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TSBB06 Multi-Dimensional Signal Analysis, Solutions 2021-10-25

1. (a) We simply compute the two distances,

$$\left|(\text{norm}_D\,\mathbf{p})^\top(\text{norm}_P\,\mathbf{x}_1)\right| = \left|\frac{-1}{3}(2,-1,-2,6)\cdot(3,1,1,1)\right| = 3,$$

and

$$\left|(\text{norm}_D\,\mathbf{p})^\top(\text{norm}_P\,\mathbf{x}_2)\right| = \left|\frac{-1}{3}(2,-1,-2,6)\cdot(1,-1,0,1)\right| = 3.$$

Answer: Both distances are 3.

(b) For $\mathbf{p}'$ to be parallel to $\mathbf{p}$, it must have the same normal. Thus, we must have $\mathbf{p}' = (2,-1,-2,\xi)$ for some number ξ. Since $\mathbf{x}_1$ should lie on $\mathbf{p}'$, we get

$$0 = (\mathbf{p}')^\top\mathbf{x}_1 = (2,-1,-2,\xi)\cdot(3,1,1,1) = 3+\xi \iff \xi = -3.$$

Answer: The plane is given by $\mathbf{p}' = (2,-1,-2,-3)$.

(c) **Alternative 1:** Every point on the line can be written as $\mathbf{x}(\lambda_1,\lambda_2) = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$, where $\lambda_1^2+\lambda_2^2 \neq 0$. For the intersection point $\mathbf{x}_0$, we have

$$0 = \mathbf{p}^\top\mathbf{x}_0 = \mathbf{p}^\top\mathbf{x}(\lambda_1,\lambda_2) = \mathbf{p}^\top(\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2) = \lambda_1\mathbf{p}^\top\mathbf{x}_1 + \lambda_2\mathbf{p}^\top\mathbf{x}_2 = 9\lambda_1 + 9\lambda_2,$$

which means that $\lambda_2 = -\lambda_1$ at $\mathbf{x}_0$. More concretely, we have

$$\mathbf{x}_0 \sim \mathbf{x}(\lambda_1,-\lambda_1) = \lambda_1(\mathbf{x}_1-\mathbf{x}_2) \sim \mathbf{x}_1-\mathbf{x}_2 = (2,2,1,0).$$

Alternative 2: By computing the signed distances in (a), i.e., without the absolute values, it is clear that $\mathbf{x}_1$ and $\mathbf{x}_2$ lie on the same side of $\mathbf{p}$ as well as at the same distance. The geometrical situation is that the line is parallel to the plane, which means that the intersection point cannot be a proper (finite) point. The intersection must occur at the ideal point of the line, and this is given by

$$\mathbf{x}_0 \sim \text{norm}_P\,\mathbf{x}_1 - \text{norm}_P\,\mathbf{x}_2 = \mathbf{x}_1-\mathbf{x}_2 = (2,2,1,0).$$

Alternative 3: The intersection point is given $\mathbf{x}_0 \sim \mathbf{L}\mathbf{p}$, where $\mathbf{L}$ represents the Plücker coordinates of the line. Using this approach, we have

$$\mathbf{x}_0 \sim (\mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top)\mathbf{p} = \mathbf{x}_1\underbrace{\mathbf{x}_2^\top\mathbf{p}}_{=9} - \mathbf{x}_2\underbrace{\mathbf{x}_1^\top\mathbf{p}}_{=9} \sim \mathbf{x}_1-\mathbf{x}_2 = (2,2,1,0).$$

Answer: The intersection point is $\mathbf{x}_0 = (2,2,1,0)$.

(d) It is readily verified that $(\mathbf{p}')^\top \mathbf{x}_0 = 0$. (In fact, $\mathbf{x}_0$ will lie on *every* plane that is parallel to $\mathbf{p}$.)

2. (a) The points are clearly not collinear either before or after the transformation, which means that they are in *general position*.

An affine transformation can bring any set of three (proper) points into any other set of three (proper) points, as long as both sets contain points in general position. The transformation is therefore clearly *affine*, which is a special case of a *homography* transformation.

To see whether or not it is also a *rigid* transformation, we can check whether it preserves distances. The three distances before the transformation are

$$\begin{aligned} d_{\mathrm{PP}}(\mathbf{x}_1, \mathbf{x}_2) &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1 - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_2\| = \sqrt{5^2 + 1^2} = \sqrt{26} \\ d_{\mathrm{PP}}(\mathbf{x}_1, \mathbf{x}_3) &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1 - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_3\| = \sqrt{(5-12)^2 + (1-5)^2} = \sqrt{65} \\ d_{\mathrm{PP}}(\mathbf{x}_2, \mathbf{x}_3) &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_2 - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_3\| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13, \end{aligned}$$

and after the transformation they are

$$\begin{aligned} d_{\mathrm{PP}}(\mathbf{x}_1', \mathbf{x}_2') &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1' - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_2'\| = \sqrt{(-1-(-2))^2 + (2-(-3))^2} = \sqrt{26} \\ d_{\mathrm{PP}}(\mathbf{x}_1', \mathbf{x}_3') &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1' - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_3'\| = \sqrt{(-1-(-2))^2 + (2-10)^2} = \sqrt{65} \\ d_{\mathrm{PP}}(\mathbf{x}_2', \mathbf{x}_3') &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_2' - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_3'\| = \sqrt{(-3-10)^2} = \sqrt{169} = 13. \end{aligned}$$

Additionally, the points are oriented in the same way before and after the transformation, which can be verified by computing the determinants

$$\begin{vmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{vmatrix} = -13 \qquad \text{and} \qquad \begin{vmatrix} \mathbf{x}_1' & \mathbf{x}_2' & \mathbf{x}_3' \end{vmatrix} = -13.$$

Answer: Yes, the transformation $\mathscr{T}$ can be a rigid transformation. It will also be both affine and a homography.

(b) No three points are collinear before or after the transformation, which means that the four points uniquely determine a homography transformation.

To see that $\mathscr{T}$ can no longer be a rigid transformation, we note that the distances

$$\begin{aligned} d_{\mathrm{PP}}(\mathbf{x}_1, \mathbf{x}_4) &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1 - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_4\| = \sqrt{(5-1)^2 + (1-2)^2} = \sqrt{17} \\ d_{\mathrm{PP}}(\mathbf{x}_1', \mathbf{x}_4') &= \|\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_1' - \mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_4'\| = \sqrt{(-1-1)^2 + 2^2} = 2\sqrt{2} \end{aligned}$$

are different.

If $\mathscr{T}$ were to be affine, it would mean, in particular, that

$$\mathbf{x}_2' = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \mathbf{x}_2 \iff \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \mathbf{t} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}.$$

This makes it easy to solve for $\mathbf{A}$ using, e.g., $\mathbf{x}_1' \leftrightarrow \mathbf{x}_1$ and $\mathbf{x}_3' \leftrightarrow \mathbf{x}_3$. We subtract $\mathbf{t}$ from the Euclidean coordinates of $\mathbf{x}_1'$ and $\mathbf{x}_2'$, and solve

$$\begin{pmatrix} -1-(-2) & -2-(-2) \\ 2-(-3) & 10-(-3) \end{pmatrix} = \mathbf{A} \begin{pmatrix} 5 & 12 \\ 1 & 5 \end{pmatrix},$$

which gives

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} 5 & 12 \\ 1 & 5 \end{pmatrix}^{-1} = \frac{1}{13} \begin{pmatrix} 1 & 0 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} 5 & -12 \\ -1 & 5 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 5 & -12 \\ 12 & 5 \end{pmatrix}.$$

This is not consistent with $\mathbf{x}_4' \leftrightarrow \mathbf{x}_4$, since

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \mathbf{x}_4 = \begin{pmatrix} \frac{17}{13} - 2 \\ \frac{7}{13} - 3 \\ 1 \end{pmatrix} \neq \mathbf{x}_4'.$$

Answer: In this case, $\mathscr{T}$ can neither be a rigid or an affine transformation, but it can be a homography.

3. (a) On the x-axis we have proper points $(x, 0, 1)$ as well as an ideal point $(1, 0, 0)$. For such points, clearly

$$\begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & a \end{pmatrix} \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ax \\ 0 \\ a \end{pmatrix} \sim \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(b) We note that

$$\mathbf{H}_2\mathbf{H}_1 = \begin{pmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & 0 \\ 0 & d_2 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & 0 \\ 0 & d_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_1a_2 & b_1a_2 + c_1b_2 & 0 \\ 0 & c_1c_2 & 0 \\ 0 & c_1d_2 + d_1a_2 & a_1a_2 \end{pmatrix} \in \mathscr{H}_x.$$

Answer: Yes, $\mathscr{H}_x$ is closed under composition.

(c) We recall (or derive) the expression for the cross product matrix,

$$[\mathbf{x}_k']_\times = \begin{pmatrix} 0 & -1 & y_k' \\ 1 & 0 & -x_k' \\ -y_k' & x_k' & 0 \end{pmatrix}.$$

Bearing this in mind, the DLT constraint $(\mathbf{x}_k^\top \otimes [\mathbf{x}_k']_\times)\,\mathrm{vec}\,\mathbf{H}$ becomes

$$\begin{pmatrix} x_k \begin{pmatrix} 0 & -1 & y_k' \\ 1 & 0 & -x_k' \\ -y_k' & x_k' & 0 \end{pmatrix} & y_k \begin{pmatrix} 0 & -1 & y_k' \\ 1 & 0 & -x_k' \\ -y_k' & x_k' & 0 \end{pmatrix} & 1 \begin{pmatrix} 0 & -1 & y_k' \\ 1 & 0 & -x_k' \\ -y_k' & x_k' & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \\ b \\ c \\ d \\ 0 \\ 0 \\ a \end{pmatrix} = \mathbf{0}.$$

Removing the superfluous columns (i.e., the ones meeting the zeroes), and adding the two columns meeting a, we obtain

$$\underbrace{\begin{pmatrix} y'_k & 0 & -y_k & y_k y'_k \\ x_k - x'_k & y_k & 0 & -y_k x'_k \\ -x_k y'_k & -y_k y'_k & y_k x'_k & 0 \end{pmatrix}}_{=\mathbf{A}_k} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

A suitable data matrix will then be

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}.$$

Answer: See above.

4. (a) First, we can find dual homogeneous coordinates $\mathbf{p}$ of the plane π. The condition for π to contain the three points is

$$\begin{pmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{pmatrix} \mathbf{p} = \mathbf{0}.$$

We can find such $\mathbf{p}$, either by Gaußian elimination, or by computing a singular value decomposition

$$\begin{pmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{pmatrix} = \mathbf{USV}^\top$$

and setting $\mathbf{p}$ equal to the rightmost column in $\mathbf{V}$.
The horizon line of π can now be computed as the intersection of π and the plane at infinity, represented by $\mathbf{p}_\infty = (0, 0, 0, 1)$. (The assumption that not all three points are ideal points is necessary to ensure that $\mathbf{p}$ and $\mathbf{p}_\infty$ are distinct.)
If we represent the horizon line using its dual Plücker coordinates $\tilde{\mathbf{L}}$, we have

$$\tilde{\mathbf{L}} = \mathbf{p}\mathbf{p}_\infty^\top - \mathbf{p}_\infty \mathbf{p}^\top.$$

Another option is to find two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ to

$$\begin{pmatrix} \mathbf{p}^\top \\ \mathbf{p}_\infty^\top \end{pmatrix} \mathbf{y} = \mathbf{0},$$

and parametrising the horizon line as $\lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2$.

Answer: See above.

(b) The algebraic cost function can be rewritten as

$$\varepsilon_A(\mathbf{p}) = \|\mathbf{Ap}\|^2 = \left\| \begin{pmatrix} \mathbf{x}_1^\top \mathbf{p} \\ \mathbf{x}_2^\top \mathbf{p} \\ \mathbf{x}_3^\top \mathbf{p} \end{pmatrix} \right\|^2 = \sum_{k=1}^{3} (\mathbf{x}_k^\top \mathbf{p})^2.$$

The only change that is needed to make this into a geometric cost function is to P-normalise and D-normalise appropriately, i.e.

$$\varepsilon_G(\mathbf{p}) = \sum_{k=1}^{3} \left((\mathrm{norm}_{\mathrm{P}}\, \mathbf{x}_k{}^{\top} (\mathrm{norm}_{\mathrm{D}}\, \mathbf{p}) \right)^2 .$$

Answer: The cost function can be made into a geometric cost function by ensuring that the points are P-normalised and the plane is D-normalised. See the expression for $\varepsilon_G(\mathbf{p})$ above.

5. (a) A scalar product must satisfy $\langle \mathbf{u} \mid \mathbf{u} \rangle \geq 0$ with equality precisely when $\mathbf{u} = \mathbf{0}$. With this in mind, it is easy to find counterexamples for f_1 and f_2. For example, if we let $\mathbf{u} = (u_1, u_2) = (-1, 1)$, we see that

$$f_1(\mathbf{u}, \mathbf{u}) = -1 \cdot 1 - 1 \cdot 1 = -2 < 0,$$
$$f_2(\mathbf{u}, \mathbf{u}) = -1 \cdot 1 + 1 \cdot (-1) = -2 < 0,$$

which disqualifies f_1 and f_2 from being a scalar product.
Alternatively, one could verify that f_3 is a scalar product, since we can write

$$\begin{aligned} f_3(\mathbf{u}, \mathbf{v}) &= \mathbf{v}^{\top}\mathbf{u} + (v_1 + v_2)(u_1 + u_2) = \\ &= \mathbf{v}^{\top}\mathbf{I}\mathbf{u} + \mathbf{v}^{\top} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{u} = \\ &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{v}^{\top}\mathbf{G}_0\mathbf{u} \end{aligned}$$

for a symmetric and positive definite matrix $\mathbf{G}_0$.

Answer: The valid scalar product is f_3. See above for an explanation of why f_1 and f_2 are not valid scalar products.

(b) The Gram matrix $\mathbf{G}$ is defined to have its entries $G_{ij} = \langle \mathbf{b}_k \mid \mathbf{b}_j \rangle$, and, by letting $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}$, it can be computed as

$$\begin{aligned} \mathbf{G} &= \mathbf{B}^{*}\mathbf{G}_0\mathbf{B} = [\text{in the real valued case}] = \mathbf{B}^{\top}\mathbf{G}_0\mathbf{B} = \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \neq \mathbf{I}. \end{aligned}$$

This shows that the basis is orthogonal but *not* orthonormal in this scalar product.

Answer: The Gram matrix is $\mathbf{G} = \mathrm{diag}(6, 2)$. The basis is not orthonormal in the chosen scalar product (but it is orthogonal).

(c) Since we already have the Gram matrix, the dual basis vectors can readily be obtained as the columns of

$$\tilde{\mathbf{B}} = \mathbf{B}\mathbf{G}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{12} \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 2 & 6 \\ 2 & -6 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 3 \\ 1 & -3 \end{pmatrix}.$$

Answer: The dual basis vectors are $\tilde{\mathbf{b}}_1 = \left(\frac{1}{6}, \frac{1}{6}\right)$ and $\tilde{\mathbf{b}}_2 = \left(\frac{1}{2}, -\frac{1}{2}\right)$.

6. (a) The frame operator is

$$\mathbf{F} = \mathbf{B}\mathbf{B}^\top \mathbf{G}_0 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -3 \\ 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 0 \\ 0 & 15 \end{pmatrix}.$$

Answer: The frame operator is $\mathbf{F} = \operatorname{diag}(30, 15)$.

(b) The lower frame bound is the smallest eigenvalue of $\mathbf{F}$, and the upper frame bound is the largest eigenvalue of $\mathbf{F}$. In this case, when $\mathbf{F}$ is diagonal, we can read the eigenvalues off of the diagonal, giving $L = 15$ and $U = 30$.

Answer: The lower frame bound is $L = 15$ and the upper frame bound $U = 30$.

(c) The frame operator associated with a set of frame vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is defined, for an arbitrary vector $\mathbf{v}$, as

$$\mathbf{F}\mathbf{v} = \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \mathbf{b}_k.$$

To show that $\mathbf{F}$ is self-adjoint, i.e. that $\langle \mathbf{F}\mathbf{u} \mid \mathbf{v} \rangle = \langle \mathbf{u} \mid \mathbf{F}\mathbf{v} \rangle$ for all $\mathbf{u}$ and $\mathbf{v}$, we verify that

$$\begin{aligned}
\langle \mathbf{F}\mathbf{v} \mid \mathbf{u} \rangle &= \left\langle \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \mathbf{b}_k \,\middle|\, \mathbf{u} \right\rangle = \sum_{k=1}^{n} \left\langle \langle \mathbf{v} \mid \mathbf{b}_k \rangle \mathbf{b}_k \,\middle|\, \mathbf{u} \right\rangle = \\
&= \sum_{k=1}^{n} \langle \mathbf{v} \mid \mathbf{b}_k \rangle \langle \mathbf{b}_k \mid \mathbf{u} \rangle = \sum_{k=1}^{n} \left\langle \mathbf{v} \,\middle|\, \langle \mathbf{b}_k \mid \mathbf{u} \rangle^* \mathbf{b}_k \right\rangle = \\
&= \sum_{k=1}^{n} \left\langle \mathbf{v} \,\middle|\, \langle \mathbf{u} \mid \mathbf{b}_k \rangle \mathbf{b}_k \right\rangle = \left\langle \mathbf{v} \,\middle|\, \sum_{k=1}^{n} \langle \mathbf{u} \mid \mathbf{b}_k \rangle \mathbf{b}_k \right\rangle = \langle \mathbf{v} \mid \mathbf{F}\mathbf{u} \rangle.
\end{aligned}$$

7. (a) First of all, to be a valid fundamental matrix, $\mathbf{F}$ must have rank two. It is clear that $\operatorname{rank} \mathbf{F} \geq 2$, since the last two rows are linearly independent, so if we chose the constants a, b, c such that $\det \mathbf{F} = 0$, the rank will be two. This constraint results in the linear equation

$$0 = \det \mathbf{F} = \begin{vmatrix} a & b & c \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 3a + b - 2c.$$

The epipolar constraints give two additional linear equations:

$$0 = \mathbf{x}_1^\top \mathbf{F} \mathbf{x}_1' = \begin{pmatrix} 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} -a + 2b + c \\ 7 \\ 5 \end{pmatrix} =$$
$$= -5a + 10b + 5c + 12$$

and

$$0 = \mathbf{x}_2^\top \mathbf{F}\mathbf{x}_2' = \begin{pmatrix} -2 & -3 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 & 1 \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix} = -2c + 1.$$

From the last one we immediately obtain $c = \frac{1}{2}$, and inserting this into the previous two allows us to obtain $a = \frac{7}{10}$ and $b = -\frac{11}{10}$.

Answer: $(a, b, c) = \frac{1}{10}(7, 11, 5)$.

(b) **Answer:** The epipolar constraint tells us that $\mathbf{x}_1$ lies on the epipolar line $\mathbf{l}_1 \sim \mathbf{F}\mathbf{x}_1'$ in the first view, and similarly that $\mathbf{x}_1'$ lies on the epipolar line $\mathbf{l}_1' \sim \mathbf{F}^\top \mathbf{x}_1$ in the second view.

Note: The epipolar constraint *does not* tell us that the points are projections of the same scene point!

8. (a) Since

$$\begin{aligned} \varepsilon &= \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\right\|^2\right] = \mathbb{E}\left[(\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v})^\top(\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v})\right] = \\ &= \mathbb{E}\left[\mathbf{v}^\top\mathbf{v} - 2\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v} + \mathbf{v}^\top\mathbf{B}\underbrace{\mathbf{B}^\top\mathbf{B}}_{=\mathbf{I}}\mathbf{B}^\top\mathbf{v}\right] = \\ &= \mathbb{E}\left[\mathbf{v}^\top\mathbf{v} - \mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}\right] = \mathbb{E}\left[\|\mathbf{v}\|^2\right] - \mathbb{E}\left[\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}\right] = \mathbb{E}\left[\|\mathbf{v}\|^2\right] - \varepsilon_1, \end{aligned}$$

minimising ε is equivalent to maximising ε_1.

(b) The objective is to find a subspace that minimises the expected norm (squared) of the difference between a vector and its projection on the subspace. If $\mathbf{c}$ are the subspace coordinates of $\mathbf{v}$ and $\tilde{\mathbf{c}}$ are the subspace dual coordinates of $\mathbf{v}$, and $\mathbf{G}_\mathbf{B}$ is the subspace Gram matrix, then

$$\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{c}\right\|^2\right] = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{G}_\mathbf{B}^{-1}\tilde{\mathbf{c}}\right\|^2\right] = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}(\mathbf{B}^\top\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{G}_0\mathbf{v}\right\|^2\right].$$

It is also important to remember that the norm is induced by the scalar product, and thus also depends on $\mathbf{G}_0$.

Answer: The expression becomes $\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}(\mathbf{B}^\top\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{G}_0\mathbf{v}\right\|^2\right]$.