Guide* to answers for written examination in TSBB06 Multi-dimensional signal analysis, 2018-01-12

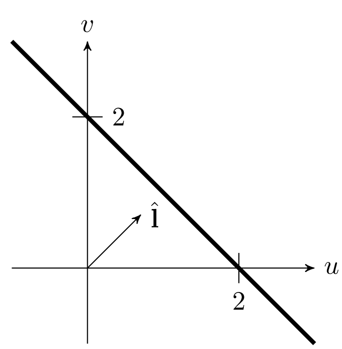
Klas Nordberg, klas.nordberg@liu.se,

PART I

Exercise 1 In order to determine the geometric characteristics of the line, its dual homogeneous coordinates need to be D-normalized:

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \qquad \Leftarrow \qquad \text{First two elements squared add to one, last element negative}$$

The line lies at distance $\sqrt{2}$ (last element of the D-normalized vector) from the origin, with $\hat{\mathbf{l}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ as a normal vector (first two elements of the D-normalized vector) that points from the origin out to the line. This is illustrated here:



Exercise 2 No. A rigid transformation preserves the distance between any pair of points. This will not happen in general when rotation of 3D points is combined with projection onto a 2D image. For example, two points at some distance d that lie on a the same projection line, are projected onto the same image point. After a 3D rotation, they could end up on different projection lines, thereby projecting onto different images points. Consequently, distance in the image plane is not preserved when 3D points are rotated.

Exercise 3 The three eigenvalues of $\mathbf{R}$ are 1, $e^{i\alpha}$, and $e^{-i\alpha}$. See the IREG compendium (section B.7.4 in version 0.33).

Exercise 4 An affine transformation preserves Euclidean points as Euclidean points, and points at infinity as points at infinity. If two lines are not parallel, they intersect

*This guide is not an authoritative description of how answers to the questions must be given in order to pass the exam. Some explanations given here may not have to be included in the answer, unless explicitly called for.

at a Euclidean point, and since it remains a Euclidean point also after the affine transformation, the two lines will be non-parallel after the transformation. A similar argument can be made to a point at infinity, which instead is the intersection of a two parallel lines.

An alternative is to do the math. The transformation matrix of an affine transformation is characterized as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$$

If we apply $\mathbf{T}$ onto the homogeneous coordinates of a proper (Euclidean) point, the intersection of two non-parallel lines, we get

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\,\mathbf{x} + \mathbf{t} \\ 1 \end{pmatrix},$$

which, again, is a proper point, and therefore the lines have been transformed to remain non-parallel. If the point instead is at infinity, the intersection of two parallel lines, we get

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\,\mathbf{x} \\ 0 \end{pmatrix},$$

which is a point that remains at infinity, again the intersection of two parallel lines.

Exercise 5 The camera matrix $\mathbf{C}$ is a projective element, which means that it has $3 \times 4 - 1 = 11$ degrees of freedom. The $\mathbf{K}$ matrix is upper triangular, with 3 elements set to zero, and is also a projective element. As a result $\mathbf{K}$ has $3 \times 3 - 3 - 1 = 5$ degrees of freedom. The rotation matrix $\mathbf{R} \in SO(3)$ has 3 degrees of freedom, which also is the case for $\mathbf{t} \in \mathbb{R}^3$. In total the internal and external parameters have $5 + 3 + 3 = 11$ degrees of freedom, which is the same and $\mathbf{C}$.

PART II

Exercise 6 The Euclidean distance between the point and the line is a typical example of a geometric error, given as:

$$d = |\text{norm}_P(\mathbf{y}) \cdot \text{norm}_D(\mathbf{l})| = \left| \frac{\mathbf{y} \cdot \mathbf{l}}{y_3 \sqrt{l_1^2 + l_2^2}} \right|$$

Exercise 7 The inhomogeneous method: set some element in $\mathbf{z}$ equal to one and solve for the remaining elements. The homogeneous method: set $\|\mathbf{z}\| = 1$ and minimize $\|\mathbf{A}\,\mathbf{z}\|$ with that constrain.

Exercise 8 For example:

- No method degeneracy. The homogeneous method cannot robustly solve problems where any $\mathbf{z}$ is the solution, while the inhomogeneous method cannot since some pre-specified element must always be $= 1$.
- The SVD provides a profile of the singular values, which characterizes the solution in terms of dimension of the solution space.

Exercise 9 OPP estimates an orthogonal transformation between the two datasets in $\mathbf{A}$ and $\mathbf{B}$, respectively. If $\det \mathbf{U}\mathbf{U}^\top = -1$ is means that the most optimal orthogonal transformation between the datasets includes a reflection. Since we want $\mathbf{R} \in SO(3)$, the optimal transformation is far from satisfying $\det \mathbf{R} = +1$, probably because large amount of noise has affected the data.

To assure $\det \mathbf{R} = +1$ for the estimated rotation, we need to flip the sign of the third column in either $\mathbf{U}$ or $\mathbf{V}$.

Exercise 10 Since the plane is determined from only three 3D points, it can be determined exactly, without any errors that should be minimized in one way or another. Consequently, we get the same solution with any reasonable method, including using Hartley-normalized coordinates.

PART III

Exercise 11

$$\langle f \mid g \rangle = \int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, dt,$$

or, more generally,

$$\langle f \mid g \rangle = \int_{-\infty}^{\infty} f(t)\, w(t)\, \overline{g(t)}\, dt,$$

where w is a positive weighting function.

Exercise 12 The dual frame vectors are generated from the original frame vectors as $\tilde{\mathbf{B}} = \mathbf{F}^{-1}$, where $\mathbf{F}$ is the frame operator $\mathbf{F} = \mathbf{B}\,\mathbf{B}^\top \mathbf{G}_0$. An alternative relation is: scalar products with the frame vectors generate coefficients that in a linear combination with the dual frame vectors reconstruct the vector: $\tilde{\mathbf{B}}\mathbf{B}^\top \mathbf{G}_0 = \mathbf{I}$.

Exercise 13 The coordinates of the orthogonal projection, $\mathbf{v}_1$ is given as

$$\mathbf{c} = \tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{v} = \left(\mathbf{B}^\top \mathbf{G}_0 \mathbf{B}\right)^{-1} \mathbf{B}^\top \mathbf{v}.$$

This leads to

$$\mathbf{v}_1 = \mathbf{B}\,\mathbf{c} = \mathbf{B}\left(\mathbf{B}^\top \mathbf{G}_0 \mathbf{B}\right)^{-1} \mathbf{B}^\top \mathbf{v}.$$

Exercise 14 The Gramian matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} \langle \mathbf{b}_1 \mid \mathbf{b}_1 \rangle & \langle \mathbf{b}_2 \mid \mathbf{b}_1 \rangle \\ \langle \mathbf{b}_1 \mid \mathbf{b}_2 \rangle & \langle \mathbf{b}_2 \mid \mathbf{b}_2 \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_2 \\ \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

From this, the dual basis matrix is computed:

$$\tilde{\mathbf{B}} = \mathbf{B}\,\mathbf{G}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

The dual basis vectors are the two columns in $\tilde{\mathbf{B}}$: $\tilde{\mathbf{b}}_1 = (1,1)/2$ and $\tilde{\mathbf{b}}_2 = (-1,1)/6$. We can check this result by verifying that two sets of basis vectors are in a dual relation:

$$\mathbf{B}^\top \mathbf{G}_0 \tilde{\mathbf{B}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (!)$$

Exercise 15 Convolution between two time-discrete functions f and g:

$$h[k] = \sum_n f[k-n]\, g[n].$$

A specific element in the resulting function, $h[k]$ at position k, is then a scalar product between two functions, $p[n] = f[k-n]$ (f after time mirroring and shift by k) and $g[n]$:

$$h[k] = \langle p \mid g \rangle.$$

As an alternative, the entire function $h[k]$ can be written as a linear combination of functions $f_n[k] = f[k-n]$ (shifted by n) together with coefficients $c_n = g[n]$:

$$h[k] = \sum_n c_n \, f_n[k].$$

PART IV

Exercise 16 Several operations that need to be performed involve scalar products of functions in the frequency (Fourier) domain, defined in terms of integrals of products of two functions. For general functions, the numerical values of these scalar products have to be approximated as Riemann sums of finite number of terms. These are computed by sampling of the frequency domain.

Exercise 17 The coordinate of a polynomial of order n is the derivative of the signal of order n (divided by $n!$):

$$f(x+\tau) = f(x)\,\tau^0 + f'(x)\,\tau + \frac{1}{2} f''(x)\,\tau^2 + \ldots$$

Exercise 18 Quantization noise, which happens when the signal is converted to a binary representation with a fixed number of bits.

Exercise 19 The residual error with N principal components is given as

$$\epsilon_N = \sum_{k=N+1}^{M} \lambda_k = \lambda_{N+1} + \lambda_{N+2} + \ldots + \lambda_M,$$

where λ_k is the k-th eigenvalue of the $M \times M$ correlation matrix $\mathbf{C}$, sorted from largest to smallest. If instead $N+1$ principal components are used, the residual error is

$$\epsilon_{N+1} = \sum_{k=N+2}^{M} \lambda_k = \lambda_{N+2} + \lambda_{N+3} + \ldots + \lambda_M,$$

The reduction in error is

$$\epsilon_N - \epsilon_{N+1} = \lambda_{N+1}.$$

Exercise 20 With ϕ as the scaling function, $\phi(t-k), k \in \mathbb{Z}$ is a basis for V_0, and $2^{1/2}\phi(2t-k), k \in \mathbb{Z}$ is a basis for V_1. This means for a general function $f(t) \in V_0$, that it can be expanded as a linear combination of the corresponding basis with coordinates c_k:

$$f(t) = \sum_k c_k \, \phi(t-k).$$

Using the same coordinates, but divided by $2^{1/2}$, in a linear combination with the basis in V_1:

$$\sum_k c_k \, \phi(2t-k) = f(2t).$$

Consequently, if $f(t) \in V_0$, then $f(2t) \in V_1$. In a similar way, we can show that if $f(2t) \in V_1$, then $f(t) \in V_0$. In summary, whatever functions we find in one of the two spaces, we will find a time-scaled version (by a factor 2 or 1/2, depending on which way we go) in the other space.