TSBB06 TEN2 Instructions

Welcome to TEN2, the final exam in TSBB06!

The written examination in TSBB06 has 2 parts: One part for the course content in HT1 and one for the content from HT2.

There are 10 problems in part I (corresponding to the midterm, KTR1), and these are grouped under the two headings Geometry, and Estimation. There are another 10 problems in part II, grouped under Linear Signal Representations, and Signal Processing Applications.

The exam problems are of two types: **Type A** are 1 point problems, and **Type B** are 2 point problems. Half points can be awarded.

The maximal score on the final exam is 28p, and grading of the exam is done based on your score as follows:

If you obtain **15p** or more you get **grade 3**.
If you obtain **18p** or more you get **grade 4**.
If you obtain **22p** or more you get **grade 5**.

All tasks should be answered on **separate sheets** (i.e. not on the printed exam). It is fine to answer multiple questions on one sheet.

Write your AID-number and the date on all paper sheets that you hand in. In addition, these sheets should be numbered in consecutive order.

Good luck!
Mårten Wadenbäck

PART I, Geometry

Problem 1 (A, 1p) Consider two points in the extended Euclidean space (3D), with homogeneous coordinates given by $\mathbf{x}_1 = (3, 0, 1, 1)$ and $\mathbf{x}_2 = (4, 1, -2, 1)$, and let ℓ be the line through $\mathbf{x}_1$ and $\mathbf{x}_2$.

Determine the *ideal point* (the point at infinity) on ℓ.

Problem 2 (A, 1p) Three points in the extended Euclidean plane, with Euclidean coordinates

$$\mathbf{x}_1 = (10, 5), \qquad \mathbf{x}_2 = (14, 2), \qquad \mathbf{x}_3 = (11, 3)$$

are transformed into new points $\mathbf{x}'_i = \mathcal{T}(\mathbf{x}_i)$, for an unspecified transformation $\mathcal{T}$, such that

$$\mathbf{x}'_1 = (-12, 24), \qquad \mathbf{x}'_2 = (-7, 24), \qquad \mathbf{x}'_3 = (-10, 23).$$

Is it possible that the transformation $\mathcal{T}$ is a *rigid* transformation?
An *affine* transformation? A *homography* transformation?

Problem 3 (A, 1p) Suppose a 3D rotation (which is not the identity rotation) is represented by the axis–angle pair $(\hat{\mathbf{n}}, \alpha)$.

What other representations $(\hat{\mathbf{n}}', \alpha')$ result in the same rotation?

Problem 4 (B, 2p) Consider a line ℓ in the extended Euclidean plane, represented by the Plücker coordinates

$$\mathbf{L} = \begin{pmatrix} 0 & 5 & 4 \\ -5 & 0 & 3 \\ -4 & -3 & 0 \end{pmatrix}.$$

The ideal point on $\mathbf{L}$ has homogeneous coordinates $\mathbf{x}_\infty = (4, 3, 0)$.

(a) Find an additional point on ℓ. (1p)

(b) Compute the shortest distance from the origin to ℓ. (1p)

Problem 5 (B, 2p) Homogeneous representations of three distinct points in the extended Euclidean space (3D) are given as the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^4$. Assume that at least one of the points is finite (i.e. not an ideal point), and let π be the plane spanned by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

Devise a method that finds the *horizon line* of π, i.e. all ideal points on π.

PART I, Estimation

Problem 6 (A, 1p) Explain the orthogonal Procrustes problem (OPP). What is known, and what is to be determined? How is this done?

Problem 7 (A, 1p) Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^4$ are homogeneous coordinates of points in 3D space. One can fit a plane to these points by minimising a suitable geometric error. Define such an error and write out the formula explicitly.

Problem 8 (A, 1p) A model is fitted to a very large amount of experimental data, and it turns out that the model fits the data *exactly* (i.e. zero error).

Explain why this most likely is a bad model.

Problem 9 (B, 2p) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be Euclidean coordinates of points in the extended Euclidean plane. Suppose the points $\mathbf{x}_i$ are transformed into new points with Euclidean coordinates $\mathbf{x}'_i$ by means of an *affine* transformation. For each $i = 1, \ldots, n$, let $\mathbf{y}_i$ be a noisy measurement of $\mathbf{x}_i$ and let $\mathbf{y}'_i$ be a noisy measurement of $\mathbf{x}'_i$, where the noise is i.i.d. (independent, idendically distributed). Now suppose we estimate a *rigid transformation* $\mathbf{T}_R$, an *affine transformation* $\mathbf{T}_A$, and a *homography* $\mathbf{T}_H$ from the correspondences $\mathbf{y}_i \leftrightarrow \mathbf{y}'_i$, in such a way as to minimise the same geometric error.

How will the values of the geometric error be related for the estimated transformations (i.e. which of them will be smallest and which will be largest)?

Problem 10 (B, 2p) A diagonal matrix $\mathbf{\Lambda} = \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & -a^2b^2 \end{pmatrix}$ can be used to represent an *ellipse* that is centered at the origin and has its semi-axes parallel to the coordinate axes. The ellipse consists of all $\mathbf{x} \sim (x, y, 1)$ satisfying the equation $\mathbf{x}^\top \mathbf{\Lambda} \mathbf{x} = 0$, or, written out explicitly,

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

Suppose that we want to estimate $\mathbf{\Lambda}$ from a number of known points $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

a) The equation of the ellipse is linear in the entries of $\mathbf{\Lambda}$. Construct a suitable data matrix, and explain how it can be used to estimate $\mathbf{\Lambda}$ from the known points $\mathbf{x}_1, \ldots, \mathbf{x}_m$. (1p)

b) What is the smallest number of points needed to determine $\mathbf{\Lambda}$? (1p)

PART II, Linear Signal Representations

Problem 11 (A, 1p) Suppose $\mathbf{B}$ is a matrix that contains a set of frame vectors as its columns, and correspondingly for the dual frame vectors in $\tilde{\mathbf{B}}$.

Describe the set of reconstructing coefficients for $\mathbf{v}$, i.e. the set of coefficients $\mathbf{c}$ for which $\mathbf{v} = \mathbf{B}\mathbf{c}$.

Problem 12 (A, 1p) Let $\mathbf{B}$ be a matrix that contains a set of frame vectors in its columns, and let $\tilde{\mathbf{B}}$ be the corresponding matrix for the dual frame. How are the two matrices related, assuming that the scalar product is defined by $\mathbf{G}_0 = \mathbf{I}$?

Problem 13 (A, 1p) Let $\langle \mathbf{u} \mid \mathbf{v} \rangle = \mathbf{v}^\top \mathbf{G}_0 \mathbf{u}$ define the scalar product in $\mathbb{R}^2$, where $\mathbf{G}_0 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ for some positive a_1, a_2. Let $\mathbf{b}_1 = (1, 0)$ and $\mathbf{b}_2 = (0, 1)$ constitute a basis, and compute the *dual basis* vectors $\tilde{\mathbf{b}}_1$ and $\tilde{\mathbf{b}}_2$.

Problem 14 (B, 2p) The discrete Fourier transform maps a set of N samples $f[k]$ in the time domain to N samples $F[l]$ in the frequency domain, and it is defined according to the equation below.

$$F[l] = \sum_{k=0}^{N-1} f[k]\, e^{-i2\pi kl/N}$$

This expression suggests that the set of all $F[l]$, $l \in [0, N-1]$ are the coordinates of the sequence $f[k]$, $k \in [0, N-1]$, relative to some set of basis functions. What basis functions are used, and what scalar product is used?

Problem 15 (B, 2p) The convolution operation can be interpreted either as computations of scalar products or as forming linear combinations. Explain these two distinct interpretations.

PART II, Signal Processing Applications

Problem 16 (A, 1p) A band-limited continuous-time signal is reconstructed from a sampled and quantized version of the signal. Under certain assumptions, the noise in the reconstructed signal can be reduced by oversampling the signal. Explain the general idea behind this.

Problem 17 (A, 1p) The epipolar constraint can formulated as $\mathbf{u}^\top \mathbf{F} \mathbf{v} = 0$, for the image coordinates, $\mathbf{u} = (u_1, u_2, 1)$, $\mathbf{v} = (v_1, v_2, 1)$. After stereo rectification, the fundamental matrix, $\mathbf{F}$ has a special form. Write down this matrix, and the constraint it encodes in terms of the image coordinates.

Problem 18 (A, 1p) Let $\mathbf{u} = (u_1, u_2, 1)$ and $\mathbf{v} = (v_1, v_2, 1)$ be image points in two distinct camera views. Given that the epipolar constraint $\mathbf{u}^\top \mathbf{F} \mathbf{v} = 0$ is satisfied, what does this tell us about $\mathbf{u}$ and $\mathbf{v}$?

Problem 19 (B, 2p) The discrete wavelet transform can be implemented in terms of an iterated filter bank. Describe how this is done, and for which values of the scale and translation parameters, a and b, the transform is computed, given that the filter bank has N levels.

Problem 20 (B, 2p) The correlation matrix $\mathbf{C}$ has been determined for a particular set of N-dimensional signals. The eigenvalues of $\mathbf{C}$ are λ_k with corresponding normalized eigenvectors $\mathbf{e}_k$, $k = 1, \ldots, N$. For a particular application, M principal components are used to represent the signal. How are they chosen? How large is the expected reconstruction error in this case?