LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Final Exam 2022-01-11

Course module: TEN2

Date & Time: 2022-01-11 14:00–18:00

Location: TER4

Examiner: Mårten Wadenbäck
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The examiner will make one visit, approximately one hour after the exam has started, and will be available on the phone the rest of the time.

Material & aids: You are allowed to use any non-electronic reference material you like, as long as you *do not bring solutions to previous exams or midterms*. This means that you may bring books, printouts, your own notes, etc.

Scoring: The maximum score on the exam is 36 points, split evenly between Part I and Part II. Grade 3 requires at least 20 points, grade 4 requires at least 25 points, and grade 5 requires at least 30 points.

Instructions: Justify your solutions and answers with clear and concise arguments. All solutions and answers should be written on dedicated pages (i.e., not on the printed exam). Write your AID-number and the exam date on all pages that you hand in. Start each numbered problem on a new page. Right before you hand in, sort your solutions in consecutive order and add page numbers in the upper right corner.

If you passed the voluntary midterm exam (i.e., obtained at least 10 points), you can use your midterm score instead of solving Part I. This is done by *not handing in solutions to any problem from Part I*.

Good luck!

Part I: Geometry and Estimation

1. Consider three lines in the extended Euclidean plane (2D), with (dual) homogeneous coordinates given by

$$\mathbf{l}_1 = (3, 1, 1), \qquad \mathbf{l}_2 = (1, -1, 0), \qquad \mathbf{l}_3 = (0, 1, 1).$$

(a) Compute the Euclidean coordinates of the intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$, and determine its distance to $\mathbf{l}_3$. (2 p)

(b) Determine (dual) homogeneous coordinates $\mathbf{l}$ of the line that passes through the intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$, and which is parallel to $\mathbf{l}_3$. (2 p)

(c) Find the ideal point on $\mathbf{l}_3$, and show that it also lies on the line $\mathbf{l}$ from (b). (1 p)

2. Let $\mathbf{a} \in \mathbb{R}^3$ be homogeneous coordinates of a fixed point in the extended Euclidean plane (2D), and let $\mathscr{H}_{\mathbf{a}}$ be the set of planar homographies $\mathbf{H}$ that leave $\mathbf{a}$ unchanged, i.e. $\mathbf{Ha} \sim \mathbf{a}$.

(a) Show that if $\mathbf{H} \in \mathscr{H}_{\mathbf{a}}$, then so is $\mathbf{H} + s[\mathbf{a}]_\times$, for all $s \in \mathbb{R}$. (1 p)

(b) Show that $\mathscr{H}_{\mathbf{a}}$ is a group (with respect to composition). (2 p)

(c) How many degrees of freedom (DoF) does a homography in $\mathscr{H}_{\mathbf{a}}$ have? (1 p)

3. Consider a 3D *rotation matrix* $\mathbf{R}$ which satisfies

$$\mathbf{R} \sim \begin{pmatrix} 1 & 8 & a \\ -4 & 4 & b \\ 8 & 1 & c \end{pmatrix} \iff \mathbf{R} = \lambda \begin{pmatrix} 1 & 8 & a \\ -4 & 4 & b \\ 8 & 1 & c \end{pmatrix}$$

for some scalars $\lambda, a, b, c \in \mathbb{R}$.

(a) Determine $a, b, c \in \mathbb{R}$ and $\lambda > 0$ which make $\mathbf{R}$ a valid rotation matrix. (2 p)

(b) Using the values for λ, a, b, c from (a), compute an axis–angle representation of the rotation represented by $\mathbf{R}$. (3 p)

4. Let $\mathbf{X}_1, \dots, \mathbf{X}_n \in \mathbb{R}^4$ be homogeneous coordinates of points in the extended Euclidean space (3D). Suppose we wish to fit a plane, with (dual) homogeneous coordinates $\mathbf{p}$, to the points $\mathbf{X}_1, \dots, \mathbf{X}_n$.

(a) If we are content to minimise an algebraic cost function, the estimation problem can be solved by constructing a data matrix $\mathbf{A}$ and taking $\mathbf{p}$ as its approximate null space. Suggest how to construct such a data matrix $\mathbf{A}$. (2 p)

(b) Formulate a *geometric cost function* $\varepsilon_G(\mathbf{p})$ for the estimation problem. (2 p)

Part II: Linear Signal Representation, Analysis, and Applications

5. Let $\mathbb{R}^3$ be equipped with the scalar product $\mathbf{G}_0 = \frac{1}{8}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$. The vectors $\mathbf{b}_1 = (1, 1, 0)$ and $\mathbf{b}_2 = (0, 0, 1)$ are normalised and mutually orthogonal with respect to $\mathbf{G}_0$.

(a) Determine a vector $\mathbf{b}_3$ such that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. (2 p)

(b) Let $\mathbf{v} = (1, 2, 3)$. Find the vector $\mathbf{u}$ in the subspace spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$ that is the closest to $\mathbf{v}$ (as measured by the norm induced by the scalar product). (2 p)

6. Let $\mathbf{B} = \begin{pmatrix} 1 & 0 & a \\ 0 & 2 & a \end{pmatrix}$ be a matrix whose columns hold a set of frame vectors in $\mathbb{R}^2$, and let $\mathbb{R}^2$ be equipped with the scalar product $\mathbf{G}_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The *dual frame vectors* will then be found as the columns of $\tilde{\mathbf{B}} = \frac{1}{10+8a^2}\begin{pmatrix} 4 + a^2 & -2a^2 & 4a \\ -2a^2 & 4 + 4a^2 & 2a \end{pmatrix}$.

(a) Compute the frame operator $\mathbf{F}$ corresponding to the frame vectors in $\mathbf{B}$. (1 p)

(b) Determine $a > 0$ such that the *lower frame bound* $L = 3$ and the *upper frame bound* $U = 6$. (2 p)

(c) Use the value of a you determined in (b), and let $\mathbf{v} = (3, 15)$. Find the reconstructing coefficients $\mathbf{c} \in \mathbb{R}^3$, with the smallest Euclidean norm, that satisfy $\mathbf{v} = \mathbf{B}\mathbf{c}$. (If you did not solve (b), and *only in that case*, you can use $a = -1$ here.) (2 p)

7. Consider two perspective pinhole cameras with camera matrices given by

$$\mathbf{C}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C}_2 = \begin{pmatrix} 2 & -3 & 0 & 5 \\ 2 & 1 & 2 & 3 \\ -4 & -1 & 1 & -2 \end{pmatrix}.$$

(a) Which of the cameras is located at $(-1, 1, -1)$? (1 p)

(b) Compute the the two epipoles (one in each view) for the camera pair. (2 p)

(c) There exists an $a \in \mathbb{R}$ such that the *fundamental matrix* $\mathbf{F}$ of this camera pair is

$$\mathbf{F} = \begin{pmatrix} 8 & -16 & a \\ -7 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine this a. (2 p)

8. In *principal component analysis* (PCA), we want to find a matrix $\mathbf{B}$ whose columns are orthonormal vectors, such that

$$\varepsilon = \mathbb{E}\left[\left\|\mathbf{v} - \mathbf{B}\mathbf{B}^\top\mathbf{v}\right\|^2\right]$$

is minimised. ($\mathbb{E}$ means taking the expectation over all vectors, and we have made the simplifying assumption that $\mathbf{G}_0 = \mathbf{I}$.) It can be shown that minimising ε is equivalent to maximising $\varepsilon_1 = \mathbb{E}[\mathbf{v}^\top\mathbf{B}\mathbf{B}^\top\mathbf{v}]$.

(a) Let us focus on the case where $\mathbf{B}$ consists of only one column, i.e. $\mathbf{B} = \mathbf{b}_1$. Use, e.g., Lagrange's method of multipliers to show that $\mathbf{b}_1$ is an eigenvector of the correlation matrix $\mathbf{C} = \mathbb{E}[\mathbf{v}\mathbf{v}^\top]$ corresponding to the largest eigenvalue λ_1. (3 p)

(b) If we only use the most significant principal component to represent our signals, what is the expected reconstruction error ε? (1 p)