LINKÖPING UNIVERSITY

TSBB06 Multi-Dimensional Signal Analysis, Solutions 2022-01-11

1. (a) The intersection point can be found as $\mathbf{x}_{12} \sim \mathbf{l}_1 \times \mathbf{l}_2 = (3,1,1) \times (1,-1,0) = (1,1,-4)$. The distance between this point and the line $\mathbf{l}_3$ is

$$\left|(\text{norm}_\text{D}\,\mathbf{l}_3)^\top(\text{norm}_\text{P}\,\mathbf{x}_{12})\right| = \left|(0,-1,-1)\cdot\left(-\frac{1}{4},-\frac{1}{4},1\right)\right| = \frac{3}{4}.$$

Answer: The Euclidean coordinates of the intersection point are $\left(-\frac{1}{4},-\frac{1}{4}\right)$, and the distance from this point to $\mathbf{l}_3$ is $\frac{3}{4}$.

(b) We can parametrise $\mathbf{l} = (0,1,s)$, and since the intersection point $(1,1,-4)$ should lie on $\mathbf{l}$, it follows that $(0,1,s)\cdot(1,1,-4) = 0 \iff s = \frac{1}{4}$.

Answer: The line is $\mathbf{l} = \left(0,1,\frac{1}{4}\right)$.

(c) The ideal point on $\mathbf{l}_3$ can be found, e.g., as the intersection of $\mathbf{l}_3$ and the line at infinity, $\mathbf{l}_\infty = (0,0,1)$. This is found as $\mathbf{x}_\infty \sim \mathbf{l}_3 \times \mathbf{l}_\infty = (1,0,0)$. It is readily verified that $\mathbf{l}^\top\mathbf{x}_\infty = 0$ (of course, since the lines are parallel).

Answer: The ideal point on $\mathbf{l}_3$ is $\mathbf{x}_\infty = (1,0,0)$.

2. (a) With the assumption that $\mathbf{Ha} \sim \mathbf{a}$, it follows that

$$\left(\mathbf{H} + s[\mathbf{a}]_\times\right)\mathbf{a} = \mathbf{Ha} + s\underbrace{[\mathbf{a}]_\times\mathbf{a}}_{=\mathbf{0}} = \mathbf{Ha} \sim \mathbf{a},$$

showing that $\mathbf{H} + s[\mathbf{a}]_\times \in \mathscr{H}_\mathbf{a}$, regardless of s.

(b) We need to check *closure*, *existence of an identity* (neutral element), *existence of inverse*, and *associativity*.

Associativity: This follows from the associativity of matrix multiplication.

Closure: If $\mathbf{H}_1 \in \mathscr{H}_\mathbf{a}$ and $\mathbf{H}_2 \in \mathscr{H}_\mathbf{a}$, then $\mathbf{H}_2\mathbf{H}_1\mathbf{a} \sim \mathbf{H}_2\mathbf{a} \sim \mathbf{a}$, so $\mathbf{H}_2\mathbf{H}_1 \in \mathscr{H}_\mathbf{a}$.

Identity: Clearly $\mathbf{I} \in \mathscr{H}_\mathbf{a}$, and for any $\mathbf{H} \in \mathscr{H}_\mathbf{a}$ it holds that $\mathbf{IH} = \mathbf{HI} = \mathbf{H}$.

Existence of inverse: Homographies are, by definition, invertible, but we still need to check that the inverse is actually in $\mathscr{H}_\mathbf{a}$. This is indeed the case, since any $\mathbf{H} \in \mathscr{H}_\mathbf{a}$ has $\mathbf{a}$ as an eigenvector with a corresponding eigenvalue $\lambda \neq 0$, and thus $\mathbf{Ha} = \lambda\mathbf{a} \iff \mathbf{H}^{-1}\mathbf{a} = \frac{1}{\lambda}\mathbf{a} \sim \mathbf{a}$.

(c) A *general* homography has eight degrees of freedom, and we know that four point correspondences (where no three points are collinear) provide sufficient constraints to uniquely determine the homography. For a homography to belong to $\mathscr{H}_{\mathbf{a}}$, it is necessary to have a correspondence $\mathbf{a} \leftrightarrow \mathbf{a}$, leaving six degrees of freedom.

Answer: The homographies in $\mathscr{H}_{\mathbf{a}}$ have six degrees of freedom.

3. (a) For $\mathbf{R}$ to be a valid rotation matrix, it must satisfy $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$. In particular, all columns must have length one, which gives $\lambda = \frac{1}{\sqrt{1^2+(-4)^2+8^2}} = \frac{1}{9}$. Now, the third column is given by

$$\frac{1}{9}(a,b,c) = \frac{1}{9}(1,-4,8) \times \frac{1}{9}(8,4,1) = \frac{1}{81}(-36,63,36) = \frac{1}{9}(-4,7,4).$$

Answer: The only possibility is to have $\lambda = \frac{1}{9}$ and $(a,b,c) = (-4,7,4)$.

(b) With the values from (a), the rotation matrix is

$$\mathbf{R} = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ -4 & 4 & 7 \\ 8 & 1 & 4 \end{pmatrix}.$$

Recall Rodrigues' formula, $\mathbf{R} = \mathbf{I} + \sin\varphi\,[\hat{\mathbf{n}}]_\times + (1-\cos\varphi)\,[\hat{\mathbf{n}}]_\times^2$. It follows that

$$\sin\varphi\,[\hat{\mathbf{n}}]_\times = \frac{\mathbf{R}-\mathbf{R}^\top}{2} = \frac{1}{9}\begin{pmatrix} 0 & 6 & -6 \\ -6 & 0 & 3 \\ 6 & -3 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}.$$

We can choose $\hat{\mathbf{n}} = -\frac{1}{3}(1,2,2)$, which is normalised, and $\sin\varphi = 1 \Rightarrow \varphi = \frac{\pi}{2} + 2k\pi$. NOTE: In general, $\sin\varphi = \sin(\pi-\varphi)$, so if $\varphi \neq \pm\frac{\pi}{2}$, we need to disambiguate the angle using the additional constraint $\cos\varphi = \frac{\operatorname{tr}\mathbf{R}-1}{2}$.

Answer: One possible axis–angle representation of the rotation is $\hat{\mathbf{n}} = -\frac{1}{3}(1,2,2)$ and $\varphi = \frac{\pi}{2}$.

4. (a) Ideally, we want $\mathbf{p}$ to satisfy $\mathbf{X}_j^\top\mathbf{p} = 0$ for $j = 1,\ldots,n$. Together, this can be formulated as $\mathbf{A}\mathbf{p} = \mathbf{0}$ with a data matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{X}_1^\top \\ \vdots \\ \mathbf{X}_n^\top \end{pmatrix}.$$

(b) **Answer:** A geometric cost function for this problem is

$$\varepsilon_{\mathrm{G}}(\mathbf{p}) = \sum_{j=1}^{n}\Big((\operatorname{norm}_{\mathrm{P}}\mathbf{X}_j)^\top(\operatorname{norm}_{\mathrm{D}}\mathbf{p})\Big)^2.$$

5. (a) We seek a vector $\mathbf{b}_3$ such that $\langle \mathbf{b}_1 \mid \mathbf{b}_3 \rangle = \langle \mathbf{b}_2 \mid \mathbf{b}_3 \rangle = 0$ and $\langle \mathbf{b}_3 \mid \mathbf{b}_3 \rangle = 1$. The two orthogonality constraints tell us that

$$\begin{cases} \mathbf{b}_3^\top \mathbf{G}_0 \mathbf{b}_1 = 0 \\ \mathbf{b}_3^\top \mathbf{G}_0 \mathbf{b}_2 = 0 \end{cases} \quad \Rightarrow \quad \mathbf{b}_3 \sim (\mathbf{G}_0 \mathbf{b}_1) \times (\mathbf{G}_0 \mathbf{b}_2) \sim (1, 1, 0) \times (0, 0, 1) \sim (1, -1, 0).$$

Now we need to normalise this with respect to the scalar product to obtain our $\mathbf{b}_3$, so we let $\mathbf{b}_3 = (s, -s, 0)$. Since

$$\frac{1}{8} \begin{pmatrix} s & -s & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} s \\ -s \\ 0 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} s & -s & 0 \end{pmatrix} \begin{pmatrix} 2s \\ -2s \\ 0 \end{pmatrix} = \frac{4s^2}{8},$$

we can choose $s = \pm\sqrt{2}$, resulting in either $\mathbf{b}_3 = (\sqrt{2}, -\sqrt{2}, 0)$ or $\mathbf{b}_3 = (-\sqrt{2}, \sqrt{2}, 0)$.

Answer: The possible choices are $\mathbf{b}_3 = (\sqrt{2}, -\sqrt{2}, 0)$ and $\mathbf{b}_3 = (-\sqrt{2}, \sqrt{2}, 0)$.

(b) The best subspace approximation $\mathbf{u}$ of $\mathbf{v}$ can be found as

$$\mathbf{u} = \langle \tilde{\mathbf{b}}_1 \mid \mathbf{v} \rangle \mathbf{b}_1 + \langle \tilde{\mathbf{b}}_2 \mid \mathbf{v} \rangle \mathbf{b}_2,$$

where $\tilde{\mathbf{b}}_1$ and $\tilde{\mathbf{b}}_2$ are the corresponding dual basis vectors to $\mathbf{b}_1$ and $\mathbf{b}_2$. However, since the basis is orthogonal, it is its own dual basis, and consequently

$$\mathbf{u} = \langle \mathbf{b}_1 \mid \mathbf{v} \rangle \mathbf{b}_1 + \langle \mathbf{b}_2 \mid \mathbf{v} \rangle \mathbf{b}_2.$$

Computing the subspace coordinates

$$\begin{cases} \langle \mathbf{b}_1 \mid \mathbf{v} \rangle = \frac{1}{8} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} = \frac{3}{2}, \\ \langle \mathbf{b}_2 \mid \mathbf{v} \rangle = \frac{1}{8} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = 3, \end{cases}$$

we finally have $\mathbf{u} = \frac{3}{2}\mathbf{b}_1 + 3\mathbf{b}_2 = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

Answer: The best approximation of $\mathbf{v}$ is $\mathbf{u} = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

6. (a) The frame operator is

$$\mathbf{F} = \mathbf{B}\mathbf{B}^\top \mathbf{G}_0 = \begin{pmatrix} 1 & 0 & a \\ 0 & 2 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ a & a \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 + 2a^2 & a^2 \\ 2a^2 & 4 + a^2 \end{pmatrix}.$$

Answer: The frame operator is $\mathbf{F} = \begin{pmatrix} 2 + 2a^2 & a^2 \\ 2a^2 & 4 + a^2 \end{pmatrix}$.

(b) The lower frame bound $L = 3$ is the smallest eigenvalue of $\mathbf{F}$, and the upper frame bound $U = 6$ is the largest eigenvalue of $\mathbf{F}$. Since $\mathbf{F}$ only has two eigenvalues (it is a 2×2 matrix), we know that

$$\begin{cases} \operatorname{tr} \mathbf{F} = L + U \\ \det \mathbf{F} = LU \end{cases} \iff \begin{cases} 2 + 2a^2 + 4 + a^2 = 3 + 6 \\ (2 + 2a^2)(4 + a^2) - 2a^4 = 3 \cdot 6 \end{cases} \iff \begin{cases} 6 + 3a^2 = 9 \\ 8 + 10a^2 = 18 \end{cases}.$$

This means that $a = 1$.

Answer: $a = 1$ gives the desired frame bounds.

(c) The reconstructing coefficients with smallest Euclidean norm are given as

$$\mathbf{c} = \tilde{\mathbf{B}}^{\top} \mathbf{G}_0 \mathbf{v} = \frac{1}{18} \begin{pmatrix} 5 & -2 \\ -2 & 8 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 15 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 0 \\ 108 \\ 54 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}.$$

Answer: The sought reconstructing coefficients are $\mathbf{c} = (0, 6, 3)$.
(If you used $a = -1$, then $\mathbf{c} = (0, 6, -3)$.)

7. (a) The homogeneous coordinates of the camera centre are found as the null space of the camera matrix. The homogeneous coordinates of $(-1, 1, -1)$ are $(-1, 1, -1, 1)$, and we see that

$$\mathbf{C}_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\mathbf{C}_2 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 0 & 5 \\ 2 & 1 & 2 & 3 \\ -4 & -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Answer: The camera $\mathbf{C}_2$ is located at $(-1, 1, -1)$.

(b) From (a), we know that the centre of the camera $\mathbf{C}_2$ is $\mathbf{n}_2 = (-1, 1, -1, 1)$, meaning that the epipole $\mathbf{e}_{12}$ is $\mathbf{e}_{12} = \mathbf{C}_1 \mathbf{n}_2 = (0, 0, 1)$. To compute the epipole in the other view, we first need to compute the camera centre of $\mathbf{C}_1$. This is found as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} -2 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

so we can take $\mathbf{n}_1 = (-2, 2, -2, 1)$. Projecting it into the second view gives the epipole $\mathbf{e}_{21} = \mathbf{C}_2 \mathbf{n}_1 = (-5, -3, 2)$.

Answer: The epipoles are $\mathbf{e}_{12} = (0, 0, 1)$ and $\mathbf{e}_{21} = (-5, -3, 2)$.

(c) The most direct way is to use the fact that $\mathbf{F}\mathbf{e}_{21} = \mathbf{0}$, which immediately gives us $a = -4$.

Another possibility is project a 3D point $\mathbf{X}$ into the two views, and then use the epipolar constraint. This will also result in $a = -4$.

Answer: For $\mathbf{F}$ to be the fundamental matrix for this camera pair, we need $a = -4$.

8. (a) In the case where $\mathbf{B} = \mathbf{b}_1$, we have

$$\varepsilon_1 = \mathbb{E}[\mathbf{v}^\top \mathbf{b}_1 \mathbf{b}_1^\top \mathbf{v}] = \mathbb{E}[\mathbf{b}_1^\top \mathbf{v}\mathbf{v}^\top \mathbf{b}_1] = \mathbf{b}_1^\top \mathbb{E}[\mathbf{v}\mathbf{v}^\top]\mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{C}\mathbf{b}_1.$$

With the constraint $g(\mathbf{b}_1) = \mathbf{b}_1^\top \mathbf{b}_1 = 1$, the constrained optima of ε_1 are found where $\nabla \varepsilon_1 \parallel \nabla g \iff 2\mathbf{C}\mathbf{b}_1 = 2\lambda \mathbf{b}_1 \iff \mathbf{C}\mathbf{b}_1 = \lambda \mathbf{b}_1$, which means that $\mathbf{b}_1$ is an eigenvalue of $\mathbf{C}$ with corresponding eigenvalue λ. The largest eigenvalue λ is the one that maximises $\varepsilon_1 = \mathbf{b}_1^\top \mathbf{C}\mathbf{b}_1 = \lambda \mathbf{b}_1^\top \mathbf{b}_1 = \lambda$.

(b) The expected reconstruction error is the sum of the eigenvalues corresponding to the unused eigenvectors of $\mathbf{C}$. Assuming the eigenvalues are ordered in decreasing order, this means that

$$\varepsilon = \sum_{j=2}^{n} \lambda_j = \operatorname{tr} \mathbf{C} - \lambda_1.$$

Answer: See above.