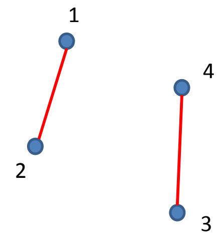
# Spectral clustering

Lecture 2

Spectral clustering:
from confusion to clarity

# A simple example

- Two ideal clusters, with two points each




$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

- Ideally permuted
- Ideal affinities



# Indicator vectors

- Each cluster has an *indicator vector*, represented by a binary vector that contains ``1´´ for points in the cluster and ``0´´ otherwise:

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Indicator vectors of distinct clusters must be orthogonal!



# A simple example

- Clearly, we can decompose **A** as

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\mathbf{c}_1$ and $\mathbf{c}_2$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$$

# Eigensystem of **A**

- An eigenvalue decomposition of **A** gives

$$\text{normalized eigenvectors} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{corresponding eigenvalues} = \begin{pmatrix} 2 & 2 & 0 & 0 \end{pmatrix}$$



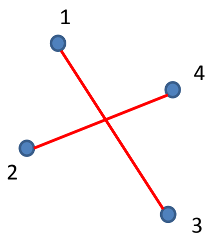
# Initial idea

- To each cluster there is a non-zero eigenvalue in **A**
  - Number of clusters = number of non-zero eigenvalues in **A**
- To each such eigenvalue/cluster, the corresponding normalized eigenvector is a scaled version of the corresponding indicator vector



# Permutations of **A**

- Two ideal cluster, with two points each




$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$



# Eigensystem of permuted **A**

- An eigenvalue decomposition of **A** gives

$$\text{normalized eigenvectors} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{corresponding eigenvalues} = \begin{pmatrix} 2 & 2 & 0 & 0 \end{pmatrix}$$

Initial idea holds: permutations of the points carries over to permutations of the elements of the eigenvectors

# Eigensystem of permuted **A**

- The goal of spectral clustering is to determine the permutation of **A** that turns it into a block diagonal form
- This is done by analyzing the eigensystem of **A**



# A glitch (I)

- In this case: the non-zero eigenvalues are equal
  - Any linear combination of the first two eigenvectors is also an eigenvector of the same eigenvalue
  - Any small perturbation of **A** can make a large change in the eigenvectors
  - Eigenvectors will not correspond to the indicator vectors



# A glitch (I)

$$\mathbf{A} = \begin{pmatrix} 1 & 0.99 & 0.01 & 0.02 \\ 0.99 & 1 & 0.01 & 0.03 \\ 0.01 & 0.01 & 1 & 0.98 \\ 0.02 & 0.03 & 0.98 & 1 \end{pmatrix}$$

Again ideally ordered but with some noise

Approximate numerical values

$$\text{normalized eigenvectors} = \begin{pmatrix} 0.53 & -0.46 & -0.28 & 0.65 \\ 0.54 & -0.46 & 0.27 & -0.65 \\ 0.46 & 0.54 & -0.65 & -0.27 \\ 0.47 & 0.53 & 0.65 & 0.27 \end{pmatrix}$$

$$\text{corresponding eigenvalues} = \begin{pmatrix} 2.02 & 1.95 & 0.02 & 0.01 \end{pmatrix}$$



# A glitch (I)

- It is still the case the there are two dominant eigenvalues, corresponding to the two separate clusters
- But the corresponding eigenvectors do not directly reveal the points of each cluster
  - A linear combination of them, however, will!

# Fixing the glitch (I)

- Define, for $n$ points and $k$ clusters:

  **U** = $n \times k$ matrix containing the normalized eigenvectors of the $k$ largest eigenvalues of **A** in its columns

  – Each row in **U** corresponds to a data point



# Fixing the glitch (I)

- In the last numerical example:

$$\begin{pmatrix} 0.53 & -0.46 & -0.28 & 0.65 \\ 0.54 & -0.46 & 0.27 & -0.65 \\ 0.46 & 0.54 & -0.65 & -0.27 \\ 0.47 & 0.53 & 0.65 & 0.27 \end{pmatrix}$$

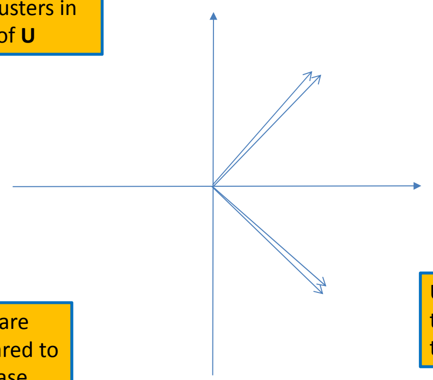
= **U**

We notice that rows of **U** corresponding to the same cluster are approximately equal



# Fixing the glitch (I)

Points belonging to the same cluster clusters in the row space of **U**



Cluster points are rotated compared to the previous case

Use *k*-means clustering to find these clusters in the row space of **U**



# A clustering algorithm, (I)

- Assume $n$ points and $k$ clusters
- Compute $n \times n$ affinity matrix **A**
- Compute the eigensystem of **A**
- There should be $k$ non-zero eigenvalues
- Set **U** to hold the corresponding normalized eigenvectors in its columns
- Apply *k*-means clustering on the row space of **U** to find the $k$ clusters

# An observation (I)

- The **self-affinity** of each point is a constant value found in the diagonal of **A**
- Changing this constant means adding a term to **A** that is proportional to the identity matrix:

$$\mathbf{A}' = \mathbf{A} + \alpha\,\mathbf{I}$$



# An observation (I)

- **A** and **A**' have the same eigenvectors but their eigenvalues differ:

$$\lambda'_k = \lambda_k + \alpha$$

$$\mathbf{A}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Same eigenvectors as before

With $\alpha$ = -1

$$\text{corresponding eigenvalues} = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$$



# An observation (I)

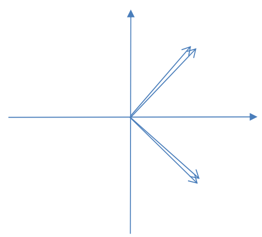
- In the literature it is common to set the self-affinity to zero
  - All diagonal elements of **A** are zero
- The phrase
  "*k* eigenvalues of **A** are non-zero"
  should then be replaced by
  "*k* eigenvalues of **A** are large"



# An observation (II)

In the previous numerical example:

- Not only are the row vectors of **U** for points in different clusters distinct, they are orthogonal



- This is not a coincidence!

# An observation (II)

- Assuming that the $k$ largest eigenvalues of **A** are approximately equal (to $\lambda$):

$$\mathbf{A} + \alpha\, \mathbf{I} = \lambda\, \mathbf{U}\, \mathbf{U}^T$$

The inner product of rows from different clusters correspond to zero affinity in an ideal **A**

In the ideal case: rows in **U** belonging to different clusters must be orthogonal
  - But not necessarily of unit length!
  - We will return to this later on!



# A clustering algorithm (II)

- Assume $n$ points and $k$ clusters
- Compute $n \times n$ affinity matrix **A** (0 in diagonal!)
- Compute eigensystem of **A**
- There should be $k$ “large” eigenvalues which are approximately equal
- Set **U** to hold the corresponding normalized eigenvectors in its columns
- Apply $k$-means clustering on the row space of **U** to find the $k$ clusters



# An observation (III)

- Using the “larger” or “significant” eigenvalues of **A** can be replaced with “equal to zero” or “close-to-zero” eigenvalues of related matrices
- We need to modify **A** accordingly
- Leads to the Laplacian **L** of **A**, and we do clustering based on the eigensystem of **L** instead of **A**



# Degree matrix

- We define

**D** = diagonal matrix { $d_{ii}$ }

where $d_{ii}$ = sum of row/column $i$ in **A**

as the *degree* matrix of **A**

# A simple example

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

**In the ideal case:**
The indicator vectors are eigenvectors both to **A** and **D** and have eigenvalues {1, 2} relative both **A** and **D**



# Laplacian

- Formally, we define

**L = D – A**

as the *Laplacian* of **A**

- The indicator vectors are eigenvectors also of **L**, with eigenvalue 0



# Properties of L

In the ideal case:

- **L** has the same eigenvectors as **A** and **D**
- **L** has eigenvalues = 0 for the indicator vectors

In general (also with noise):

$a_{ij} \geq 0$ for affinity matrix **A**

$$\mathbf{u}^T \mathbf{L}\, \mathbf{u} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (u_i - u_j)^2$$

**L** is positive semi-definite!



# Properties of L

In the general case (also with noise):

- Positive semi-definite
- Sum along rows/columns of **L** vanishes
- There is always one eigenvalue = 0 in **L**
- Corresponding eigenvector = is **1** (constant 1)
  - **1** is the sum of all indicator vectors!

# Properties of **L**

In the ideal case

- **L** has a block structure,
  - Non-zero blocks representing fully connected components
  - Zero blocks representing unconnected components



# Properties of **L**

From this follows:

1. If **u** is a cluster indicator vector $\Rightarrow$ **u** is an eigenvector of **L** with eigenvalue 0
2. If **u** is an eigenvector of **L** with eigenvalue 0 $\Rightarrow$ **u** is a linear combination of the cluster indicator vectors

From this follows:

1. The number of eigenvalues = 0 in **L** is = $k$ ($k$= number of clusters)
2. The corresponding eigenvectors span the space of indicator vectors



# A clustering algorithm (III)
## Unnormalized spectral clustering

- Assume $n$ points and $k$ clusters
- Compute $n \times n$ affinity matrix **A**
- Compute **D**, and compute **L** = **D** - **A**
- Compute eigensystem of **L**
- There should be $k$ "zero" eigenvalues
- Set **U** to hold the corresponding normalized eigenvectors in its columns
- Apply $k$-means clustering on the row space of **U** to find the $k$ clusters



# Fiedler's method for $k$ = 2

- The Laplacian **L** has always (even for noisy data) an eigenvalue $\lambda_1 = 0$
- Corresponding eigenvector $\mathbf{e}_1$ is **1**
- If $k$ = 2, there should be a second eigenvalue = 0, or at least close to zero
- Corresponding eigenvector denoted $\mathbf{e}_2$
- The row space of $\{\mathbf{e}_1, \mathbf{e}_2\}$ should form clusters in two orthogonal directions

# Fiedler's method for $k = 2$

- Consequently, the signs of the elements in $\mathbf{e}_2$ must be indicators of the two classes
- For example:

  "+" means class 1

  "-" means class 2
- We don't really need $\mathbf{e}_1$
- Only the signs of the elements in $\mathbf{e}_2$
  - $\mathbf{e}_2$ is often referred to as the *Fiedler vector*



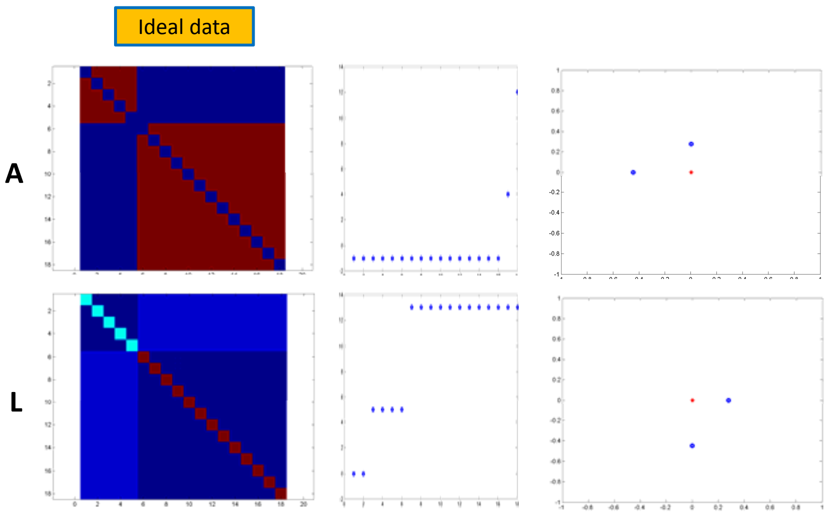
# An observation (IV)

- Should we do clustering on **A** or on **L**?
- For ideal data
  - full connections internally in each component
  - no connects between components

  there is, in general, no difference in the result
- For non-ideal data, (= in practice) the results differ
  - Often: clustering based on **L** is better!

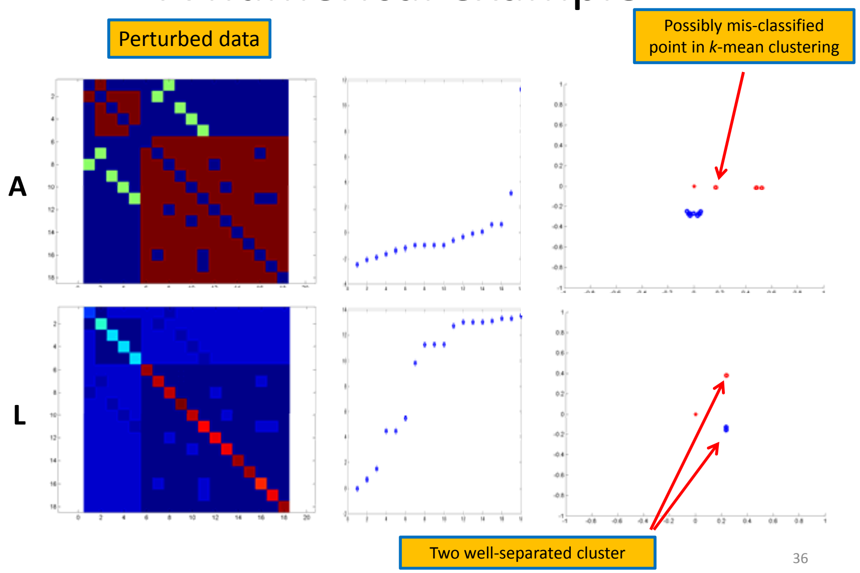


# A numerical example






# A numerical example



# Analysis

- It can be shown that the clustering on **A** is equivalent to solving the *mincut* problem of the corresponding graph [see von Luxburg]
- Prefers to cut fewer edges, even if they have higher affinity, than more edges even when each has lower affinity
- In our example: there is a risk of cutting the edge between point 1 and the rest of the points in the first cluster



# Analysis

- It can be shown that the clustering on **L** is for $k$ = 2 approximates the solution of the *Ratio-cut* problem of the corresponding graph [see von Luxburg]
- Normalizes the cost of a cut with the number of vertices of each sub-graph
- In our example: reduces the risk of cutting the edge between point 1 and the rest of the points in the first cluster



# A glitch (II)

- The last clustering algorithm works well for arbitrary $k$, but assumes that the number of points in each cluster, $n_k$, is approximately equal
- Otherwise, eigenvalues which are "zero" and "non-zero" may mix in the data of real data



# A simple example

- An ideal **A** with $k$ = 2 and $n_1$ and $n_2$ points in each cluster

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} n_1 \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} n_2 \end{matrix}$$

# A simple example

- Eigensystem of **A**

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_2 \end{matrix}$$

$$\text{corresponding eigenvalues} = \begin{pmatrix} n_1 - 1 & n_2 - 1 & \underbrace{-1 \quad \dots \quad -1}_{n\,-\,2} \end{pmatrix}$$



# A simple example

$$\mathbf{D} = \begin{pmatrix} n_1-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & n_1-1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & n_1-1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_2-1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_2-1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_2-1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_2-1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} n_2 \end{matrix}$$



# A simple example

- Eigensystem of **D**

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_2 \end{matrix}$$

$$\begin{pmatrix} n_1 - 1 & n_2 - 1 & \underbrace{n_1 - 1 \quad \dots \quad n_1 - 1}_{n_1\,-\,1} & \underbrace{n_2 - 1 \quad \dots \quad n_2 - 1}_{n_2\,-\,1} \end{pmatrix}$$



# A simple example

- Eigensystem of **L**

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n_2 \end{matrix}$$

$$\begin{pmatrix} 0 & 0 & \underbrace{n_1 \quad \dots \quad n_1}_{n_1\,-\,1} & \underbrace{n_2 \quad \dots \quad n_2}_{n_2\,-\,1} \end{pmatrix}$$

# A glitch (II)

- For this example:
  - There are 2 eigenvalues approximately = 0
  - There are $n_1$ - 1 eigenvalues approximately = $n_1$
  - There are $n_2$ - 1 eigenvalues approximately = $n_2$
- If $n_2 >> n_1$ and with sufficiently noisy data:
  - The first two types of eigenvalues can mix
  - Also their eigenvectors will mix
  - Poor clustering performance



# Fixing the glitch (II)

- There are (at least) two ways of fixing this glitch, where both **normalize** the Laplacian **L** before computing the eigensystem:
  - Normalized spectral clustering according to Shi & Malik (2000) [Not covered here!]
    - Based on EVD of $\mathbf{L}_{rw} = \mathbf{D}^{-1}\mathbf{L}$
  - Normalized spectral clustering according to Ng *et al* (2002) [Next!]



# Fixing the glitch (II)

- We define a normalized Laplacian as

$$\mathbf{L}_{sym} = \mathbf{D}^{-1/2}\,\mathbf{L}\,\mathbf{D}^{-1/2}$$

- Referred to as the *normalized symmetric* Laplacian



# Fixing the glitch (II)

- $\mathbf{L}_{sym}$ is symmetric, and (in the ideal case):
  - Diagonal elements in $\mathbf{L}_{sym}$ are all = 1
  - Off-diagonal elements sum to -1 along row and columns
  - Same number of eigenvalues = 0 as **L**
  - Same block structure as **L**
  - Same eigenvectors as **L**
  - An non-zero eigenvalue $n_k$ in **L** becomes $n_k / (n_k - 1)$ in $\mathbf{L}_{sym}$

# Fixing the glitch (II)

- The cluster indicator vectors are eigenvectors also of $\mathbf{L}_{sym}$, with eigenvalues = 0
- We can consider the eigensystem of $\mathbf{L}_{sym}$ instead!
- Better separation between "zero" and "non-zero" eigenvalues



# A glitch (III)

A simple example with three ideal clusters

- $n_1, n_2, n_3$ points each
- The indicator vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ are eigenvectors of $\mathbf{L}_{sym}$ with eigenvalue 0
- Normalized to unit norm they become

$$\hat{\mathbf{c}}_1 = \begin{pmatrix} 1/\sqrt{n_1} \\ \vdots \\ 1/\sqrt{n_1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \hat{\mathbf{c}}_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\sqrt{n_2} \\ \vdots \\ 1/\sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \hat{\mathbf{c}}_3 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1/\sqrt{n_3} \\ \vdots \\ 1/\sqrt{n_3} \end{pmatrix} \begin{matrix} \} \; n_1 \\ \} \; n_2 \\ \} \; n_3 \end{matrix}$$



# A glitch (III)

- In the practical case, these is some noise and the three eigenvectors if $\mathbf{L}_{sym}$ corresponding to eigenvalue "zero" are linear combinations of the previous vectors
  - Normalized linear combinations!
  - Correspond to rotations of the previous vectors
  - Therefore we do *k*-means clustering on the row space of **U** to find the clusters
  - If $n_1$, $n_2$, $n_3$ are of different magnitudes:
    - Clusters with many points are found close to the origin
    - (Why?)



# Fixing the glitch (III)

- We normalize the rows of **U** before the final *k*-means clustering
- The resulting rows lie on a unit hyper-sphere
- This leads to a better separation of the clusters in the row space of **U**
- We return to the issue of clustering points on a sphere in the following lecture

# A clustering algorithm (IV)
## Ng et al (2002)

- Assume *n* points and *k* clusters
- Compute $n \times n$ affinity matrix **A**, and its **D**
- Compute **L** = **D** – **A**
- Compute $\mathbf{L}_{sym} = \mathbf{D}^{-1/2}\,\mathbf{L}\,\mathbf{D}^{-1/2}$
- Compute eigensystem of $\mathbf{L}_{sym}$
- There should be *k* "zero" eigenvalues
- Set **U** to hold the corresponding normalized eigenvectors in its columns
- Set **T** = **U** but with each row normalized to unit norm
- Apply *k*-means clustering on the row space of **T** to find the *k* clusters



# Does it matter with algorithm we use?

- The unnormalized algorithm is attractive since it is simple, but
  - Use it only when you know that the clusters have the same order of points
- The two normalized methods (S-M & Ng) are approximately of the same order of additional computations
  - Von Luxburg suggests S-M before Ng method
  - In practice Ng's method appears to work as well



# Summary

- 3 basic algorithms for spectral clustering
  - Unnormalized: $\mathbf{L}\,\mathbf{u} = \lambda\,\mathbf{u}$
  - Shi-Malik
    - Solve: $\mathbf{L}_{rw}\,\mathbf{u} = \lambda\,\mathbf{u}$ , where $\mathbf{L}_{rw} = \mathbf{D}^{-1}\mathbf{L}$
  - Ng, et al:
    - Solve: $\mathbf{L}_{sym}\,\mathbf{u} = \lambda\,\mathbf{u}$ , where $\mathbf{L}_{sym} = \mathbf{D}^{-1/2}\,\mathbf{L}\,\mathbf{D}^{-1/2}$
- Spectral properties of **A**, **D**, **L**
  - Relations to the cluster indicator vectors