Guide* to answers for written examination in TSBB06 Multi-dimensional signal analysis, 2017-01-10

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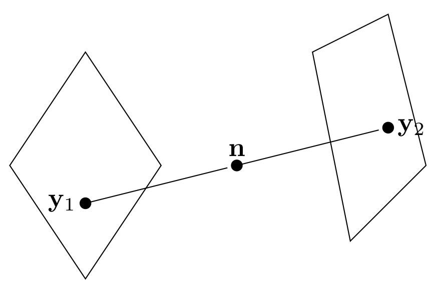
PART I

Exercise 1 The Cartesian coordinates of the two points are obtained by P-normalization:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}, \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

The distance between these two points is given, as usual, as $\sqrt{(3/2-3)^2 + (-1/2+2)^2} = 3/\sqrt{2} \approx 2.12$.

Exercise 2 A homography is defined as a mapping from a point $\mathbf{y}_1$ in one plane to a point $\mathbf{y}_2$ in another plane, which we get by drawing a line from $\mathbf{y}_1$, through a fixed point $\mathbf{n}$, and finding the intersection of this line with the second plane. That intersection point is $\mathbf{y}_2$. See the figure.



Exercise 3 The two quaternions are given as

$$\pm \begin{pmatrix} \cos\frac{\alpha}{2} \\ \hat{\mathbf{n}} \sin\frac{\alpha}{2} \end{pmatrix} = \pm \begin{pmatrix} \cos\frac{\pi}{4} \\ 0.6 \cdot \sin\frac{\pi}{4} \\ 0.8 \cdot \sin\frac{\pi}{4} \\ 0 \end{pmatrix} \approx \pm \begin{pmatrix} 0.71 \\ 0.42 \\ 0.57 \\ 0 \end{pmatrix}$$

Exercise 4 Alternative I: Form 3×4 matrix $\mathbf{X}$ that holds the homogeneous coordinates of the three points in its rows. Since $\mathbf{x}_k \cdot \mathbf{p} = 0$ for each of the three points, it then follows that $\mathbf{X}\,\mathbf{p} = \mathbf{0}$. This means that we find $\mathbf{p}$ in the null space of $\mathbf{X}$. We can determine the null space by applying the SVD to the matrix: $\mathbf{X} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^\top$. The null space is spanned by the right singular vector corresponding to the zero singular value, i.e., to the rightmost columns of $\mathbf{V}$.

*This guide is not an authoritative description of how answers to the questions must be given in order to pass the exam. Some explanations given here may not have to be included in the answer, unless explicitly called for.

Alternative II: Form the Plücker coordinates $\mathbf{L}$ of the line that passes through $\mathbf{x}_1$ and $\mathbf{x}_2$: $\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top$. By means of the duality mapping we can represent the same line as the dual Plücker coordinates $\tilde{\mathbf{L}}$. Finally, we get the plane that intersect both the line and the third point $\mathbf{x}_3$ as $\mathbf{p} = \tilde{\mathbf{L}}\,\mathbf{x}_3$. Notice that we may start with a line that passes through any pair of points, the resulting plane will be the same.

Exercise 5 The rigid transformation is represented by the transformation matrix

$$\mathbf{T} = \begin{pmatrix} \mathbf{R} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix}$$

We apply this transformation to the homogeneous coordinates

$$\mathbf{y} = \begin{pmatrix} \bar{\mathbf{y}} \\ 0 \end{pmatrix}, \quad \text{where} \quad \bar{\mathbf{y}} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and get

$$\mathbf{T}\,\mathbf{y} = \begin{pmatrix} \mathbf{R} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{R}\,\bar{\mathbf{y}} \\ 0 \end{pmatrix}.$$

This is, again, a point at infinity. It lies in a direction that has been rotated by $\mathbf{R}$ relative to the original direction.

The resulting point is unaffected by the translation part $\bar{\mathbf{t}}$. This is reasonable since a finite translation of any point that is infinitely fart from any proper point, will produce a point that still is at infinity. Hence, the translation will be ignored by this point.

An alternative argument why the point is unaffected by the translation is that a point at infinity can be seen as the intersection of two parallel lines. The translation may move these lines around, but not change their orientation. Therefore, they still intersect at the same point at infinity after the translation.

PART II

Exercise 6 The points are translated such that their mean (center of gravity) ends up at the origin. The points are then scaled such that their average distance to the origin equals $\sqrt{2}$.

Exercise 7 Algebraic errors are easy to minimize, e.g., using the homogeneous or the inhomogeneous methods. The minimizers can be used as initial solutions to iterative methods for minimization of geometric errors.

Exercise 8 The SVD profile provides information about the solution space, in particular its dimension. This is given as the number of singular values that are sufficiently small to be counted as approximately equal to zero. If the dimension is one, the error has a unique and well-defined minimum. If the dimension is larger than one, the error does not have a well-defined minimum, since it not unique. If the dimension is zero, although there exists a minimizer of the error, the error does not have a well-defined minimum, since no model can be fitted to the data without larger errors.

Exercise 9 For example:

$$\epsilon = \sum_{k=1}^{m} d_{\mathrm{PP}}(\mathbf{y}'_k, \mathbf{H}\,\mathbf{y}_k)^2 + d_{\mathrm{PP}}(\mathbf{y}_k, \mathbf{H}^{-1}\,\mathbf{y}'_k)^2$$

Exercise 10 We start with the defining relation between corresponding points and the homography:

$$\mathbf{y}'_k \sim \mathbf{H}\,\mathbf{y}_k, \quad k = 1, \dots, m.$$

This is an equivalence relation, not an equality, so the left and right-hand sides are not equal, but *equivalent* in the sense that one is a non-zero scalar multiplied with the other. The relation can be made into a proper equality, by using DLT:

$$\mathbf{0} = \mathbf{y}'_k \times \big(\mathbf{H}\,\mathbf{y}_k\big) = [\,\mathbf{y}'_k\,]_\times\,\mathbf{H}\,\mathbf{y}_k, \quad k = 1, \dots, m. \tag{1}$$

This is a set of linear and homogeneous equation in the elements of the 3×3 matrix $\mathbf{H}$. To be specific, it is a set of three equations, but since $[\,\mathbf{y}'_k\,]_\times$ has rank two, at most two of them are linearly independent. In general, we can pick any two the three equations and form a 2×9 matrix $\mathbf{A}_k$ such that Equation (1) is equivalent to

$$\mathbf{0} = \mathbf{A}_k\,\mathbf{h}, \quad k = 1, \dots, m, \tag{2}$$

where $\mathbf{h}$ is a vectorization of $\mathbf{H}$ into a 9-dimensional vector.

Equation (2) should be true for all pairs $\{\mathbf{y}_k, \mathbf{y}'_k\}$ of corresponding points with index k. To this end, we form a $2m \times 9$ data matrix $\mathbf{A}$ by stacking all the matrices $\mathbf{A}_k$ on top of each other:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{pmatrix}$$

The total set of equations, for all $k = 1, \dots, m$, is then expressed as

$$\mathbf{0} = \mathbf{A}\,\mathbf{h}. \tag{3}$$

This equations can be true only in the ideal case when the initial defining relation is satisfied exactly for all points. In practice there is measurement noise on the point coordinates, which means that the relation is only approximately true. An algebraic measure of how well this approximation can be made is given as the squared norm of the right-hand side of Equation (3):

$$\epsilon = \|\mathbf{A}\,\mathbf{h}\|^2.$$

The squaring makes it possible to easily find a minimizer $\mathbf{h}$, either using the homogeneous method or the inhomogeneous method.

PART III

Exercise 11 $\langle\, \mathbf{b}_k \,|\, \tilde{\mathbf{b}}_l \,\rangle = \delta_{kl}$. This is the defining relation for the dual basis vectors.

Exercise 12 $\mathbf{c}^\top \mathbf{c}_0 = 0$. This is a consequence of the fact that $\mathbf{c} = \tilde{\mathbf{B}}^\star \mathbf{v}$ is the vector of reconstructing coefficients that has shortest norm.

Exercise 13 For example: $h[k] = \langle\, f \mid g_0 \,\rangle$ where $g_0[l] = g[k-l]$ (g is here assumed to be a real function). This follows from

$$\begin{aligned} h[k] &= \big/\text{definition of convolution}\big/ = \sum_l f[l]\, g[l-k] = \\ &= \big/\text{use } g_0 \text{ as defined above}\big/ = \sum_l f[l]\, g_0[l] = \\ &= \big/\text{use definition of scalar prod. between two discrete functions}\big/ = \langle\, f \mid g_0 \,\rangle. \end{aligned}$$

Exercise 14 If $\mathbf{x}$ is an element of the subspace spanned by $\mathbf{B}$ then it is true that $\tilde{\mathbf{B}}^\star\mathbf{x}$ are the coordinates of $\mathbf{x}$ relative to the basis in $\mathbf{B}$. In the general case, $\mathbf{x}$ may not be an element of this subspace, and then $\tilde{\mathbf{B}}^\star\mathbf{x}$ instead are the coordinates of the orthogonal projection of $\mathbf{x}$ into the subspace (in fact, this is always true, also when $\mathbf{x}$ lies in the subspace).

Exercise 15 The dual frame vectors are given as $\tilde{\mathbf{b}}_k = \mathbf{F}^{-1}\mathbf{b}_k$. Any $\mathbf{v} \in V$ can be reconstructed from scalar products with the frame vectors by a linear combination with the corresponding frame vectors:

$$\sum_k \langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle\, \tilde{\mathbf{b}}_k = \sum_k \langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle\, \mathbf{F}^{-1}\mathbf{b}_k = \mathbf{F}^{-1} \underbrace{\left(\sum_k \langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle\, \mathbf{b}_k \right)}_{\mathbf{F}\,\mathbf{v}} =$$

$$= \mathbf{F}^{-1}\mathbf{F}\,\mathbf{v} = \mathbf{v}.$$

This derivation also works the other way around: to reconstruct $\mathbf{v}$ from scalar products with the dual frame vectors by a linear combination with the corresponding frame vectors.

PART IV

Exercise 16 The noise should have zero-mean and independent between the different samples.

Exercise 17 $f_i[k] = a[-k] \cdot b_i[-k]$, i.e., the applicability function a is multiplied point-wise on each basis function b_i, and the resulting function is then time-reversed to give the filters f_i.

Exercise 18 Advantage: the resulting filter has fewer coefficients and convolving this filter to the signal can therefore be implemented using fewer multiplications and additions. Disadvantage: since the resulting filter has fewer coefficients, it is likely to also have a larger error with the ideal filter in the frequency domain.

Exercise 19 Given is a set of n-dimensional vectors $\{\mathbf{v}_k \in \mathbb{R}^n, k = 1, \ldots, p\}$, often observations of some data or some process. What we seek is a subspace of dimension $m << n$, such that we loose "as little as possible" of these vectors when they are projected onto the subspace. Formally, this is described by a cost function

$$\epsilon = \sum_{k=1}^{p} \|\mathbf{v}_k - \mathbf{P}\,\mathbf{v}_k\|^2,$$

where $\mathbf{P}$ is a projection operator into the subspace. In practice, the subspace is represented by an orthonormal subspace basis $\mathbf{B}$ (an $n \times m$ basis matrix, with the basis vectors in its columns), such that $\mathbf{P} = \mathbf{B}\,\mathbf{B}^\top$. In the end, we seek an orthonormal basis of an m-dimensional subspace such that

$$\epsilon = \sum_{k=1}^{p} \|\mathbf{v}_k - \mathbf{B}\,\mathbf{B}^\top\,\mathbf{v}_k\|^2,$$

is minimal.

This subspace basis is found by first computing the correlation matrix $\mathbf{C}$ of the data:

$$\mathbf{C} = \sum_{k=1}^{p} \mathbf{v}_k\mathbf{v}_k^\top.$$

The basis is given as an ON-basis of eigenvectors of $\mathbf{C}$ corresponding to the m largest eigenvalues (including multiplicities). This implies that the initial data may instead be given as the matrix $\mathbf{C}$, instead of the set of vectors described above, since this is sufficient to solve the problem.

Exercise 20 It must satisfy the orthogonality (O) condition, which means that the filter h itself has unit norm, and is orthogonal to every shift of itself by an even number of samples:

$$\sum_l h[l]\, h[l-2k] = \delta[k]. \tag{4}$$

In the frequency domain this can, equivalently, be described as

$$|H(u)|^2 + |H(u+\pi)|^2 = 2.$$

An example of a filter h that satisfy the O-condition is given by the filter vector $h = [11]/\sqrt{2}$. It has unit norm: $\|h\| = 1$, which means that Equation (4) is satisfied for $k = 0$. Since the filter length is 2, it means that Equation (4) is satisfied also for $k \neq 0$.