LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Solutions 2021-01-12

1. (a) We compute the distances

$$d_{\text{PD}}(\mathbf{x}_1, \mathbf{l}) = \left|(\text{norm}_\text{P}\, \mathbf{x}_1)^\top(\text{norm}_\text{D}\, \mathbf{l})\right| = \left|(1,2,1)\cdot\frac{1}{13}(12,-5,26)\right| = \frac{28}{13},$$
$$d_{\text{PD}}(\mathbf{x}_2, \mathbf{l}) = \left|(\text{norm}_\text{P}\, \mathbf{x}_2)^\top(\text{norm}_\text{D}\, \mathbf{l})\right| = \left|(2,-1,1)\cdot\frac{1}{13}(12,-5,26)\right| = \frac{55}{13},$$
$$d_{\text{PD}}(\mathbf{x}_3, \mathbf{l}) = \left|(\text{norm}_\text{P}\, \mathbf{x}_3)^\top(\text{norm}_\text{D}\, \mathbf{l})\right| = \left|(-1,5,1)\cdot\frac{1}{13}(12,-5,26)\right| = \frac{11}{13}.$$

Since $\mathbf{x}_4$ is an ideal point and $\mathbf{l}^\top\mathbf{x}_4 \neq 0$, it lies infinitely far away from the line.

Answer: Of the four points, the point $\mathbf{x}_3$ lies closest to the line $\mathbf{l}$.

(b) All points on the line are formed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$, and since both of those are proper points, the ideal point can be computed as

$$\mathbf{x}_\infty \sim \text{norm}_\text{P}\, \mathbf{x}_2 - \text{norm}_\text{P}\, \mathbf{x}_1 = (1,-3,0).$$

Answer: The ideal point on the line through $\mathbf{x}_1$ and $\mathbf{x}_2$ is $\mathbf{x}_\infty = (1,-3,0)$.

(c) The points (in the extended Euclidean plane) with homogeneous coordinates $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ lie on a line if and only if their *triple product* is zero, i.e. $\mathbf{x}_3^\top(\mathbf{x}_1 \times \mathbf{x}_2) = 0$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, then $\mathbf{l}_{12} = \mathbf{x}_1 \times \mathbf{x}_2$ represents the line through $\mathbf{x}_1$ and $\mathbf{x}_2$, and $\mathbf{x}_3^\top \mathbf{l}_{12} = \mathbf{x}_3^\top(\mathbf{x}_1 \times \mathbf{x}_2) = 0$ means that $\mathbf{x}_3$ also lies on this line. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are not distinct, the three points automatically lie on a line, and the triple product also clearly becomes zero.
Using the three points in the problem text, we get

$$\mathbf{x}_3^\top(\mathbf{x}_1 \times \mathbf{x}_2) = (-1,5,1)\cdot\big((1,2,1)\times(2,-1,1)\big) = (-1,5,1)\cdot(3,1,-5) = -3 \neq 0,$$

so the points do not lie on a line.

Answer: Three points lie on a line if their triple product is zero.

2. (a) **Answer:** A general element $\mathbf{T} \in \mathscr{A}$ can be written as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix},$$

where $\mathbf{A}$ is a non-singular 3×3 matrix and $\mathbf{t} \in \mathbb{R}^3$. It follows that $\mathbf{T}$ has $3\cdot3+3 = 12$ degrees of freedom.

(b) If points are transformed by the transformation $\mathbf{T}$, then planes are transformed using the *dual transformation*

$$\tilde{\mathbf{T}} = \mathbf{T}^{-\top} = \begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{t} \\ \mathbf{o}^\top & 1 \end{pmatrix}^\top = \begin{pmatrix} \mathbf{A}^{-\top} & \mathbf{o} \\ -\mathbf{t}^\top\mathbf{A}^{-\top} & 1 \end{pmatrix}.$$

Transforming the plane at infinity, $\mathbf{p}_\infty = (\mathbf{o}, 1)$, we get $\tilde{\mathbf{T}}\mathbf{p}_\infty = \mathbf{p}_\infty$, as desired.

(c) Each point correspondence $\mathbf{x}_j \leftrightarrow \mathbf{x}'_j$ gives rise to four linear equations,

$$\operatorname{norm}_{\mathrm{P}} \mathbf{x}'_j = \mathbf{T} \operatorname{norm}_{\mathrm{P}} \mathbf{x}_j,$$

where only the three first add anything useful. This means that we need at least four point correspondences to determine the $4 \cdot 3 = 12$ parameters in $\mathbf{T}$.

Answer: Four is the smallest number of point correspondences that are needed to uniquely determine $\mathbf{T}$.

3. (a) **Answer:** The common geometric cost function in this case sums the squared point-to-line distances, i.e.

$$\varepsilon(\mathbf{l}) = \sum_{k=1}^{n} \left((\operatorname{norm}_{\mathrm{P}} \mathbf{x}_k)^\top (\operatorname{norm}_{\mathrm{D}} \mathbf{l}) \right)^2.$$

(b) To simplify the notation, assume that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are already P-normalised so that all $\mathbf{x}_k = (\bar{\mathbf{x}}_k, 1)$, where $\bar{\mathbf{x}}_k = (x_k, y_k)$ are the Euclidean coordinates. Assume we use the D-normalised representation $\mathbf{l} = (\hat{\mathbf{l}}, -\Delta)$, where $\|\hat{\mathbf{l}}\| = 1$ and $\Delta \geq 0$.
We can rewrite the cost function as

$$\varepsilon(\mathbf{l}) = \sum_{k=1}^{n} \left(\mathbf{x}_k^\top \mathbf{l} \right)^2 = \sum_{k=1}^{n} \left(\bar{\mathbf{x}}_k^\top \hat{\mathbf{l}} - \Delta \right)^2.$$

The optimal value for Δ is found by setting $\frac{\partial \varepsilon}{\partial \Delta} = 0$, which results in $\Delta = \hat{\mathbf{l}}^\top \bar{\mathbf{s}}$, where $\bar{\mathbf{s}} = \frac{1}{n} \sum_{k=1}^{n} \bar{\mathbf{x}}_k$ is the centroid of the points.
For the optimal Δ, we get

$$\varepsilon(\mathbf{l}) = \sum_{k=1}^{n} \left(\bar{\mathbf{x}}_k^\top \hat{\mathbf{l}} - \Delta \right)^2 = \sum_{k=1}^{n} \left((\bar{\mathbf{x}}_k - \bar{\mathbf{s}})^\top \hat{\mathbf{l}} \right)^2 = \hat{\mathbf{l}}^\top \mathbf{A}\mathbf{A}^\top \hat{\mathbf{l}},$$

where $\mathbf{A} = \begin{pmatrix} \bar{\mathbf{x}}_1 - \bar{\mathbf{s}} & \ldots & \bar{\mathbf{x}}_n - \bar{\mathbf{s}} \end{pmatrix}$. Minimising $\hat{\mathbf{l}}^\top \mathbf{A}\mathbf{A}^\top \hat{\mathbf{l}}$ over $\|\hat{\mathbf{l}}\| = 1$ corresponds to computing the rightmost singular vector of $\mathbf{A}^\top$.

4. (a) **Answer:** Collecting the entries in $\mathbf{C}$ into

$$\mathbf{c} = (c_{11}, c_{12}, c_{13}, c_{14}, c_{22}, c_{23}, c_{24}, c_{33}, c_{34}, c_{44})$$

and defining

$$\mathbf{A}_k = \begin{pmatrix} x_k^2 & 2x_ky_k & 2x_kz_k & 2x_k & y_k^2 & 2y_kz_k & 2y_k & z_k^2 & 2z_k & 1, \end{pmatrix}$$

we can estimate **c** as the null space to the data matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_m \end{pmatrix} \in \mathbb{R}^{m \times 10}.$$

(b) We want the data matrix to have a one-dimensional null space in order to have a unique (up to scale) solution. Since $\operatorname{rank} \mathbf{A} \leq \min(m, 10)$, the smallest number of points needed to determine $\mathbf{C}$ is $m = 9$.

Answer: The smallest number of points needed to determine $\mathbf{C}$ is $m = 9$.

5. (a) For $\mathbf{G}_0$ to define a valid scalar product, it must be symmetric and positive definite. Here, $\mathbf{G}_0$ is clearly symmetric, which means that we only need to find out which a make it positive definite. This can be done in several ways, for example by rewriting (i.e., expanding and then completing the squares)

$$\mathbf{u}^\top \mathbf{G}_0 \mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & a \\ 0 & a & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} u_1 \\ 2u_2 + au_3 \\ au_2 + 2u_3 \end{pmatrix} =$$

$$= u_1^2 + 2u_2^2 + 2au_2u_3 + 2u_3^2 = u_1^2 + 2\left(u_2 + \frac{a}{2}u_3\right)^2 + \left(2 - \frac{a^2}{2}\right)u_3^2.$$

We thus need $2 - \frac{a^2}{2} > 0 \iff |a| < 2$.

Answer: The matrix $\mathbf{G}_0$ defines a valid scalar product precisely when $|a| < 2$.

(b) We compute the scalar products (we only need six of them, since $\mathbf{G}$ is symmetric), resulting in

$$\begin{aligned} \langle \mathbf{b}_1 \mid \mathbf{b}_1 \rangle &= \ldots = 1 \\ \langle \mathbf{b}_2 \mid \mathbf{b}_2 \rangle &= \ldots = 6 \\ \langle \mathbf{b}_3 \mid \mathbf{b}_3 \rangle &= \ldots = 6 \\ \langle \mathbf{b}_1 \mid \mathbf{b}_2 \rangle &= \ldots = 0 \\ \langle \mathbf{b}_1 \mid \mathbf{b}_3 \rangle &= \ldots = 0 \\ \langle \mathbf{b}_2 \mid \mathbf{b}_3 \rangle &= \ldots = 3 \end{aligned}$$

The Gram matrix is

$$\mathbf{G} = \begin{pmatrix} \langle \mathbf{b}_1 \mid \mathbf{b}_1 \rangle & \langle \mathbf{b}_2 \mid \mathbf{b}_1 \rangle & \langle \mathbf{b}_3 \mid \mathbf{b}_1 \rangle \\ \langle \mathbf{b}_1 \mid \mathbf{b}_2 \rangle & \langle \mathbf{b}_2 \mid \mathbf{b}_2 \rangle & \langle \mathbf{b}_3 \mid \mathbf{b}_2 \rangle \\ \langle \mathbf{b}_1 \mid \mathbf{b}_3 \rangle & \langle \mathbf{b}_2 \mid \mathbf{b}_3 \rangle & \langle \mathbf{b}_3 \mid \mathbf{b}_3 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{pmatrix}.$$

Answer: The Gram matrix is $\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{pmatrix}$.

(c) Let $\mathbf{B} = \begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{pmatrix}$. The vector $\mathbf{u} \in U$ closest to $\mathbf{v}$ is given by

$$\mathbf{u} = \mathbf{B}\mathbf{c} = \mathbf{B}\mathbf{G}_{\mathbf{B}}^{-1}\tilde{\mathbf{c}} = \mathbf{B}(\mathbf{B}^\top\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{G}_0\mathbf{v},$$

where $\mathbf{c}$ are the subspace coordinates of $\mathbf{v}$ and $\tilde{\mathbf{c}}$ are the subspace dual coordinates of $\mathbf{v}$, and $\mathbf{G}_{\mathbf{B}} = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$ is the subspace Gram matrix with respect to $\mathbf{B}$. Carrying out the computations, we have

$$\mathbf{u} = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} =$$

$$= \frac{1}{6}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}6 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 7\\ 8\end{pmatrix} =$$

$$= \frac{1}{6}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}6 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 15\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}6\\ 15\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\ 5\\ 5\end{pmatrix}.$$

Answer: The vector $\mathbf{u} \in U$ closest to $\mathbf{v}$ is $\mathbf{u} = \left(1, \frac{5}{2}, \frac{5}{2}\right)$.

6. (a) **Answer:** The frame operator is

$$\mathbf{F} = \mathbf{B}\mathbf{B}^\top\mathbf{G}_0 = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}6 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 6\end{pmatrix}\begin{pmatrix}6 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}11 & -1\\ 0 & 5\end{pmatrix}.$$

(b) The lower frame bound is the smallest eigenvalue of $\mathbf{F}$, and the upper frame bound is the largest eigenvalue of $\mathbf{F}$. In this case, when $\mathbf{F}$ is triangular, we can read the eigenvalues off of the diagonal, giving $L = 5$ and $U = 11$.

Answer: The lower frame bound is $L = 5$ and the upper frame bound $U = 11$.

(c) **Answer:** The dual frame vectors are given by

$$\tilde{\mathbf{B}} = \mathbf{F}^{-1}\mathbf{B} = \frac{1}{55}\begin{pmatrix}5 & 1\\ 0 & 11\end{pmatrix}\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 2\end{pmatrix} = \frac{1}{55}\begin{pmatrix}5 & 6 & 1 & 2\\ 0 & 11 & 11 & 22\end{pmatrix}.$$

7. (a) The only requirement here is the internal constraint $\operatorname{rank}\mathbf{F} = 2$, which implies $\det\mathbf{F} = 0$. It is clear by inspection that $\operatorname{rank}\mathbf{F} \geq 2$ for all values of a and b (the first two columns, for example, cannot be made parallel). This gives the constraint

$$\det\mathbf{F} = \begin{vmatrix}1 & 0 & -1\\ a & 2 & b\\ a & 1 & 3\end{vmatrix} = 6 - a - b + 2a = 6 + a - b = 0.$$

Answer: The values of a and b which make $\mathbf{F}$ a valid fundamental matrix are those satisfying $6 + a - b = 0$.

(b) If $\mathbf{x}_1$ and $\mathbf{x}'_1$ satisfy the epipolar constraint, this means that

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ a & 2 & b\\ a & 1 & 3\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\ a+4+b\\ a+2+3\end{pmatrix} = 2a+b+9 = 0.$$

Together with the internal constraint $6 + a - b = 0$, this yields $a = -5$ and $b = 1$. The epipolar line $\mathbf{l}_2$ is

$$\mathbf{l}_2 = \mathbf{F}\mathbf{x}'_2 = \begin{pmatrix}1 & 0 & -1\\ -5 & 2 & 1\\ -5 & 1 & 3\end{pmatrix}\begin{pmatrix}1\\4\\1\end{pmatrix} = \begin{pmatrix}0\\4\\2\end{pmatrix},$$

and we finally compute the distance as

$$d_{\mathrm{PD}}(\mathbf{x}_2, \mathbf{l}_2) = \left|(\mathrm{norm}_{\mathrm{P}}\,\mathbf{x}_2)^\top(\mathrm{norm}_{\mathrm{D}}\,\mathbf{l}_2)\right| = \left|\frac{1}{2}\begin{pmatrix}0 & 3 & 1\end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix}\right| = \frac{7}{2}.$$

Answer: The distance between $\mathbf{x}_2$ and the corresponding epipolar line is $\frac{7}{2}$.

8. (a) Since

$$\begin{aligned}\varepsilon &= \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\right\|^2\right] = \mathbb{E}\left[(\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v})^\top(\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v})\right] = \\ &= \mathbb{E}\Big[\mathbf{v}^\top\mathbf{v} - 2\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v} + \mathbf{v}^\top\mathbf{B}\underbrace{\mathbf{B}^\top\mathbf{B}}_{=\mathbf{I}}\mathbf{B}^\top\mathbf{v}\Big] = \\ &= \mathbb{E}\left[\mathbf{v}^\top\mathbf{v} - \mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}\right] = \mathbb{E}\left[\|\mathbf{v}\|^2\right] - \mathbb{E}\left[\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}\right] = \mathbb{E}\left[\|\mathbf{v}\|^2\right] - \varepsilon_1,\end{aligned}$$

minimising ε is equivalent to maximising ε_1.

(b) The objective is to find a subspace that minimises the expected norm (squared) of the difference between a vector and its projection on the subspace. If $\mathbf{c}$ are the subspace coordinates of $\mathbf{v}$ and $\tilde{\mathbf{c}}$ are the subspace dual coordinates of $\mathbf{v}$, and $\mathbf{G}_{\mathbf{B}}$ is the subspace Gram matrix, then

$$\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{c}\right\|^2\right] = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{G}_{\mathbf{B}}^{-1}\tilde{\mathbf{c}}\right\|^2\right] = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}(\mathbf{B}^\top\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{G}_0\mathbf{v}\right\|^2\right].$$

It is also important to remember that the norm is induced by the scalar product, and thus also depends on $\mathbf{G}_0$.

Answer: The expression becomes $\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}(\mathbf{B}^\top\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{G}_0\mathbf{v}\right\|^2\right]$.