LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Solutions 2020-08-28

1. (a) Since

$$\mathbf{p}^\top \mathbf{x}_1 = \mathbf{p}^\top \mathbf{x}_2 = \mathbf{p}^\top \mathbf{x}_4 = 0 \qquad \text{and} \qquad \mathbf{p}^\top \mathbf{x}_3 = 9,$$

it must be $\mathbf{x}_3$ that lies outside the plane. The distance between $\mathbf{p}$ and $\mathbf{x}_3$ is

$$d_{\text{PD}}(\mathbf{x}_3, \mathbf{p}) = \left|\text{norm}_\text{D}(\mathbf{p})^\top \, \text{norm}_\text{P}(\mathbf{x}_3)\right| = \left|\frac{-1}{\sqrt{(-5)^2 + (-1)^2 + (-1)^2}} \cdot \mathbf{p}^\top \mathbf{x}_3\right| = \sqrt{3}.$$

Answer: The point $\mathbf{x}_3$ lies at a distance $\sqrt{3}$ from the plane.

(b) All points on ℓ can be written as $\mathbf{x}(\lambda_2, \lambda_3) = \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3$, where $\lambda_2^2 + \lambda_3^2 \neq 0$. Any representation on this form that has zero as its final coordinate represents the ideal point, e.g. $\mathbf{x}_\infty \sim \mathbf{x}(1, -2) = (3, 3, 0, 0) \sim (1, 1, 0, 0)$.

Answer: The ideal point on ℓ is $\mathbf{x}_\infty \sim (1, 1, 0, 0)$.

(c) The ideal point can be read off as the rightmost column in the Plücker coordinates of the line, which can be shown as foolows.

Consider two arbitrary distinct points $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$. The line through $\mathbf{x}$ and $\mathbf{y}$ will have Plücker coordinates $\mathbf{L} = \mathbf{x}\mathbf{y}^\top - \mathbf{y}\mathbf{x}^\top$. This means that the columns in $\mathbf{L}$ are linear combinations of $\mathbf{x}$ and $\mathbf{y}$, and thus represent points on the line. The rightmost column of $\mathbf{L}$ is $y_4\mathbf{x} - x_4\mathbf{y}$, and it clearly has its last coordinate equal to zero, which means that it represents an ideal point.

2. (a) The origin is represented by $\mathbf{x}_0 = (0, 0, 1)$, and the requirement that an element $\mathbf{H} \in \mathscr{H}_0$ maps the origin to itself can be expressed as $\mathbf{H}\mathbf{x}_0 = \lambda \mathbf{x}_0$ for some $\lambda \neq 0$ (i.e. $\mathbf{x}_0$ is an eigenvector of $\mathbf{H}$). We see that

$$\mathbf{H}\mathbf{x} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{13} \\ h_{23} \\ h_{33} \end{pmatrix},$$

so we have the constraints $h_{13} = h_{23} = 0$. (Additionally, for $\mathbf{H}$ to be invertible, we must have $h_{33} \neq 0$ and $h_{11}h_{22} - h_{12}h_{21} \neq 0$.) Since $\mathbf{H}$ is only determined up to scale, we can set $h_{33} = 1$.

Answer: A general element $\mathbf{H} \in \mathscr{H}_0$ can be written as $\mathbf{H} = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & 1 \end{pmatrix}$.

(b) Let $\mathbf{x}_j = (u_j, v_j, w_j)$ and $\mathbf{x}'_j = (u'_j, v'_j, w'_j)$. Each correspondence $\mathbf{x}_j \leftrightarrow \mathbf{x}'_j$ gives rise to a constraint consisting of three linear equations, namely

$$\mathbf{x}'_j \sim \mathbf{H}\mathbf{x}_j \iff \mathbf{x}'_j \times \mathbf{H}\mathbf{x}_j = \mathbf{0} \iff [\mathbf{x}'_j]_\times \mathbf{H}\mathbf{x}_j = \mathbf{0} \iff$$

$$\begin{pmatrix} 0 & -w'_j & v'_j \\ w'_j & 0 & -u'_j \\ -v'_j & u'_j & 0 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & 1 \end{pmatrix} \begin{pmatrix} u_j \\ v_j \\ w_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\begin{pmatrix} 0 & -w'_j & v'_j \\ w'_j & 0 & -u'_j \\ -v'_j & u'_j & 0 \end{pmatrix} \begin{pmatrix} u_j h_{11} + v_j h_{12} \\ u_j h_{21} + v_j h_{22} \\ u_j h_{31} + v_j h_{32} + w_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\begin{pmatrix} -u_j w'_j h_{21} - v_j w'_j h_{22} + u_j v'_j h_{31} + v_j v'_j h_{32} + w_j v'_j \\ u_j w'_j h_{11} + v_j w'_j h_{12} - u_j u'_j h_{31} - v_j u'_j h_{32} - w_j u'_j \\ -u_j v'_j h_{11} - v_j v'_j h_{12} + u_j u'_j h_{21} + v_j u'_j h_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\underbrace{\begin{pmatrix} 0 & -u_j w'_j & u_j v'_j & 0 & -v_j w'_j & v_j v'_j & w_j v'_j \\ u_j w'_j & 0 & -u_j u'_j & v_j w'_j & 0 & -v_j u'_j & -w_j u'_j \\ -u_j v'_j & u_j u'_j & 0 & -v_j v'_j & v_j u'_j & 0 & 0 \end{pmatrix}}_{\mathbf{A}_j} \begin{pmatrix} h_{11} \\ h_{21} \\ h_{31} \\ h_{12} \\ h_{22} \\ h_{32} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(Those who know the Kronecker product and the vec operation can go directly from $[\mathbf{x}'_j]_\times \mathbf{H}\mathbf{x}_j = \mathbf{0}$ to the final step!) A suitable data matrix $\mathbf{A}$ for estimating $\mathbf{H} \in \mathscr{H}_0$ is

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}.$$

Since $\operatorname{rank} \mathbf{A}_j = 2$ (because $\operatorname{rank} [\mathbf{x}'_j]_\times = 2$), it follows that $\operatorname{rank} \mathbf{A} = \min(2n, 7)$. To uniquely determine $\mathbf{H}$, we thus need at least $n = 3$ point correspondences.

Answer: A suitable data matrix $\mathbf{A}$ for estimating $\mathbf{H} \in \mathscr{H}_0$ from point correspondences $(u_j, v_j, w_j) \leftrightarrow (u'_j, v'_j, w'_j)$ is

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}, \quad \text{where} \quad \mathbf{A}_j = \begin{pmatrix} 0 & -u_j w'_j & u_j v'_j & 0 & -v_j w'_j & v_j v'_j & w_j v'_j \\ u_j w'_j & 0 & -u_j u'_j & v_j w'_j & 0 & -v_j u'_j & -w_j u'_j \\ -u_j v'_j & u_j u'_j & 0 & -v_j v'_j & v_j u'_j & 0 & 0 \end{pmatrix}.$$

At least three point correspondences are needed to uniquely determine $\mathbf{H}$.

(c) To show that $\mathscr{H}_0$ is a group with respect to composition, we need to show *closure*, *associativity*, *existence of identity*, and *existence of inverse*.

Closure: Let

$$\mathbf{H}_1 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{c}_1^\top & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_2 = \begin{pmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{c}_2^\top & 1 \end{pmatrix}$$

be two homographies in $\mathscr{H}_0$. We now verify that

$$\mathbf{H}_1\mathbf{H}_2 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{c}_1^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{c}_2^\top & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1\mathbf{A}_2 & \mathbf{0} \\ \mathbf{c}_1^\top \mathbf{A}_2 + \mathbf{c}_2^\top & 1 \end{pmatrix} \in \mathscr{H}_0,$$

since $\det(\mathbf{A}_1\mathbf{A}_2) = \det \mathbf{A}_1 \cdot \det \mathbf{A}_2 \neq 0$.

Associativity: This follows from the associativity of matrix multiplication.

Existence of identity: The identity homography $\mathbf{H}_{\text{id}} \sim \mathbf{I}$ exists and maps *all* points to themselves. In particular, it maps the origin to itself, so $\mathbf{H}_{\text{id}} \in \mathscr{H}_0$.

Existence of inverse: It is clear *by definition* that every $\mathbf{H} \in \mathscr{H}_0$ is invertible (it is a homography), but it is *not* immediately clear that the inverse also lies in $\mathscr{H}_0$. Let $\mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{c}^\top & 1 \end{pmatrix} \in \mathscr{H}_0$ and let $\mathbf{H}^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{v} \\ \mathbf{w}^\top & z \end{pmatrix}$. Since

$$\mathbf{I}_3 = \mathbf{H}\mathbf{H}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{c}^\top & 1 \end{pmatrix}\begin{pmatrix} \mathbf{U} & \mathbf{v} \\ \mathbf{w}^\top & z \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{U} & \mathbf{A}\mathbf{v} \\ \mathbf{c}^\top\mathbf{U} + \mathbf{w}^\top & \mathbf{c}^\top\mathbf{v} + z \end{pmatrix} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix},$$

it follows that $\mathbf{v} = \mathbf{0}$, $z = 1$, $\mathbf{U} = \mathbf{A}^{-1}$, and $\mathbf{w}^\top = -\mathbf{c}^\top\mathbf{A}^{-1}$. Thus we must have $\mathbf{H}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -\mathbf{c}^\top\mathbf{A}^{-1} & 1 \end{pmatrix} \in \mathscr{H}_0$.

3. (a) The centroid of the point set is $\bar{\mathbf{x}} = (\mathbf{x}_1 + \mathbf{x}_2)/2 = (-1, 2, 1)$. Now, the mean distance from the origin to the centred points is

$$\frac{1}{2}\left(\|\mathbf{x}_1 - \bar{\mathbf{x}}\| + \|\mathbf{x}_2 - \bar{\mathbf{x}}\|\right) = \frac{1}{2}\left(\sqrt{3^2 + 1^2} + \sqrt{(-3)^2 + (-1)^2}\right) = \sqrt{10}.$$

To transform this distance to $\sqrt{2}$, we need to apply the scale factor $\frac{\sqrt{2}}{\sqrt{10}} = \frac{1}{\sqrt{5}}$ in both the x direction and the y direction. Thus,

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} \frac{2-(-1)}{\sqrt{5}} \\ \frac{3-2}{\sqrt{5}} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{x}}_2 = \begin{pmatrix} \frac{-4-(-1)}{\sqrt{5}} \\ \frac{1-2}{\sqrt{5}} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-3}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 1 \end{pmatrix}.$$

Answer: The normalised points are $\tilde{\mathbf{x}}_1 = \left(\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1\right)$ and $\tilde{\mathbf{x}}_2 = \left(-\frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 1\right)$.

(b) **Answer:** The transformation in question is a similarity transformation, formed as a composition of the centring translation and the uniform scaling, as

$$\mathbf{T} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & 0 & 1 \end{pmatrix}.$$

4. (a) The product $(\mathbf{l}_1^\top\mathbf{x}) \cdot (\mathbf{l}_2^\top\mathbf{x}) = 0$ precisely when $\mathbf{x}$ lies on (at least) one of the lines. A possible algebraic cost function is therefore

$$f_A(\mathbf{l}_2) = \sum_{j=1}^{n}(\mathbf{l}_1^\top\mathbf{x}_j)^2 \cdot (\mathbf{l}_2^\top\mathbf{x}_j)^2.$$

Answer: One can use the algebraic cost function $f_A(\mathbf{l}_2) = \sum_{j=1}^{n}(\mathbf{l}_1^\top\mathbf{x}_j)^2 \cdot (\mathbf{l}_2^\top\mathbf{x}_j)^2$.

(b) **Answer:** To measure geometric distances, we need to appropriately normalise all points and the two lines. Additionally, in a geometric cost function it makes sense to measure the only distance to the closest line for each point. One option is thus to use

$$f_G(\mathbf{l}_2) = \sum_{j=1}^{n} \min\left((\mathrm{norm}_\mathrm{D}\,\mathbf{l}_1)^\top(\mathrm{norm}_\mathrm{P}\,\mathbf{x}_j), (\mathrm{norm}_\mathrm{D}\,\mathbf{l}_2)^\top(\mathrm{norm}_\mathrm{P}\,\mathbf{x}_j)\right)^2.$$

5. (a) The Gramian $\mathbf{G}_0$ must be positive definite, which is equivalent to its two eigenvalues being positive (since it is symmetric, and hence diagonalisable). In the 2×2 case, this is clearly equivalent to

$$\begin{cases} \lambda_1 + \lambda_2 > 0 \\ \lambda_1\lambda_2 > 0 \end{cases} \iff \begin{cases} \operatorname{tr}\mathbf{G}_0 > 0 \\ \det\mathbf{G}_0 > 0 \end{cases} \iff \begin{cases} 2a > 0 \\ a^2 - 1 > 0 \end{cases} \iff a > 1.$$

Answer: For $\mathbf{G}_0$ to define a scalar product, we must have $a > 1$.

(b) If we let $\mathbf{B} = \begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{pmatrix}$ and $\tilde{\mathbf{B}} = \begin{pmatrix}\tilde{\mathbf{b}}_1 & \tilde{\mathbf{b}}_2\end{pmatrix}$, then $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ is the dual basis of $\{\mathbf{b}_1, \mathbf{b}_2\}$ precisely when

$$\tilde{\mathbf{B}}^\top\mathbf{G}_0\mathbf{B} = \mathbf{I} \iff \mathbf{B}^\top\mathbf{G}_0\tilde{\mathbf{B}} = \mathbf{I} \iff \tilde{\mathbf{B}} = (\mathbf{B}^\top\mathbf{G}_0)^{-1}.$$

Substituting the given values, we get

$$\tilde{\mathbf{B}} = \left(\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}\right)^{-1} = \begin{pmatrix}6 & 6\\ 8 & -8\end{pmatrix}^{-1} = \frac{1}{48}\begin{pmatrix}4 & 3\\ 4 & -3\end{pmatrix}.$$

Answer: The dual basis vectors are $\tilde{\mathbf{b}}_1 = \left(\frac{1}{12}, \frac{1}{12}\right)$ and $\tilde{\mathbf{b}}_2 = \left(\frac{1}{16}, -\frac{1}{16}\right)$.

(c) **Alternative 1:** Let $\mathbf{u} = \lambda\mathbf{b}_1 = \lambda(1, 1)$, and let

$$\varepsilon(\lambda) = \|\mathbf{u} - \mathbf{v}\|^2_{\mathbf{G}_0} = \langle\mathbf{u} - \mathbf{v} \mid \mathbf{u} - \mathbf{v}\rangle = (\mathbf{u} - \mathbf{v})^\top\mathbf{G}_0(\mathbf{u} - \mathbf{v}) = \ldots = 12\lambda^2 - 24\lambda + 28.$$

This clearly has a minimum when $\varepsilon'(\lambda) = 0 \iff \lambda = 1$, so $\mathbf{u} = (1, 1)$.

Alternative 2: We seek the orthogonal projection of $\mathbf{v} = (2, 0)$ on the subspace spanned by $\mathbf{b}_1$, which is given by

$$\mathbf{u} = \frac{\mathbf{b}_1^\top\mathbf{G}_0\mathbf{v}}{\mathbf{b}_1^\top\mathbf{G}_0\mathbf{b}_1}\mathbf{b}_1 = \ldots = \mathbf{b}_1 = \begin{pmatrix}1\\1\end{pmatrix}.$$

Answer: The vector $\mathbf{u}$ parallel to $\mathbf{b}_1$ that is closest to $\mathbf{v}$ is $\mathbf{u} = \mathbf{b}_1 = (1, 1)$.

6. (a) **Answer:** Since $\mathbf{G}_0 = \mathbf{I}$, the frame operator is given by

$$\mathbf{F} = \mathbf{B}\mathbf{B}^\top = \begin{pmatrix}1 & -1 & 2\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\\ 2 & 1\end{pmatrix} = \begin{pmatrix}6 & 2\\ 2 & 3\end{pmatrix}.$$

(b) **Answer:** The dual frame vectors are given by the columns of the matrix

$$\tilde{\mathbf{B}} = \mathbf{F}^{-1}\mathbf{B} = \frac{1}{14}\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & -5 & 4 \\ 4 & 8 & 2 \end{pmatrix}.$$

(c) The frame operator associated with a set of frame vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is defined, for an arbitrary vector $\mathbf{v}$, as

$$\mathbf{F}\mathbf{v} = \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \, \mathbf{b}_k.$$

To show that $\mathbf{F}$ is self-adjoint, i.e. that $\langle \mathbf{F}\mathbf{u} \mid \mathbf{v} \rangle = \langle \mathbf{u} \mid \mathbf{F}\mathbf{v} \rangle$ for all $\mathbf{u}$ and $\mathbf{v}$, we verify that

$$\begin{aligned}
\langle \mathbf{F}\mathbf{v} \mid \mathbf{u} \rangle &= \left\langle \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \, \mathbf{b}_k \,\middle|\, \mathbf{u} \right\rangle = \sum_{k=1}^{n} \left\langle \langle \mathbf{v} \mid \mathbf{b}_k \rangle \, \mathbf{b}_k \,\middle|\, \mathbf{u} \right\rangle = \\
&= \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \langle \mathbf{b}_k \mid \mathbf{u} \rangle = \sum_{k=1}^{n} \left\langle \mathbf{v} \,\middle|\, \langle \mathbf{b}_k \mid \mathbf{u} \rangle^* \, \mathbf{b}_k \right\rangle = \\
&= \sum_{k=1}^{n} \left\langle \mathbf{v} \,\middle|\, \langle \mathbf{u} \mid \mathbf{b}_k \rangle \, \mathbf{b}_k \right\rangle = \left\langle \mathbf{v} \,\middle|\, \sum_{k=1}^{n} \langle \mathbf{u} \mid \mathbf{b}_k \rangle \, \mathbf{b}_k \right\rangle = \langle \mathbf{v} \mid \mathbf{F}\mathbf{u} \rangle.
\end{aligned}$$

7. By performing *principal component analysis* to a large number of signals in $\mathbb{R}^4$, it has been found that the main part of the signals always lies in the subspace spanned by $\mathbf{b}_1 = (1, 1, -1, -1)/2$ and $\mathbf{b}_2 = (1, -1, -1, 1)/2$.

(a) The principal components must make up an orthonormal basis, so $\mathbf{b}_3$ must be orthogonal to $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_4$. Letting $\mathbf{b}_3 = (x, y, z, w)$, we have

$$\begin{pmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \\ \mathbf{b}_4^\top \end{pmatrix} \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\iff \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{for any } \lambda \in \mathbb{R}.$$

We need to choose λ such that $\|\mathbf{b}_3\| = 1$, and there are two valid choices which achieve this, namely $\lambda = \pm\frac{1}{2}$.

Answer: The second-but-least significant principal component can be chosen as either $\mathbf{b}_3 = (-1, 1, -1, 1)/2$ or $\mathbf{b}_3 = (1, -1, 1, -1)/2$.

(b) (We assume here that $\mathbf{b}_3 = (-1, 1, -1, 1)/2$.) Let $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{pmatrix}$. The coefficients of $\mathbf{v}$ (with respect to the principal component vectors) are the coordinates $\mathbf{c}$ with respect to the basis in $\mathbf{B}$, given by

$$\mathbf{c} = \mathbf{B}^\top \mathbf{v} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7 \\ 5 \\ -1 \\ 3 \end{pmatrix}.$$

This means that the projection of $\mathbf{v}$ on the subspace spanned by the first two principal components is

$$\mathbf{v}_2 = \frac{7}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 = \frac{7}{4}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} + \frac{5}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}12\\2\\-12\\-2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}6\\1\\-6\\-1\end{pmatrix}.$$

The error (measured by the norm) when using this representation is

$$\varepsilon(\mathbf{v}) = \|\mathbf{v} - \mathbf{v}_2\| = \left\|\left(1, \frac{1}{2}, 1, \frac{1}{2}\right)\right\| = \frac{1}{2}\|(2,1,2,1)\| = \frac{\sqrt{2^2+1^2+2^2+1^2}}{2} = \frac{\sqrt{10}}{2}.$$

Answer: The coefficients are $\mathbf{c} = \frac{1}{2}(7,5,-1,3)$, and the error when $\mathbf{v}$ is represented only using the first two principal components is $\varepsilon(\mathbf{v}) = \frac{\sqrt{10}}{2}$.

8. (a) We recall that the null vector of a camera matrix is a homogeneous representation of its camera centre (i.e., the camera centre is the only point in the extended Euclidean space which does not have a well defined image). Letting $\mathbf{n} = (1,0,0,1)$ be a homogeneous representation of the 3D point $(1,0,0)$, it is readily verified that $\mathbf{C}_1\mathbf{n} = (2,0,5)$ and $\mathbf{C}_2\mathbf{n} = (0,0,0)$.

Answer: It is camera $\mathbf{C}_2$ that has its centre at $(1,0,0)$.

(b) Let $\mathbf{n}_1 = (x,y,z,w)$ be a homogeneous representation of the centre of $\mathbf{C}_1$. We seek the epipole $\mathbf{e}_{21} \sim \mathbf{C}_2\mathbf{n}_1$, but to compute it we need first to find $\mathbf{n}_1$. As $\mathbf{n}_1$ represents the null space of $\mathbf{C}_1$, we have

$$\begin{pmatrix}1&0&-2&1\\-1&1&-1&1\\4&-2&-3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{pmatrix}1&0&-2&1\\0&1&-3&2\\0&-2&5&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff$$

$$\iff \begin{pmatrix}1&0&-2&1\\0&1&-3&2\\0&0&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \text{for any } \lambda \in \mathbb{R}.$$

Now,

$$\mathbf{e}_{21} \sim \mathbf{C}_2\mathbf{n}_1 \sim \begin{pmatrix}0&2&-2&0\\-1&1&2&1\\2&0&3&-2\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\3\\3\end{pmatrix} \sim \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Answer: The sought epipole is $\mathbf{e}_{21} \sim (0,1,1)$.