Optimization

Computer Vision, Lecture 14
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Why Optimization?

- Computer vision algorithms are usually very complex
  - Many parameters (dependent)
  - Data dependencies (non-linear)
  - Outliers and occlusions (noise)
- Classical approach
  - Trial and error (hackers' approach)
  - Encyclopedic knowledge (recipes)
  - Black-boxes + glue (hide problems)

Why Optimization?

- Establishing CV as scientific discipline
  - Derive algorithms from first principles (*optimal solution*)
  - Automatic choice of parameters (*parameter free*)
  - Systematic evaluation (*benchmarks on standard datasets*)

Optimization: howto

1. Choose a *scalar* measure (objective function) of success
   - From the benchmark
   - Such that optimization becomes *feasible*
   - Project functionality onto *one dimension*
2. Approximate the world with a model
   - Definition: allows to make *predictions*
   - Purpose: makes optimization *feasible*
   - Enables: *proper* choice of dataset

Similar to economics (money rules)

Optimization: howto

3. Apply suitable framework for model fitting
   - This lecture
   - Systematic part (1 & 2 are ad hoc)
   - Current focus of research
4. Analyze resulting algorithm
   - Find *appropriate* dataset
   - Ignore runtime behavior (*highly non-optimized Matlab code*) ;-)

Examples

- Relative pose (F-matrix) estimation:
  - Algebraic error (quadratic form)
  - Linear solution by SVD
  - Robustness by random sampling (RANSAC)
  - Result: F and inlier set
- Bundle adjustment
  - Geometric (reprojection) error (quadratic error)
  - Iterative solution using LM
  - Result: camera pose and 3D points

Taxonomy

- Objective function
  - Domain/manifold (algebraic error, geometric error, data dependent)
  - Robustness (explicitly in error norm, implicitly by Monte-Carlo approach)
- Model / simplification
  - Linearity (limited order), Markov property, regularization
- Algorithm
  - Approximate / analytic solutions (minimal problem)
  - Minimal solutions (over-determined)

Taxonomy example: KLT

- Objective function
  - Domain/manifold: grey values / RGB / …
  - Robustness: no (quadratic error, no regularization)

$$\varepsilon(\mathbf{d}) = \sum_{\mathbf{x}\in\mathcal{N}} w(\mathbf{x})|f(\mathbf{x}-\mathbf{d}) - g(\mathbf{x})|^2$$

- Model: Brightness constancy, image shift

$$f(\mathbf{x}-\mathbf{d}) = g(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{N}$$

  - local linearization (Taylor expansion)

$$f(\mathbf{x}-\mathbf{d}) \approx f(\mathbf{x}) - \mathbf{d}^T \nabla f(\mathbf{x}) \qquad \nabla f = \left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right]^T$$

Taxonomy: KLT

- Algorithm
  - iterative solution of normal equations (Gauss-Newton)

$$\left(\sum_{\mathcal{R}} w(\mathbf{x})\nabla f(\mathbf{x})\nabla^T f(\mathbf{x})\right)\mathbf{d} = \sum_{\mathcal{R}} w(\mathbf{x})\nabla f(\mathbf{x})(f(\mathbf{x}) - g(\mathbf{x}))$$

$$\mathbf{Td} = \mathbf{r}$$

  - **T**: structure tensor (orientation tensor from outer product of gradients)

$$\nabla f \nabla^T f = \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix}$$

- Block matching: same cost & model, but discretized shifts

Regularization and MAP

- In maximum a-posteriori (MAP), the objective (or loss) ε consists of a data term (fidelity) and a prior

$$\min_{\mathbf{d}} \varepsilon_{\text{data}}(f(\mathbf{d}), g) + \varepsilon_{\text{prior}}(\mathbf{d})$$

$$\Leftrightarrow \max_{\mathbf{d}} \exp(-\varepsilon_{\text{data}}(f(\mathbf{d}), g))\exp(-\varepsilon_{\text{prior}}(\mathbf{d}))$$

$$\Leftrightarrow \max_{\mathbf{d}} P(g|\mathbf{d})P(\mathbf{d})$$

$$\Leftrightarrow \max_{\mathbf{d}} P(\mathbf{d}|g)$$

- A common prior is a smoothness term (regularizer)

MAP Example: KLT

- Assume a prior probability for the displacement: $P(\mathbf{d})$ (e.g. Gaussian distribution from a motion model)
- In logarithmic domain, we now have two terms in the cost function:

$$\varepsilon(\mathbf{d}) = \sum_{\mathbf{x}\in\mathcal{N}} w(\mathbf{x})|f(\mathbf{x}-\mathbf{d}) - g(\mathbf{x})|^2 + \lambda\|\mathbf{d} - \mathbf{d}_{\text{pred}}\|^2$$

  - The standard KLT term
  - A term that *drags* the solution towards the predicted displacement (like Kalman filtering)

Demo: KLT

- KLTdemo.m

Image Reconstruction

- Assume that $\mathbf{f}$ is an unknown image that is observed through the linear operator $\mathbf{G}$: $\mathbf{f}_0 = \mathbf{G}\mathbf{f} + \text{noise}$
- Example: blurring, linear projection
- Goal is to minimize the error $\mathbf{f}_0 - \mathbf{G}\mathbf{f}$
- Example: squared error
- Assume that we have a prior probability for the image: $P(\mathbf{f})$
- Example: we assume that the image should be smooth (small gradients)

Image Reconstruction

- Minimizing

$$\varepsilon(\mathbf{f}) = \frac{1}{2}(|\mathbf{G}\mathbf{f} - \mathbf{f}_0|^2 + \lambda(|\mathbf{D}_x\mathbf{f}|^2 + |\mathbf{D}_y\mathbf{f}|^2))$$

- Gives the normal equations

$$\mathbf{G}^T\mathbf{G}\mathbf{f} - \mathbf{G}^T\mathbf{f}_0 + \lambda(\mathbf{D}_x^T\mathbf{D}_x\mathbf{f} + \mathbf{D}_y^T\mathbf{D}_y\mathbf{f}) = 0$$

- Such that

$$\mathbf{f} = (\mathbf{G}^T\mathbf{G} + \lambda(\mathbf{D}_x^T\mathbf{D}_x + \mathbf{D}_y^T\mathbf{D}_y))^{-1}\mathbf{G}^T\mathbf{f}_0$$

Gradient Operators

- Taylor expansion of image gives

$$f(x+h,y) = f(x,y) + hf_x(x,y) + \mathcal{O}(h^2)$$
$$f(x-h,y) = f(x,y) - hf_x(x,y) + \mathcal{O}(h^2)$$

- Finite left/right differences give

$$\partial_x^+ f = \frac{f(x+h,y) - f(x,y)}{h} + \mathcal{O}(h^2)$$
$$\partial_x^- f = \frac{f(x,y) - f(x-h,y)}{h} + \mathcal{O}(h^2)$$

- Often needed: products of derivative operators

Gradient Operators

- Squaring left (right) difference $(\partial_x^+)^2 f$ gives linear error in h
- Squaring central difference $\frac{f(x+h,y)-f(x-h,y)}{2h}$ gives a quadratic error in h, but leaves out every second sample
- Multiplying left and right difference

$$\partial_x^+ \partial_x^- f = \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} = \Delta_x f$$

gives quadratic error in h (usual discrete Laplace operator)

Demo: Image Reconstruction

- IRdemo.m

Robust error norms

- Alternative to RANSAC (Monte-Carlo)
- Assume quadratic error: *influence* of change f to $f+\partial f$ to the estimate is linear (why?)
- Result on set of measurements: mean
- Assume absolute error: influence of change is constant (why?)
- Result on set of measurements: median / median filter
- In general: sub-linear influence leads to robust estimates, but *non-linear*

Smoothness / regularizer

- Quadratic smoothness term: influence linear with height of edge
- Total variation (TV) smoothness (absolute value of gradient): influence constant
- With quadratic measurement error: Rudin-Osher-Fatemi (ROF) model (Physica D, 1992)

$$\min_f \frac{\|f - f_0\|^2}{2\lambda} + \sum_{i,j} |(\nabla f)_{i,j}|$$

TV Image Inpainting / Convex Optimization

- Note that many problems (including quadratic and TV) are convex optimization problems
- A good first approach: map these problems to a standard solver, e.g. CVXPY (S. Diamond & S. Boyd)
- Example: minimize the total variation (TV) of an image

$\sum_{i,j} |(\nabla f)_{i,j}|$ under the constraint of a subset of known image values f

```
prob=Problem(Minimize(tv(X)),[X[known] == MG[known]])
opt_val = prob.solve()
```

Demo: TV Inpainting

- inpaint.py

Algorithmic Taxonomy

- Minimal problems (e.g. 5 point algorithm)
  - Fully determined solution(s)
  - Analytic solvers (polynomials, Gröbner bases)
  - Numerical methods (Dogleg, Newton-Raphson)
- Overdetermined problems (e.g. KLT, BA, IR)
  - Minimization problem
  - Numerical solvers
    - Levenberg-Marquardt (interpolation Gauss-Newton and gradient descent / trust region)
  - Graph-based approaches

Non-linear LS, Dog Leg

- For comparison: LM $\mathbf{r}(\mathbf{x}+\boldsymbol{\delta}) \approx \mathbf{r}(\mathbf{x}) + \mathbf{J}\boldsymbol{\delta}$

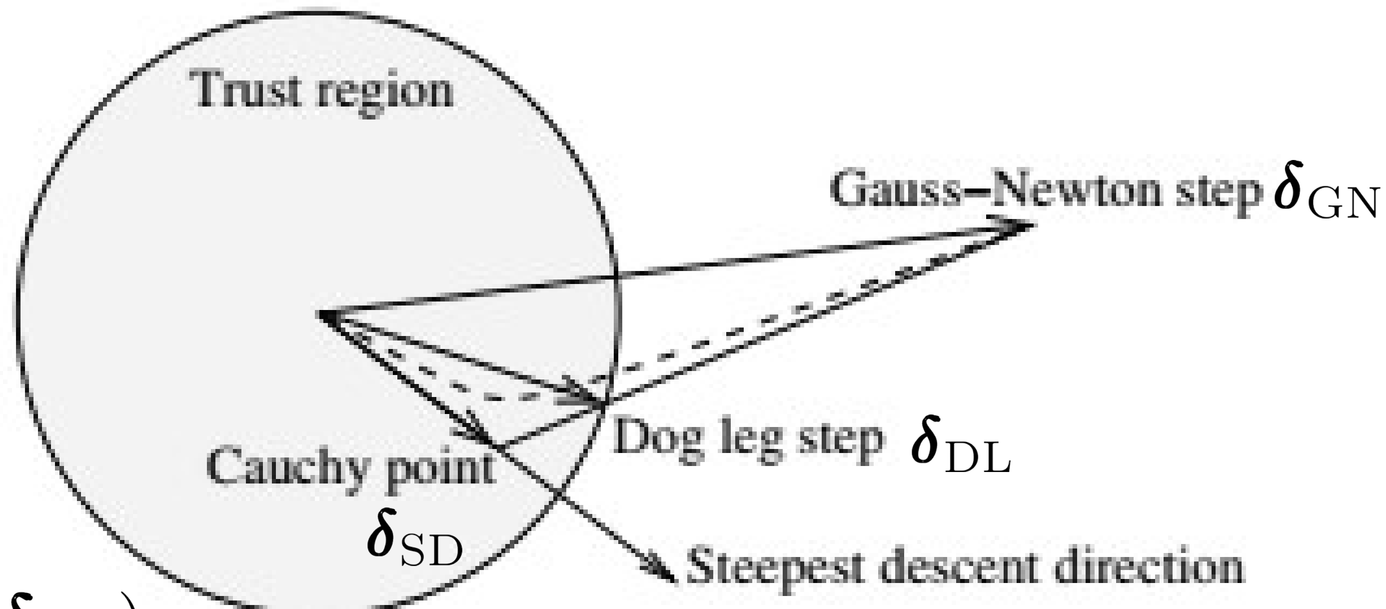
$$(\mathbf{J}^T\mathbf{J} + \lambda\,\mathbf{diag}(\mathbf{J}^T\mathbf{J}))\,\boldsymbol{\delta} = \mathbf{J}^T\mathbf{r}(\mathbf{x})$$

$$x_j \mapsto x_j + \delta_j \qquad J_{ij} = \frac{\partial r_i}{\partial x_j}$$

- More efficient: replace damping factor λ with trust region radius Δ

method	abbr.	properties
steepest descent	SD	$\boldsymbol{\delta} = \mathbf{J}^T\mathbf{r}$
Gauss-Newton	GN	$\mathbf{J}^T\mathbf{J}\boldsymbol{\delta} = \mathbf{J}^T\mathbf{r}$
Levenberg-Marquardt	LM	combines SD and GN by damping factor
Dog Leg	DL	combines SD and GN by trust region radius Δ

Dog Leg (basic idea)



1. initialize radius Δ =1
2. compute gain factor

3. if gain factor >0

$$\mathbf{x}_{\mathrm{new}} = \mathbf{x} + \underbrace{\boldsymbol{\delta}_{\mathrm{SD}} + \alpha(\boldsymbol{\delta}_{\mathrm{GN}} - \boldsymbol{\delta}_{\mathrm{SD}})}_{\boldsymbol{\delta}_{\mathrm{DL}}}$$

$$\|\boldsymbol{\delta}_{\mathrm{SD}}\| \le \Delta\ , \quad 0 \le \alpha \le 1\ , \quad \|\boldsymbol{\delta}_{\mathrm{DL}}\| = \Delta$$

4. grow/shrink Δ and update gain factor
5. if update and residual nonzero goto 3

Graph Algorithms

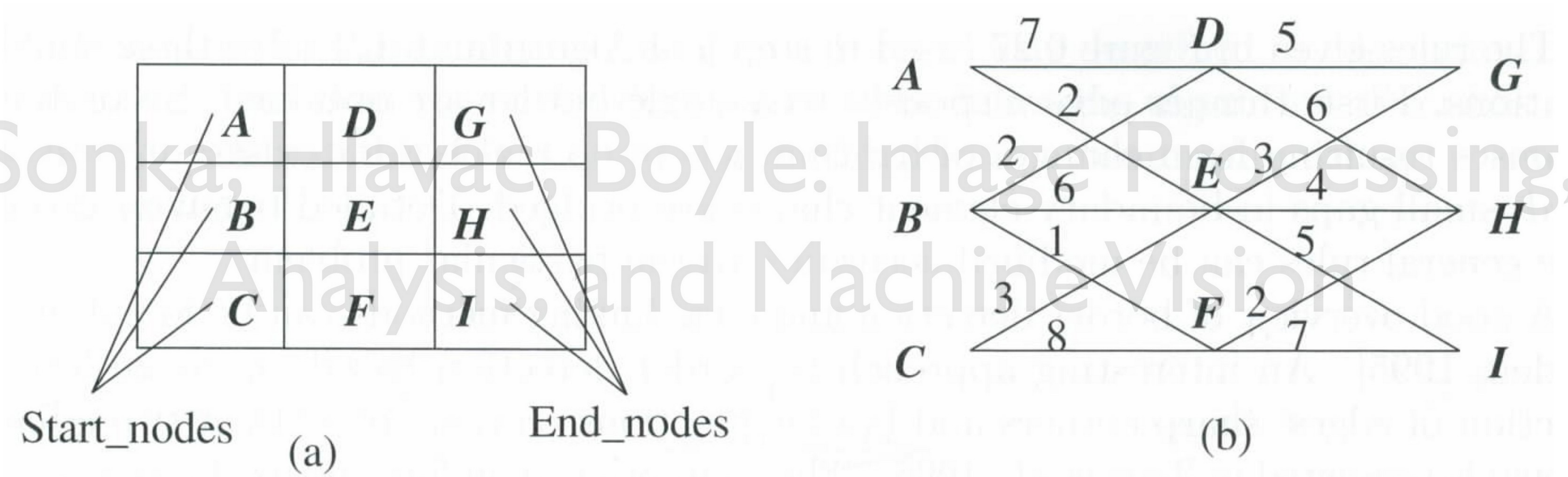
- All examples so far: vectors as solutions, i.e. finite set of (pseudo) continuous values
- Now: discrete (and binary) values
- Directly related to (labeled) graph-based optimization
- In probabilistic modeling (on regular grid): Markov random fields

Graphs

- Graph: algebraic structure $G=(V, E)$
- Nodes $V=\{v_1,v_2,\ldots,v_n\}$
- Arcs $E=\{e_1,e_2,\ldots,e_m\}$, where e_k is incident to
  - an unordered pair of nodes $\{v_i,v_j\}$
  - an ordered pair of nodes (v_i,v_j) (directed graph)
  - degree of node: number of incident arcs
- Weighted graph: costs assigned to nodes or arcs

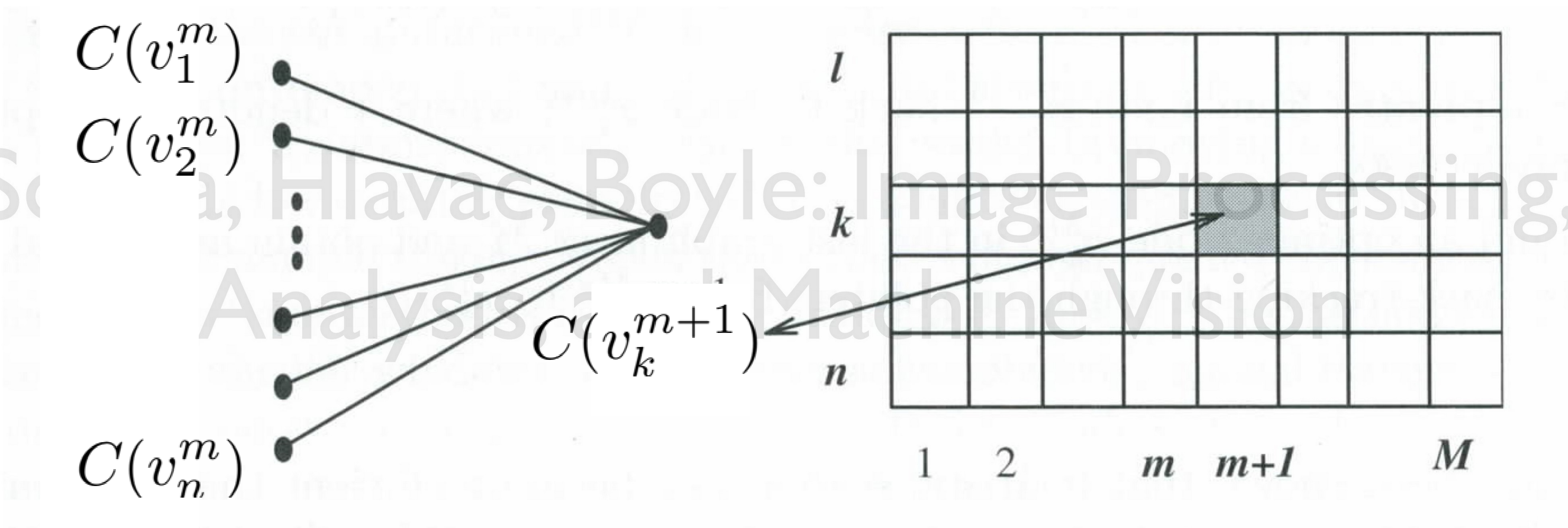
1D: Dynamic Programming

- Problem: find optimal path from source node *s* to sink note *t*
- Principle of Optimality: If the optimal path *s*-*t* goes through *r*, then both *s*-*r* and *r*-*t*, are also optimal



1D: Dynamic Programming

- $C(v_k^{m+1})$ is the new cost assigned to node v_k
- $g^m(i,k)$ is the partial path cost between nodes v_i and v_k



1D: Dynamic Programming

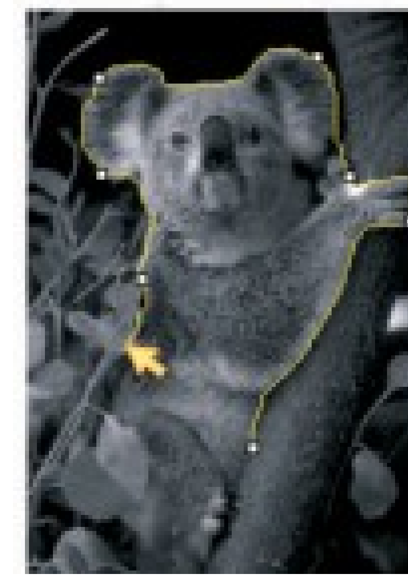
- $C(v_k^{m+1})$ is the new cost assigned to node v_k
- $g^m(i,k)$ is the partial path cost between nodes v_i and v_k

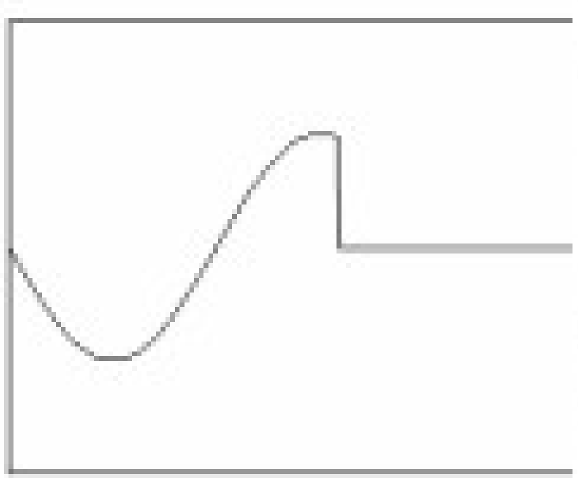
$$C(v_k^{m+1}) = \min_i \left(C(v_i^m) + g^m(i,k) \right)$$

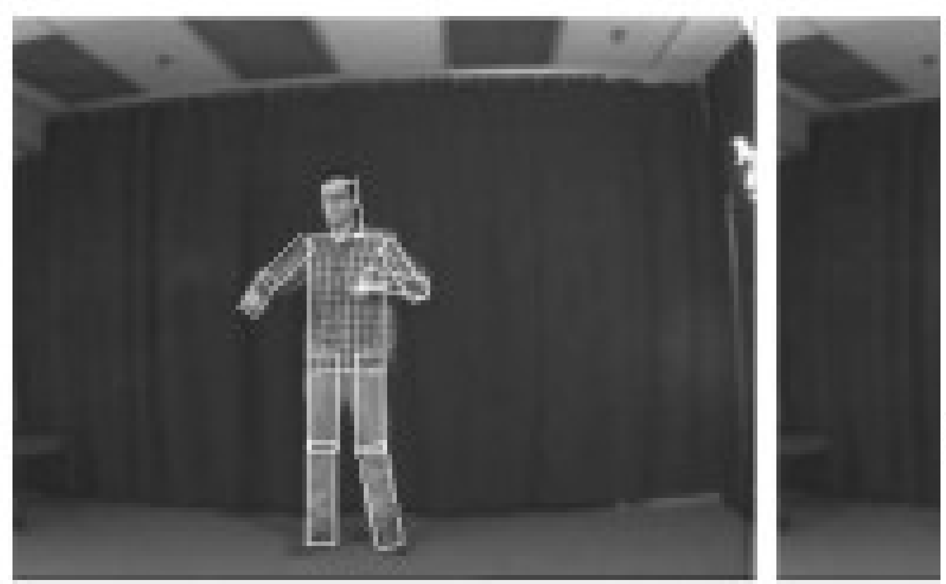
$$\min \left(C(v^1, v^2, \ldots, v^M) \right) = \min_{k=1,\ldots,n} \left(C(v_k^M) \right)$$

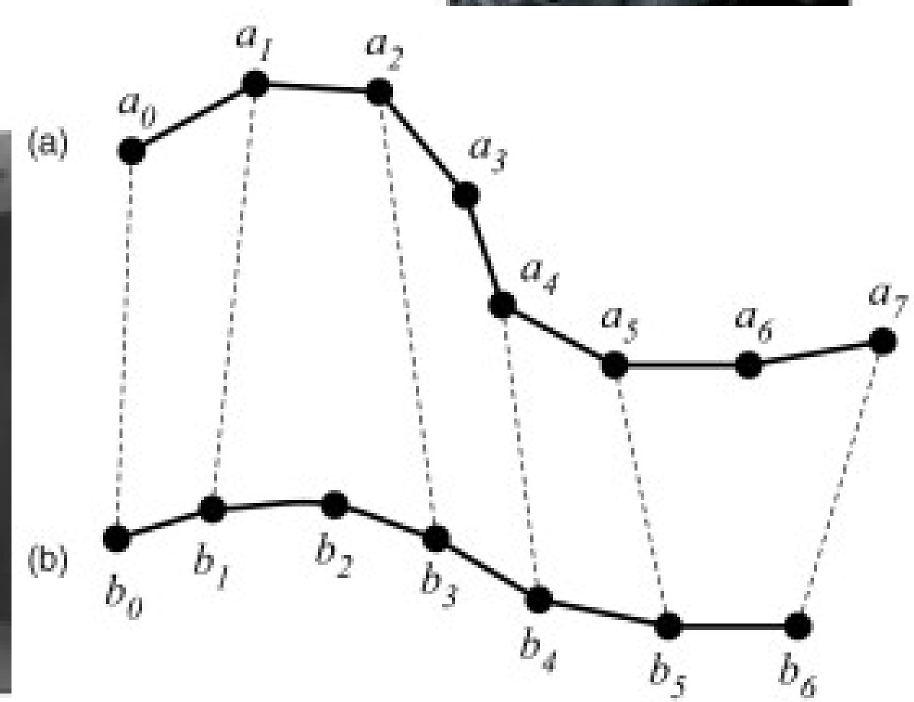
Examples

- Shortest path computation (contours / intelligent scissors)
- 1D signal restoration (denoising)
- Tree labeling (pictorial structures)
- Matching of sequences (curves)







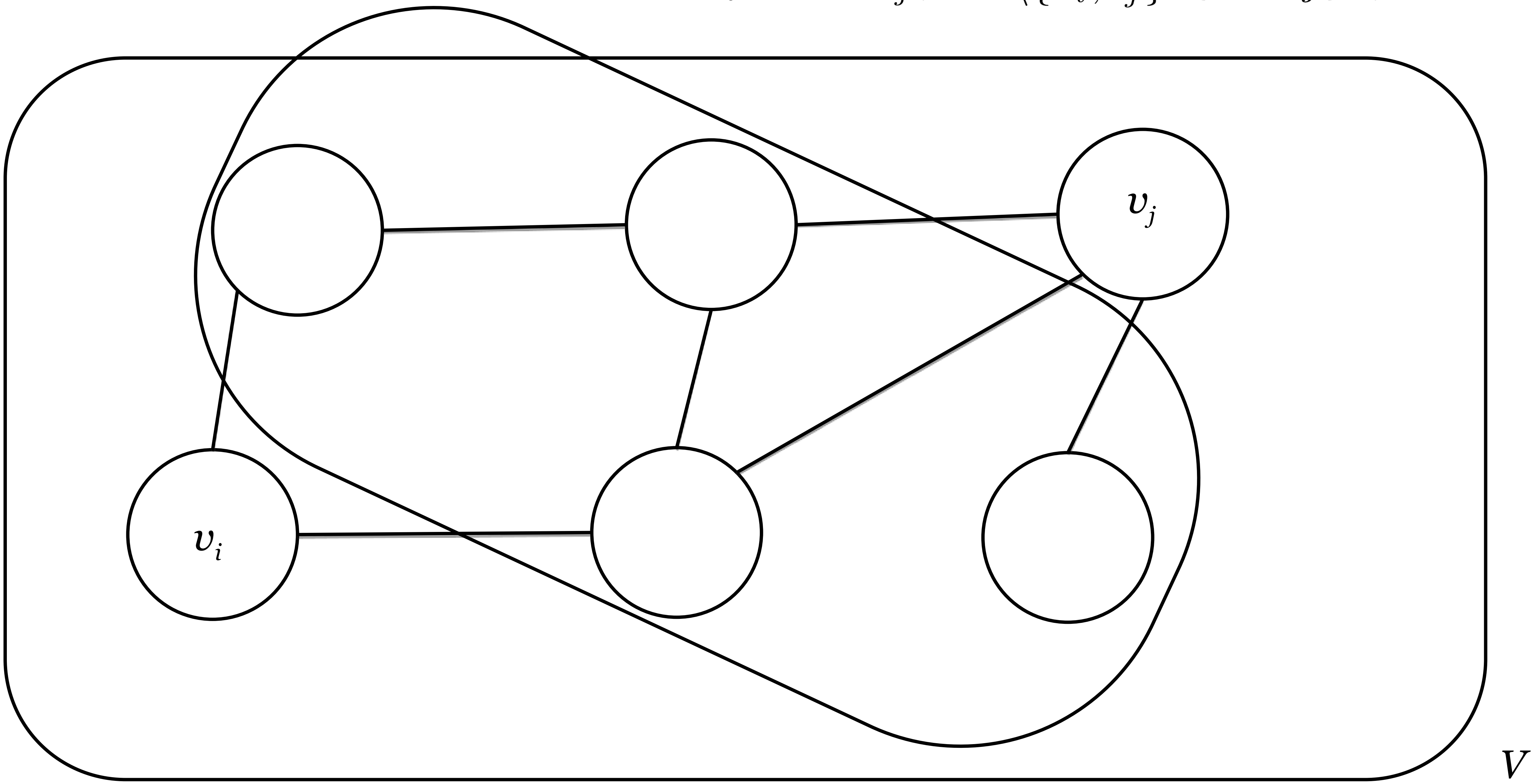


Markov property

- Markov chain: memoryless process with r.v. X
- Markov random field (undirected graphical model): random variables (e.g. labels) over nodes with Markov property (conditional independence)
  - Pairwise $X_{v_i} \perp\!\!\!\perp X_{v_j} | X_{V \setminus \{v_i, v_j\}} \quad \{v_i, v_j\} \notin E$
  - Local $X_v \perp\!\!\!\perp X_{V \setminus (\{v\} \cup N(v))} | X_{N(v)}$
  - Global $X_A \perp\!\!\!\perp X_B | X_S$ where every path from a node in A to node in B passes through S

Conditional Independence

$$X_{v_i} \perp\!\!\!\perp X_{v_j} | X_{V \setminus \{v_i, v_j\}} \quad \{v_i, v_j\} \notin E$$



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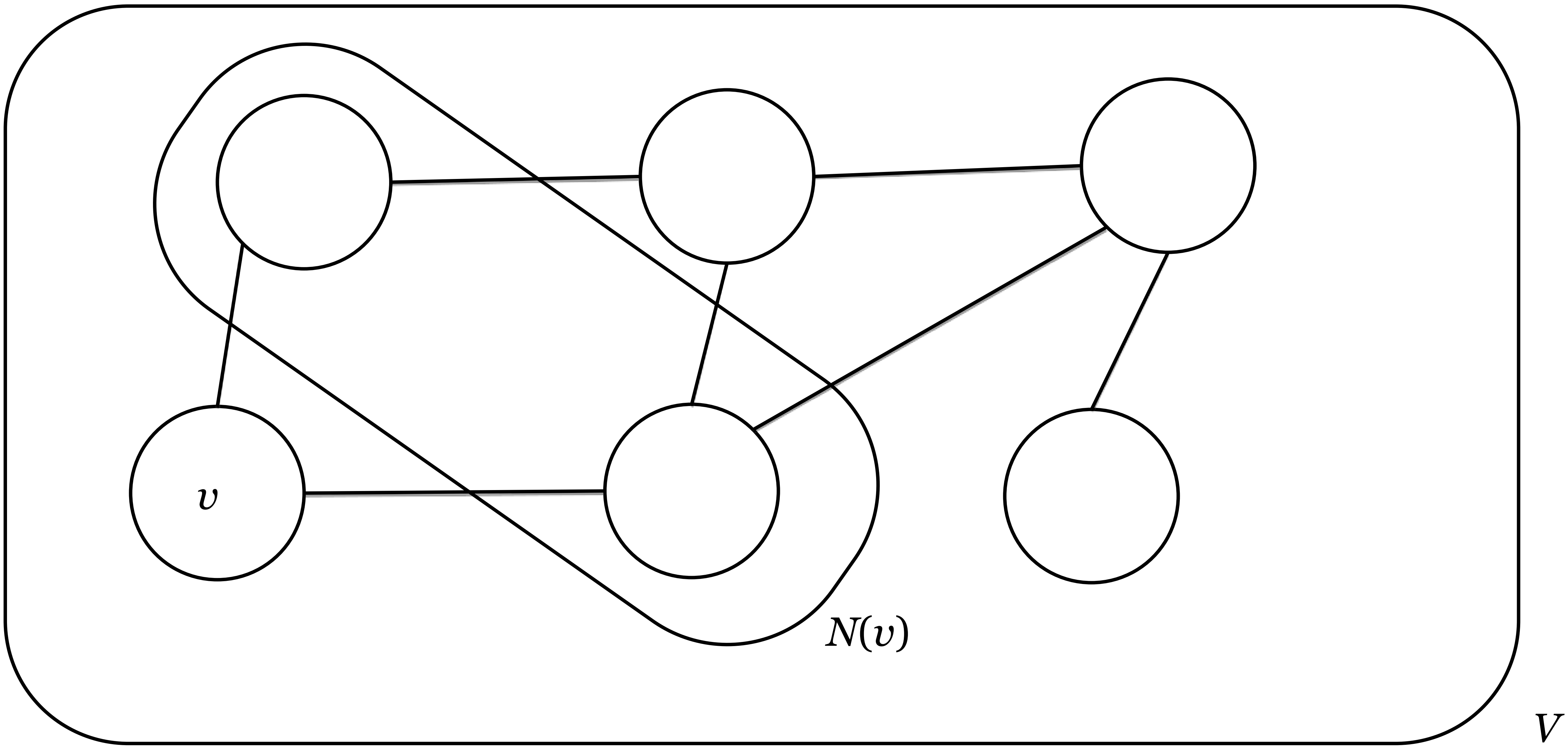
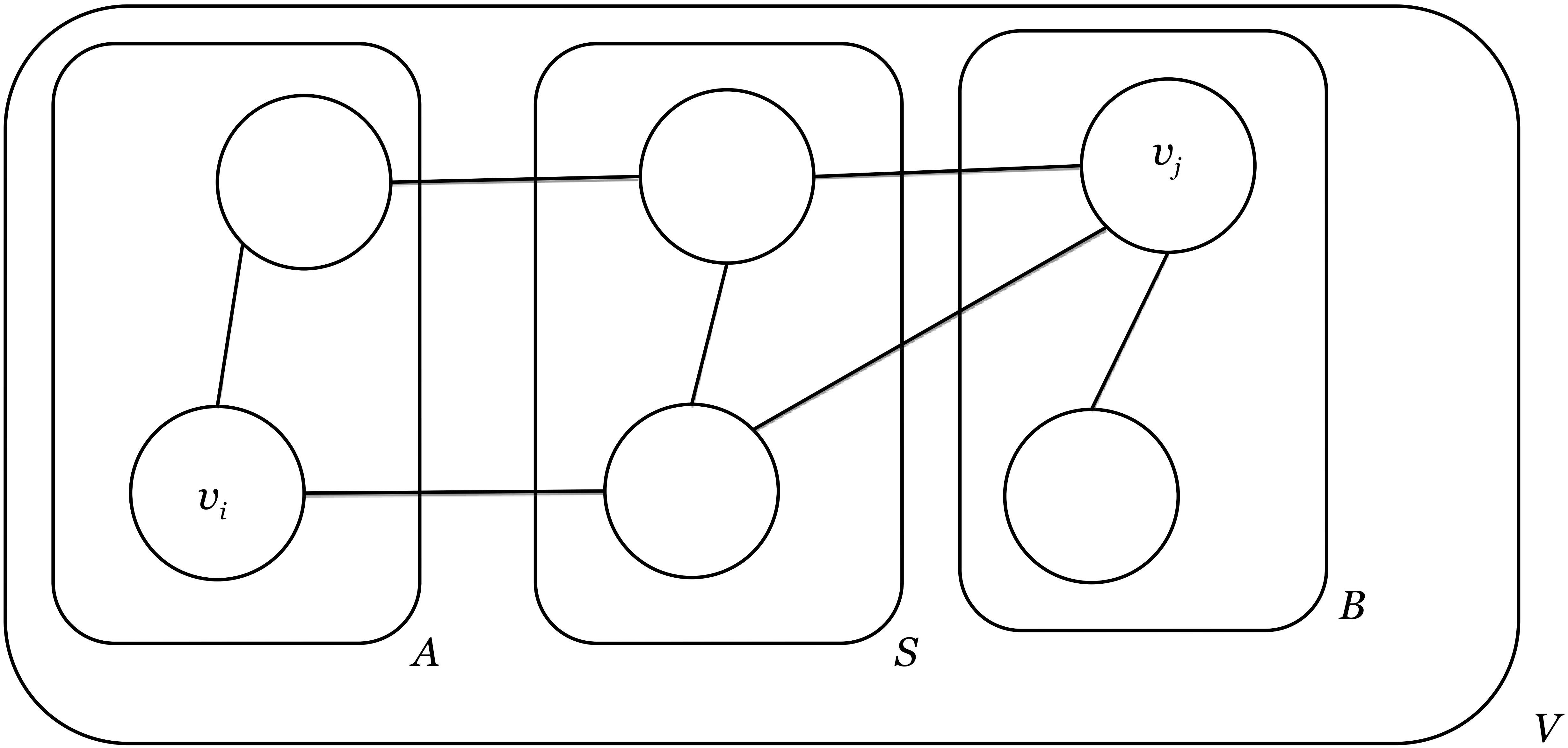
Conditional Independence

$$X_v \perp\!\!\!\perp X_{V\setminus(\{v\}\cup N(v))} | X_{N(v)}$$



Conditional Independence

$$X_A \perp\!\!\!\perp X_B | X_S$$



Terminology

- If joint density strictly positive: Gibbs RF
- Ising model (interacting magnetic spins), energy given as Hamiltonian function

$$\varepsilon(X_V) = -\sum_{e_k=\{v_i,v_j\}\in E} J_{e_k} X_{v_i} X_{v_j} - \sum_{v_j} h_{v_j} X_{v_j}$$

- General form

$$\varepsilon(X_V) = \lambda \sum_{e_k=\{v_i,v_j\}\in E} V(X_{v_i}, X_{v_j}) + \sum_{v_j} D(X_{v_j})$$

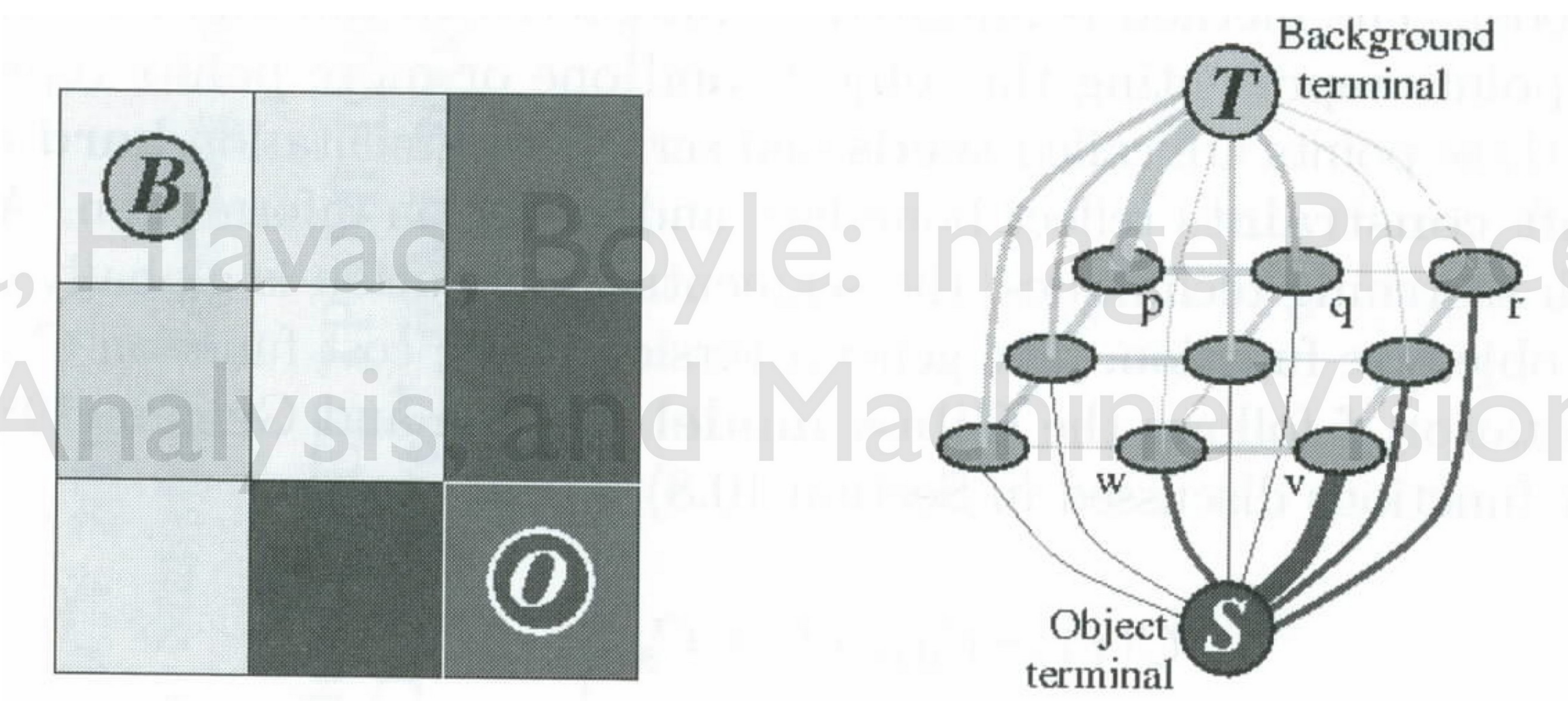
- Configuration probability $P(X_V) \propto \exp(-\varepsilon(X_V))$

Gibbs Model / Markov Random Field

- Attempts to generalize dynamic programming to higher dimensions unsuccessful
- Minimize $C(f) = C_{\mathrm{data}}(f) + C_{\mathrm{smooth}}(f)$ using arc-weighted graphs $G_{\mathrm{st}} = (V \cup \{s, t\}, E)$
- Two special terminal nodes, source *s* (e.g. object) and sink *t* (e.g. background) hard-linked with seed points

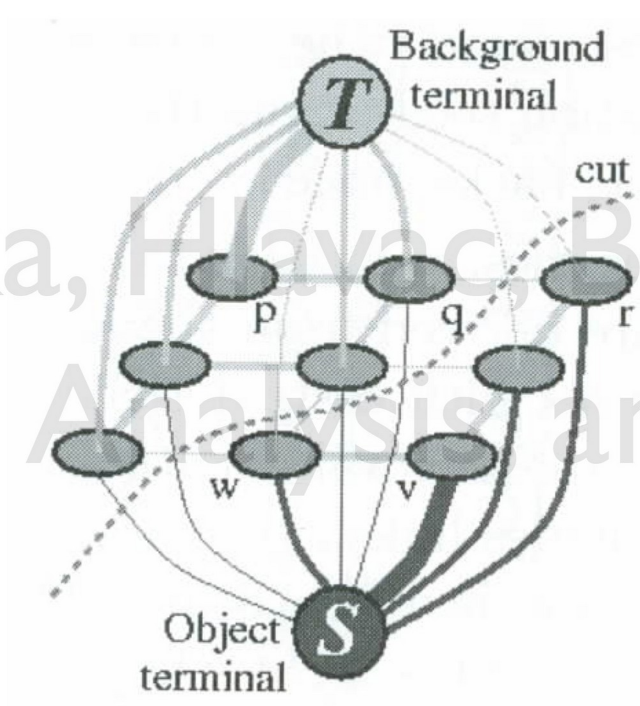
Graph Cut: Two types of arcs

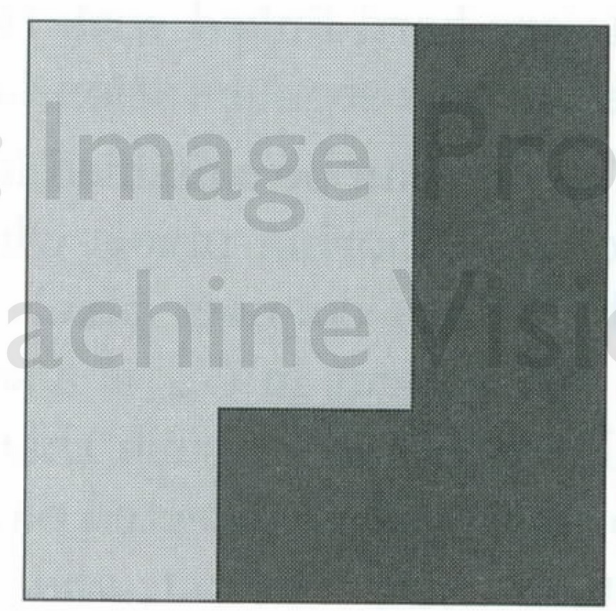
- n-links: connecting neighboring pixels, cost given by the smoothness term *V*
- t-links: connecting pixels and terminals, cost given by the data term *D*



Graph Cut

- *s*-*t* cut: set of arcs, such that the nodes and the remaining arcs form two disjoint graphs with points sets *S* and *T*
- cost of cut: sum of arc cost
- minimum *s*-*t* cut problem (dual: maximum flow problem)





Graph Cut

- n-link costs: large if two nodes belong to same segment, e.g. inverse gradient magnitude, Gaussian function, Potts model
- t-link costs:
  - *K* for hard-linked seed points (*K* > maximum sum of data terms)
  - 0 for the opposite seed point
- Submodularity

$$V(\alpha,\alpha) + V(\beta,\beta) \leq V(\alpha,\beta) + V(\beta,\alpha)$$

Demo: Graph cut

- gc_example.m

Examples / Discussion

- Binary problems solvable in polynomial time (albeit slow)
  - Binary image restoration
  - Bipartite matching (perfect assignment of graphs)
- N-ary problems (more than two terminals) are NP-hard and can only be approximated (e.g. α-expansion move)
- Continuous relaxation methods: move from hard constraints to functions
  - Continuous labels or continuous differences between labels (derivatives)