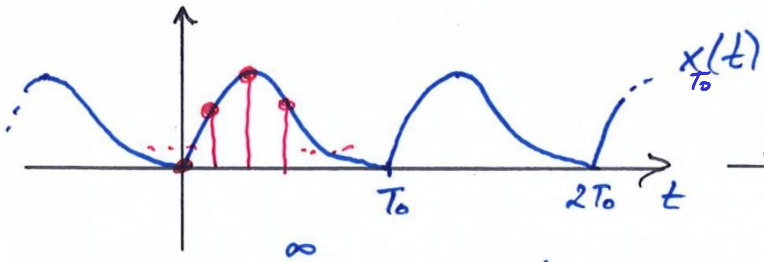
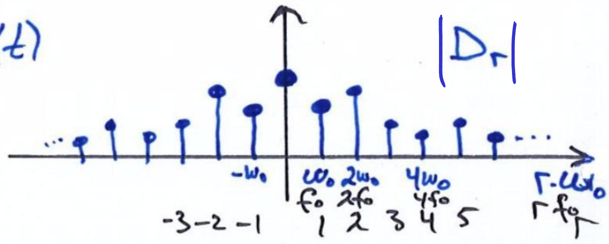
Signaler & System – Föreläsning 13:
Fouriertransformanalys av tidsdiskreta signaler & system

Frekvensegenskap för en tidsdiskret N_0-periodisk signal $x_{N_0}[n]$:



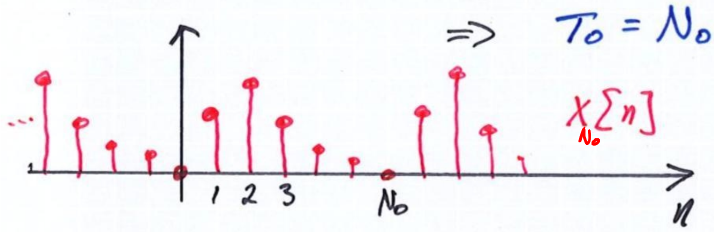


$$x_{T_0}(t) = \sum_{r=-\infty}^{\infty} D_r e^{r\omega_0 t} \quad \text{där} \quad D_r = \frac{1}{T_0}\int_{T_0} x_{T_0}(t)\, e^{-jr\omega_0 t}\, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

Sample $x(t)$: $\quad x_{N_0}[n] = x_{T_0}(nT)$; N_0 punkter / period

$$\Rightarrow \quad T_0 = N_0 \cdot T$$

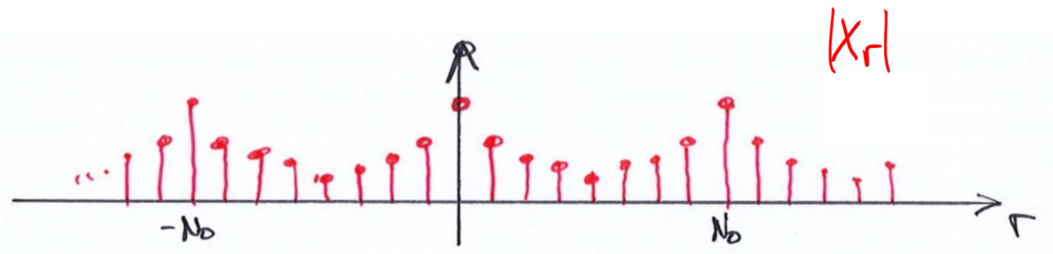


Från Föreläsning 7 – Diskreta FourierTransformen, DFT:

$$x_n = x_{N_0}[n] = x_{T_0}(nT) = \sum_{r=-\infty}^{\infty} D_r e^{jr\omega_0 nT} = \Big/ \ \Omega_0 = \omega_0 T = \frac{2\pi}{T_0}\cdot T = \frac{2\pi}{N_0} \ \Big/ = \ldots$$

$$= \sum_{r=0}^{N_0-1}\left(\sum_{m=-\infty}^{\infty} D_{r-mN_0}\right) e^{jr\Omega_0 n} = \Big/ \ X_r = N_0 \sum_{m=-\infty}^{\infty} D_{r-mN_0} \ \Big/$$

$$= \boxed{\frac{1}{N_0}\sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n} = IDFT\{X_r\}} \quad \text{där} \quad \boxed{X_r = DFT\{x_n\} = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}}$$



Fourieranalys av N_0-periodiska signaler (kap. 9.1)

Om $x_{No}[n]$ är N_0-periodisk, dvs. $x_{No}[n] = x_{No}[n+N_0]$

$\Rightarrow$ **fourierserieutveckla** (**DTFS**, Discrete-Time Fourier Series):

$$x_{N_0}[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \quad \text{där} \quad \mathcal{D}_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x_{N_0}[n] e^{-jr\Omega_0 n}, \quad \Omega_0 = \frac{2\pi}{N_0}$$

(N_0-periodiska komplexa fourierseriekoefficienter)

$$X_r = N_0 \cdot \mathcal{D}_r \Rightarrow \boxed{x_{N_0}[n] = N_0 \cdot \mathrm{IDFT}\{\mathcal{D}_r\}} \quad \boxed{\mathcal{D}_r = \frac{1}{N_0} \cdot \mathrm{DFT}\{x_{N_0}[n]\}}$$

$\left(\Rightarrow x_n = x_{N_0}[n]\right)$

$\left(\text{Notera kursbokens def: } \mathcal{D}_r = X_r \;\Rightarrow\; x_{N_0}[n] = N_0 x_n\right)$

$\Rightarrow$ Använd **DFT** för frekvensanalys av *tidsdiskreta periodiska signaler*!

(*som beräknas med någon lämplig FFT-algoritm*)

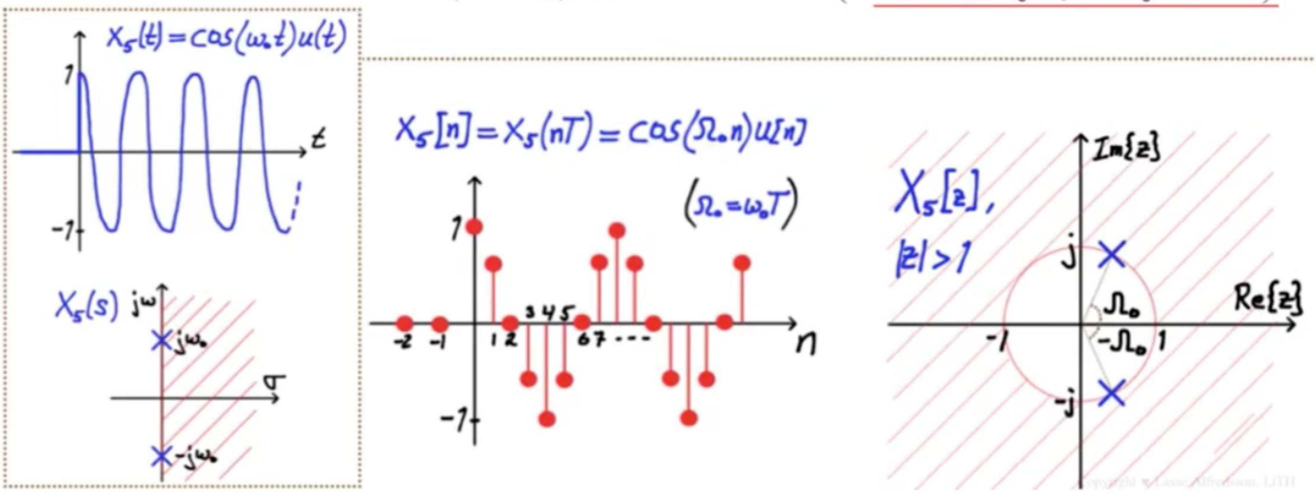
$\Rightarrow$ *I kursen tar vi därför **inte** upp fourierserier för tidsdiskreta signaler…*

Tidskontinuerliga vs. tidsdiskreta frekvenssignaler

VIDEON, Del 1

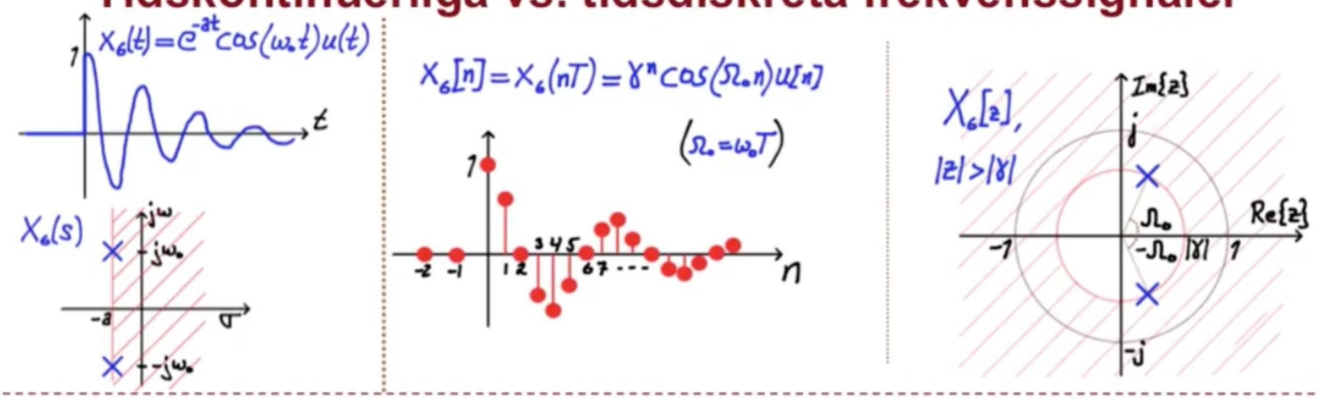
$$\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s}{s^2+\omega_0^2};\ \ \mathrm{Re}\{s\}>0 \qquad \left(s^2+\omega_0^2=0 \Rightarrow \underline{s=\pm j\omega_0}\right)$$

$$\cos(\Omega_0 t)u[n] \Leftrightarrow \frac{z(z-\cos\Omega_0)}{z^2-2(\cos\Omega_0)\cdot z+1};\ \ |z|>1 \qquad \left(\begin{array}{c} z^2-2(\cos\Omega_0)\cdot z+1=0 \\ \Rightarrow \underline{z=\cos\Omega_0 \pm j\sin\Omega_0 = e^{\pm j\Omega_0}} \end{array}\right)$$



Tidskontinuerliga vs. tidsdiskreta frekvenssignaler

VIDEON, Del 2



$$e^{-at}\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2};\ \ \mathrm{Re}\{s\}>-a \qquad \left((s+a)^2+\omega_0^2=0 \Rightarrow \underline{s=-a\pm j\omega_0}\right)$$

$$\gamma^n\cos(\Omega_0 n)u[n] \Leftrightarrow \frac{z(z-\gamma\cos\Omega_0)}{z^2-2\gamma(\cos\Omega_0)z+\gamma^2};\ \ |z|>|\gamma| \qquad \left(\begin{array}{c} z^2-2\gamma(\cos\Omega_0)\cdot z+\gamma^2=0 \\ \Rightarrow \underline{z=|\gamma|e^{\pm j\Omega_0}} \end{array}\right)$$

VIDEON, Del 3

Fouriertransformen till tidsdiskret signal

- Om ***jω*** **-axeln** ligger i konvergensområdet för $X(s)$

$$\Rightarrow \quad X(\omega) = X(s)\big|_{s=j\omega} = X(j\omega) \qquad \left(\mathcal{F}\{x(t)\} = \mathcal{L}\{x(t)\}\big|_{s=j\omega}\right)$$

- Om **enhetscirkeln** ($|z|=1$) ligger i konvergensområdet för $X[z]$

$$\Rightarrow \quad X[\Omega] = X[z]\big|_{z=e^{j\Omega}} = X\left[e^{j\Omega}\right] \qquad \left(\mathcal{F}\{x[n]\} = \mathcal{Z}\{x[n]\}\big|_{z=e^{j\Omega}}\right)$$

Fouriertransformen till $x[n]$ (*Eng*: *"DTFT, Discrete-Time Fourier Transform"*):

$$\boxed{X[\Omega] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}\bigg|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}}$$

Existensvillkor:

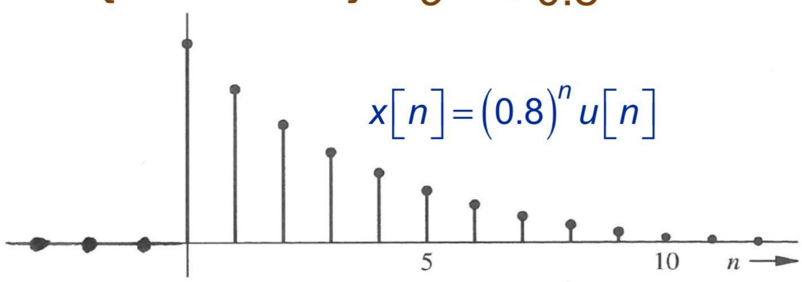
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Boken: $X(\Omega) = \text{DTFT}\{x[n]\}$

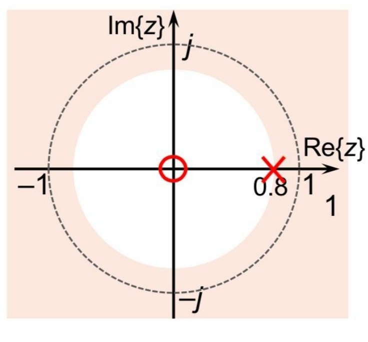
VIDEON, Del 4

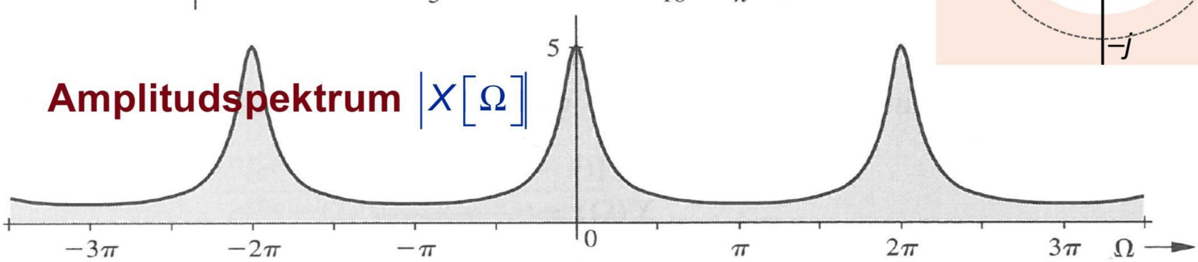
$$\mathcal{F}\left\{(0.8)^n u[n]\right\} = \frac{e^{j\Omega}}{e^{j\Omega} - 0.8}$$

$$X[z] = \frac{z}{z-0.8}, \quad |z| > 0.8$$

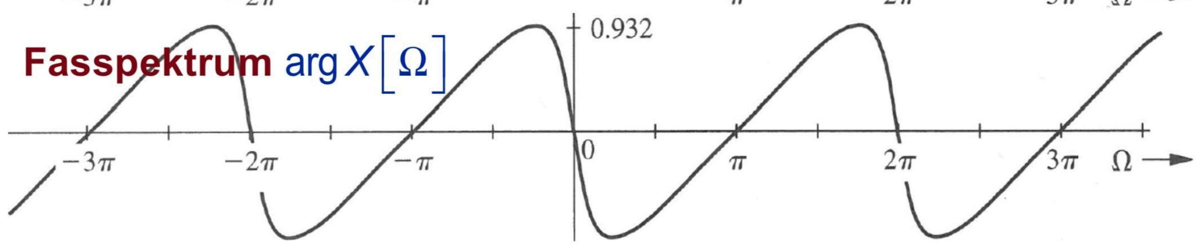


$x[n] = (0.8)^n u[n]$





Amplitudspektrum $|X[\Omega]|$



Fasspektrum $\arg X[\Omega]$

Fouriertransformen (DTFT, Discrete-Time Fourier Transform)

- Fouriertransformen till energisignal $x[n]$:

$$\boxed{\mathcal{F}\{x[n]\} = X[\Omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}}$$

Boken:
$X(\Omega) = \text{DTFT}\{x[n]\}$

- Inversa fouriertransformen till $X[\Omega]$ ($X(\Omega)$):

$$\boxed{\mathcal{F}^{-1}\{X[\Omega]\} = x[n] = \frac{1}{2\pi}\int_{2\pi} X[\Omega]e^{j\Omega n}\,d\Omega}$$

Boken:
$x[n] = \text{IDTFT}\{X(\Omega)\}$

Existensvillkor: $\boxed{\mathcal{F}\{x[n]\}\ \exists \text{ om } \sum_{n=-\infty}^{\infty} |x[n]| < \infty}$

Fouriertransformen kan erhållas/härledas:

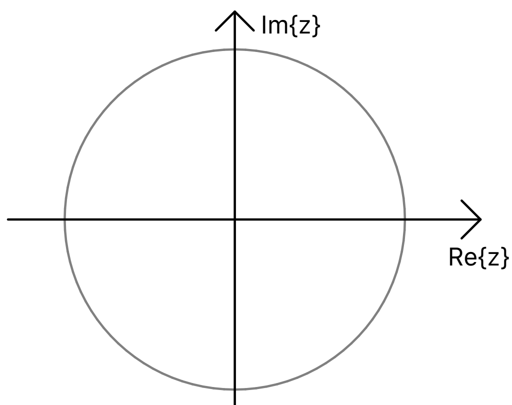
1. Genom utvidgning av $x_{N_0}[n]$ = en N_0-periodisk upprepning av $x[n]$ $\Rightarrow N_0\mathcal{D}_r = X[r\cdot\Omega_0]$ $\left(\Omega_0 = \dfrac{2\pi}{N_0}\right)$

   Låt $N_0 \to \infty \quad \Rightarrow \quad x_{N_0}[n] \to x[n], \quad r\cdot\Omega_0 \to \Omega, \quad \underline{\underline{N_0\mathcal{D}_r \to X[\Omega]}}$

2. Som z-transformen längs enhetscirkeln (*då den ligger i konvergensområdet!*):

$$X[e^{j\Omega}] = X[z]\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}\Bigg|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X[\Omega]$$

$$x[n] = \frac{1}{2\pi j}\oint_C X[z]z^{n-1}dz = \Bigg/ \begin{matrix} z = e^{j\Omega} \Rightarrow \dfrac{dz}{d\Omega} = j\cdot e^{j\Omega} \\ C = e^{j\Omega},\ \ \Omega: \alpha \text{ till } \alpha+2\pi \end{matrix} \Bigg/ = \frac{1}{2\pi}\int_{2\pi} X[e^{j\Omega}]e^{j\Omega n}\,d\Omega$$



Några centrala fouriertransformpar

$\boxed{\mathcal{F}\{x[n]\} \text{ är } 2\pi\text{-periodisk!}}$

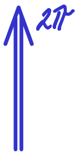
$$\delta[n] \Leftrightarrow 1$$

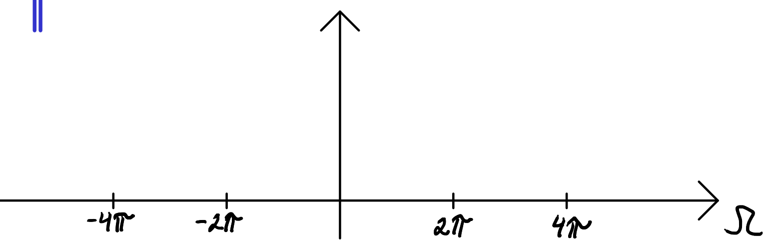
$$\delta[n-k] \Leftrightarrow e^{-jk\Omega}$$

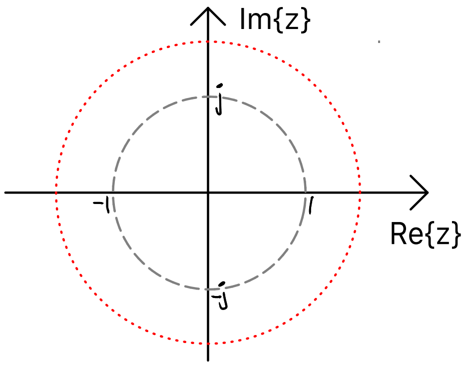
$$1 \Leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k \cdot 2\pi)$$

$$\gamma^n u[n] \Leftrightarrow \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}; \qquad |\gamma| < 1$$

$$-\gamma^n u[-n-1] \Leftrightarrow \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}; \qquad |\gamma| > 1$$



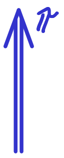


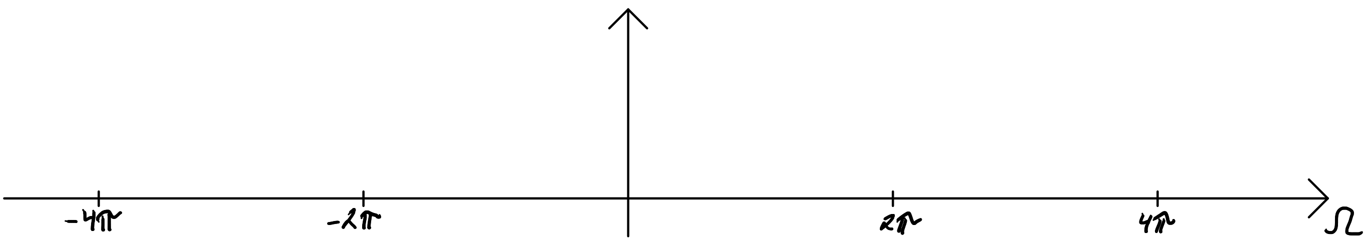


$$u[n] \Leftrightarrow \mathrm{vp}\left\{\frac{e^{j\Omega}}{e^{j\Omega} - 1}\right\} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k \cdot 2\pi)$$

$$\cos(\Omega_0 n) \Leftrightarrow \pi \sum_{k=-\infty}^{\infty} \left(\delta(\Omega + \Omega_0 - k \cdot 2\pi) + \delta(\Omega - \Omega_0 - k \cdot 2\pi)\right)$$

$$\cos(\Omega_0 n) u[n] \Leftrightarrow \mathrm{vp}\left\{\frac{e^{j2\Omega} - e^{j\Omega}\cos\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1}\right\} + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \left(\delta(\Omega + \Omega_0 - k \cdot 2\pi) + \delta(\Omega - \Omega_0 - k \cdot 2\pi)\right)$$





Exemplet i den förberedande video:

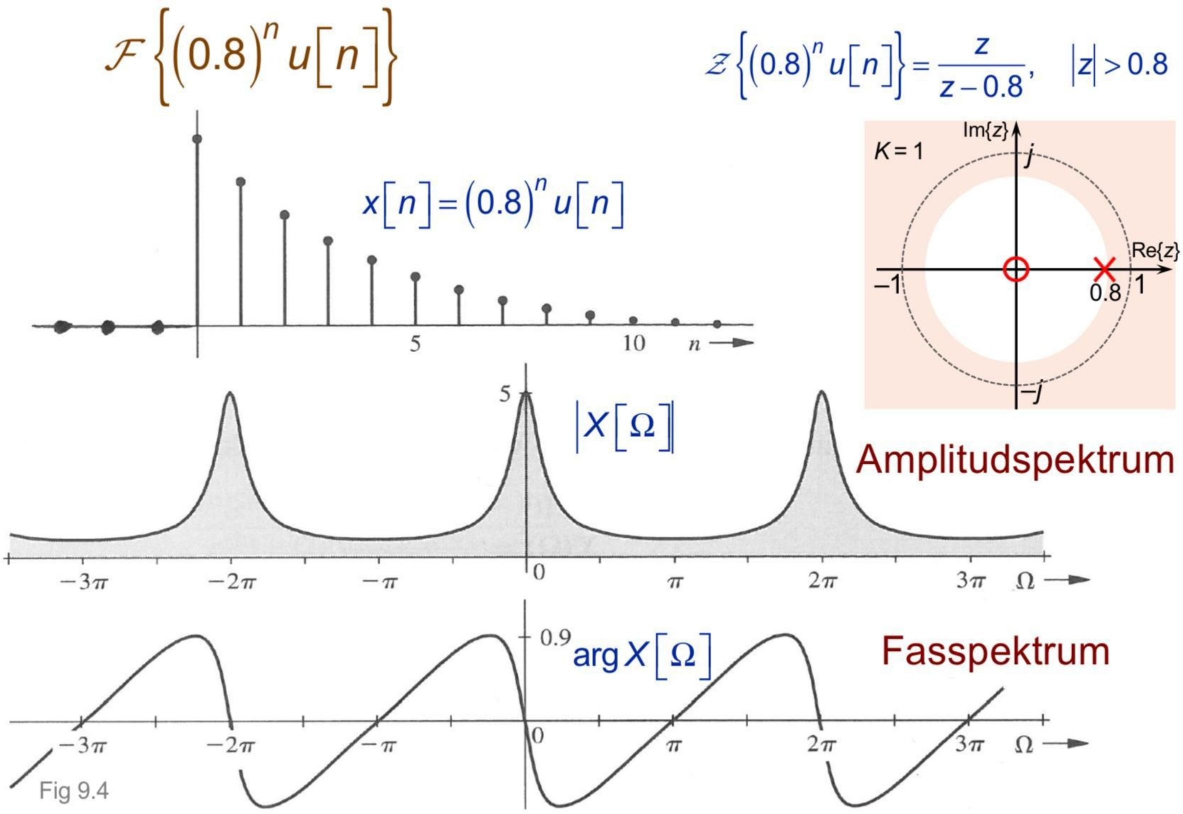


Fig 9.4

Poissons summationsformel – *igen*

$$x(t) \longrightarrow \boxed{\text{Sampling}} \longrightarrow x[n] = x(nT)$$

$$\bar{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \quad (1)$$

Kap. 8 $\Rightarrow$
Poissons summationsformel:

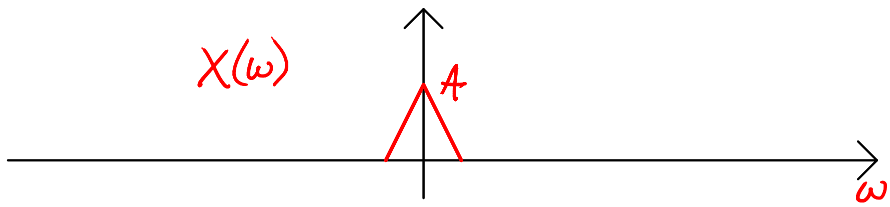
$$\boxed{\bar{X}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\cdot\omega_s)}$$

Från DFT-videon inför föreläsning 7: $\bar{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT}$

Låt $\Omega = \omega T$ $\Rightarrow$ $\bar{X}\left(\frac{\Omega}{T}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X[\Omega]$

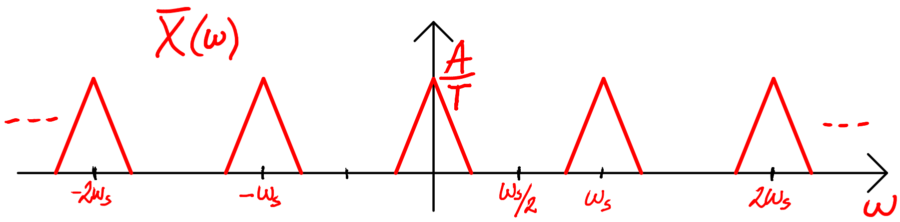
$\Rightarrow$ Alternativ Poissons summationsformel: $\boxed{X[\Omega] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\frac{\Omega - n\cdot 2\pi}{T}\right)}$

$x(t)$ $\Rightarrow$



$\downarrow$ Sampla $\Rightarrow$ Periodisering:

$$\begin{cases} \bar{x}(t) = x(t)\cdot\delta_T(t) \\ x[n] = x(nT) \end{cases}$$

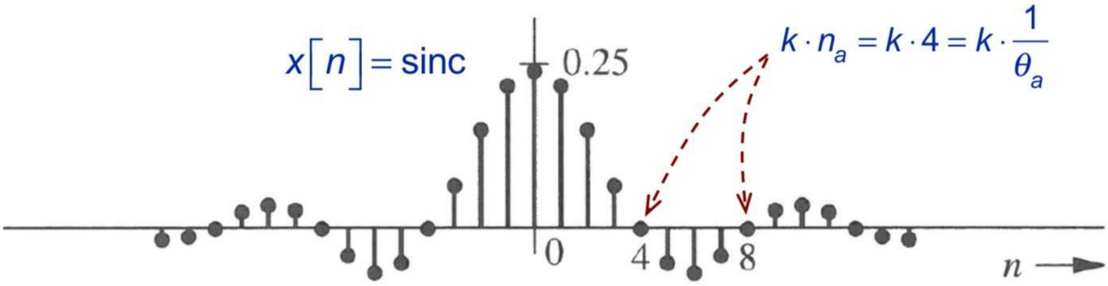


Samband, sinc $\Leftrightarrow$ rect

Nollgenomgångar:

$k \cdot n_a = k \cdot 4 = k \cdot \dfrac{1}{\theta_a}$

$x[n] = \text{sinc}$



$X[\Omega] = \sum \text{rect}$

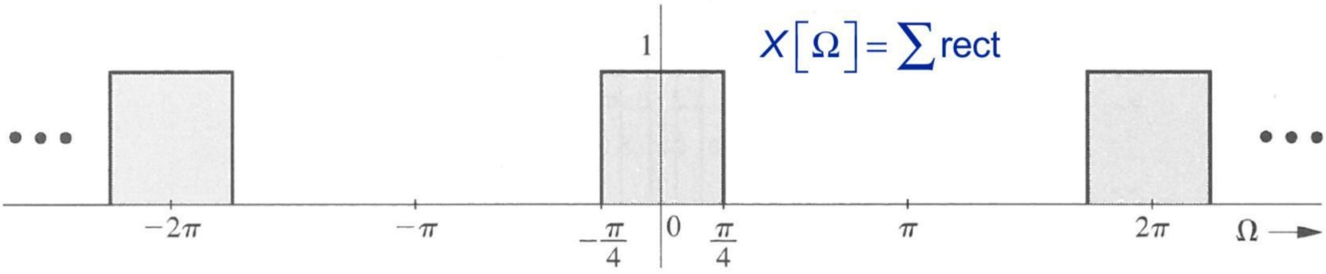
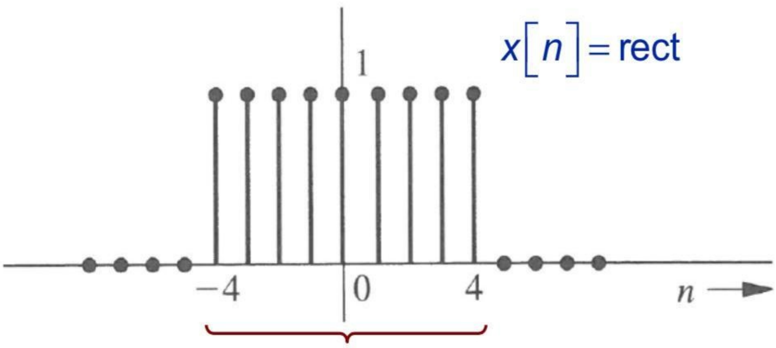


Fig 9.7

Bredd $\Omega_a = \pi/2 \Rightarrow \theta_a = 1/4 \quad (= 1/n_a)$

Samband, rect $\Leftrightarrow$ sinc

$x[n] = \text{rect}$



Bredd (längd) $n_b = 9 = \dfrac{1}{\theta_b}$

Nollgenomgångar:

$k \cdot \theta_b = k \cdot \dfrac{1}{n_b} = k \cdot \dfrac{1}{9}$, dvs

$k \cdot \Omega_b = k \cdot \dfrac{2\pi}{9}$

$X[\Omega] = \sum \text{sinc}$

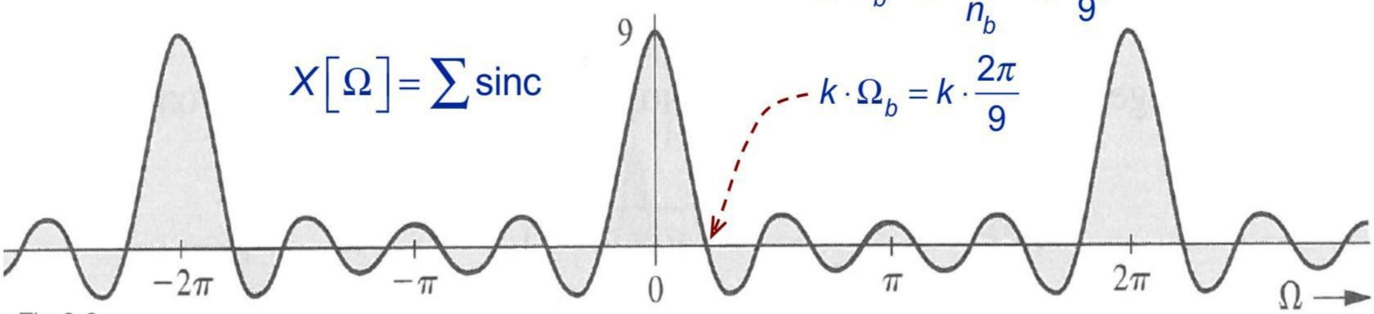


Fig 9.6

Centrala fouriertransformegenskaper (*F.s. Tab. 7*)

Tidsskiftning: $x[n-n_0] \Leftrightarrow X[\Omega]e^{-j\Omega n_0}$

Frekvensskiftning: $x[n]e^{j\Omega_0 n} \Leftrightarrow X[\Omega-\Omega_0]$

spec.fall: $x[n](-1)^n \Leftrightarrow X[\Omega-\pi]$

Spegling: $x[-n] \Leftrightarrow X[-\Omega]$

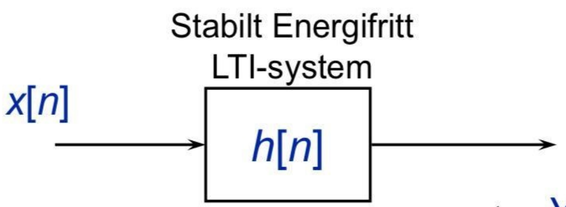
Mult. med n: $n \cdot x[n] \Leftrightarrow j\dfrac{dX[\Omega]}{d\Omega}$

Symmetri, $x[n]$ reell: $X[-\Omega] = X^*[\Omega]$

Faltning: $x_1[n] * x_2[n] \Leftrightarrow X_1[\Omega]X_2[\Omega]$

Multiplikation: $x_1[n]x_2[n] \Leftrightarrow \dfrac{1}{2\pi}X_1[\Omega] \otimes X_2[\Omega]$

Systemanalys & utsignalsberäkning



$$y[n] = x[n] * h[n]$$

$$\Rightarrow \quad Y[\Omega] = X[\Omega]H[\Omega]$$

Frekvensfunktionen: $H[\Omega] = \mathcal{F}\{h[n]\} = H[z]\big|_{z=e^{j\Omega}} = H\left[e^{j\Omega}\right]$

Frekvensspektrum:

$X[\Omega] = \mathcal{F}\{x[n]\} = |X[\Omega]|e^{j\arg X[\Omega]}$, $\quad Y[\Omega] = \mathcal{F}\{y[n]\} = |Y[\Omega]|e^{j\arg Y[\Omega]}$

$\Longrightarrow$ Utsignalens frekvensspektrum:

- Amplitudspektrum: $|Y[\Omega]| = |X[\Omega]| \cdot |H[\Omega]|$
- Fasspektrum: $\arg Y[\Omega] = \arg X[\Omega] + \arg H[\Omega]$

Exempel:

Ett stabilt energifritt LTI-system med insignal $x[n]$ och utsignal $y[n]$ beskrivs av följande differensekvation: $y[n]-0.5y[n-1]=x[n]$

Beräkna systemets frekvensfunktion $H[\Omega]$ och impulssvar $h[n]$

samt utsignalen för insignalen $x[n]=0.8^{n}u[n]$

Lösningsgång:

$$\begin{tikzcd}
 & & \underline{h[n]} & \\
\text{Differensekv.} \arrow[r, "\mathcal{F}"] & \underline{H[\Omega]} \arrow[ur, "\mathcal{F}^{-1}"] \arrow[dr] & & \\
x[n] \arrow[r, "\mathcal{F}"] & X[\Omega] \arrow[r] & Y[\Omega]=X[\Omega]H[\Omega] \arrow[r, "\mathcal{F}^{-1}"] & \underline{y[n]}
\end{tikzcd}$$