. (11)

Eq. (12)

Given \( m \) points in \( n \) images, we can stack all equations together to obtain in total \( 2mn \) equations, or in matrix form as

\[
Dk = d
\]

where \( k = [k_1, k_2]^T \).

The linear least-square solution is given by:

\[
k = (D^T D)^{-1} D^T d
\]

Eq. (13)

6) Refine the parameter estimation in a non-linear minimization algorithm

\[
s(u, v, 1)^T = A[Rt] \cdot (X, Y, 1)^T
\]

Eq. (24)

Magnusson’s notation:

\[
\bar{m} = A[Rt] \cdot \bar{M}
\]

Eq. (1)

Can be solved by the Levenberg-Marquardt algorithm, lsqnonlin in Matlab

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \left\| m_{ij} - \hat{m}(A, k_1, k_2, R, t, M_j) \right\|^2
\]

Eq. (14)

Correction for radial distortion (in the report by Zhang)

OpenCV: s extended version of Zhang’s method

- Contains a more advanced model for radial distortion:

\[
\begin{align*}
\bar{x} &= x \cdot \frac{1 + k_1 r^2 + k_2 r^4 + k_3 r^6}{1 + k_4 r^2 + k_5 r^4 + k_6 r^6} + 2p_1 xy + p_2 (r^2 + 2x^2) \\
\bar{y} &= y \cdot \frac{1 + k_1 r^2 + k_2 r^4 + k_3 r^6}{1 + k_4 r^2 + k_5 r^4 + k_6 r^6} + p_1 (r^2 + 2y^2) + 2p_2 xy
\end{align*}
\]

- \( k_1 \) and \( k_2 \) are Zhang’s original coefficients for radial distortion
- \( p_1 \) and \( p_2 \) are tangential distortion
- For barrel distortion, typically \( k_1 > 0 \)
- For pincushion distortion, typically \( k_1 < 0 \)
Tangential distortion

- Tangential distortion occurs when the lens and the image plane are not parallel. The tangential distortion coefficients $p_1$ and $p_2$ model this type of distortion.

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Alternative model for radial distortion: The arctan model

Let the image be described in polar coordinates: $(r, \theta)$.

Then

$$r_{out} = \frac{\arctan(r_{in} \cdot \gamma)}{\gamma}$$

$\gamma$ is small, e.g. $\gamma=0.001$.
Where is the camera center in a real lens?

- The camera center is at EP (the entrance pupil) i.e. the apparent position of the aperture.

Reference: Theory of the "No-Parallax" Point in Panorama Photography
Version 1.0, February 6, 2006
Rik Littlefield (rj.littlefield@computer.org)