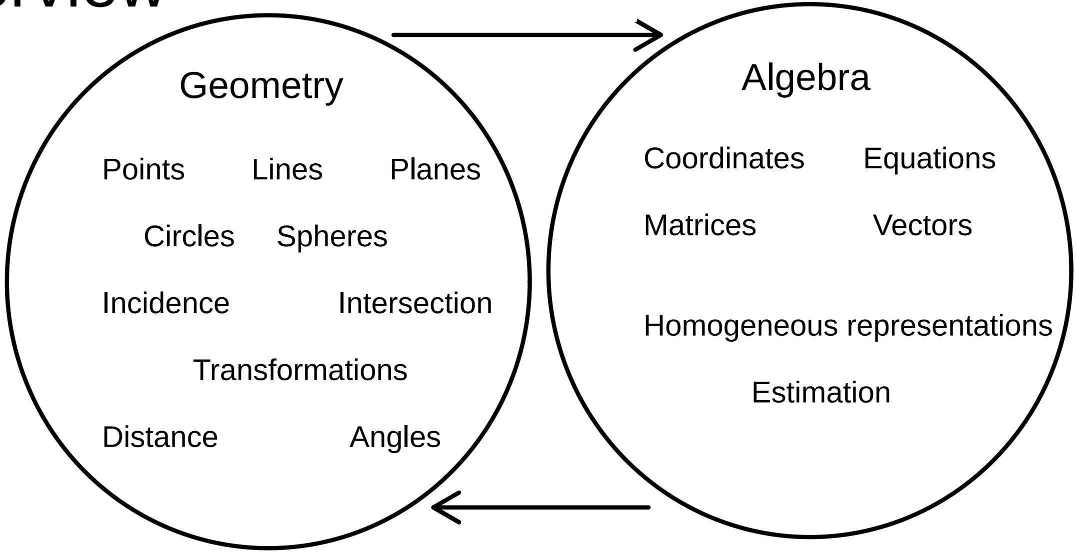
Overview


Geometry
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Linear Algebra Recap

Vectors & Vector Spaces

- Commutativity and associativity of addition.
- Existence of additive identity ('zero vector').
- Existence of additive inverse.
- Associativity and distributivity for multiplication with scalars.
- Multiplication by the scalar identity (i.e. 1) gives the same vector.

comm.
$u+v=v+u$

assoc.
$(u+v)+w=u+(v+w)$

$\exists v: \ u+v=0$

Scalar Product (Inner Product) & Norm

$u = (u_1, \cdots, u_n)$

$v = (v_1, \cdots, v_n)$

Some vector spaces are equipped with a scalar product.

- Usually in $\mathbb{R}^n$ we use $u^T v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.
- Two vectors are orthogonal (perpendicular) if $u^T v = 0$.

Some vector spaces are equipped with a norm.

- Usually in $\mathbb{R}^n$ we use the Euclidean norm, $\|u\| = \sqrt{u^T u}$, which is often called the length (or magnitude, or norm).

Scalar product:

$\langle u | v \rangle = v^T G_0 u$

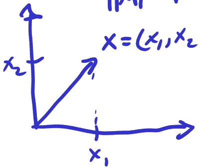
$\uparrow$

positive definite symmetric matrix

Norm: $\|u\| = \sqrt{\langle u | u \rangle}$

$\|x\| = \sqrt{x_1^2 + x_2^2}$

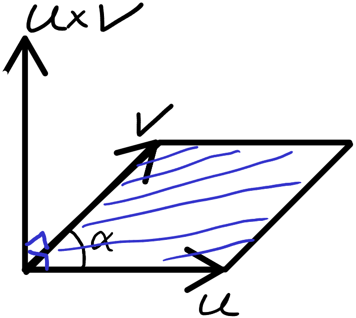
$x = (x_1, x_2)$



The Cross Product

In $\mathbb{R}^3$ we can take the cross product of two vectors, resulting in a new vector orthogonal to the first two.

- Orthogonality: $u^T(u \times v) = 0, \; v^T(u \times v) = 0$
- Length: $\|u \times v\| = \|u\| \cdot \|v\| \sin\alpha$
- Positively oriented (right handed system).



Algebraically,

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = u \times v$$

The Cross Product operator

It is often very useful to define the cross product operator (or cross product matrix) of a vector:

$$[u]_\times = \frac{\partial}{\partial v}(u \times v) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

$[u]_\times v = u \times v$

$[u]_\times u = 0$

$[u]_\times^T = -[u]_\times$ antisymmetric

All non-zero anti-symmetric 3x3 matrices are cross product operator matrices!

$A \in \mathbb{R}^{3\times3}$ with $A^T = -A$ (anti-symmetric)

$$\Rightarrow A + A^T = 0 \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow a_{ii} = 0$

$a_{ij} = -a_{ji}$

$$\Rightarrow A = \begin{pmatrix} 0 & -a_{21} & a_{13} \\ a_{21} & 0 & -a_{23} \\ -a_{13} & a_{23} & 0 \end{pmatrix}$$

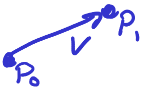
$$\Rightarrow A = [(a_{23}, a_{13}, a_{21})]_\times$$

Points vs Vectors

Both points and vectors can be represented using n-tuples $(x_1, x_2, \dots, x_n)$,
but they permit different types of operations:

$\begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$

i) vector + vector = vector
ii) point + vector = point
iii) point - point = vector
iv) point + point = undefined
v) scalar · vector = vector
vi) scalar · point = undefined



$V = P_1 - P_0$

$P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ $P_3 = P_1 + P_2 = (x_1 + x_2, y_1 + y_2)$?

Points vs Vectors (contd.)

For the pedantic, $\mathbb{R}^n$ is a vector space (that contains vectors), and $\mathbb{E}^n$ is an affine space (that contains $\mathbb{R}^n$ and also points).

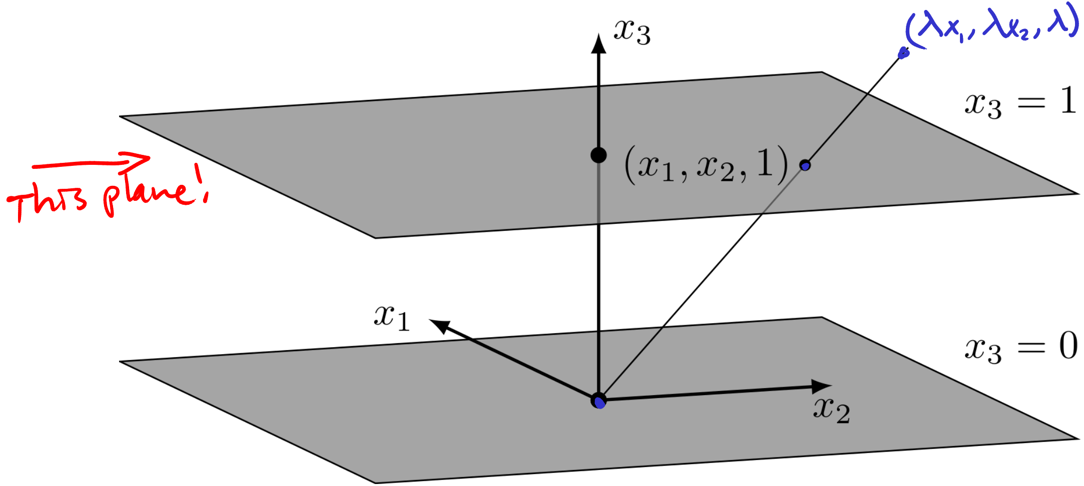
We will sometimes be careless and talk about 'points' in $\mathbb{R}^n$ when we do not at the moment make use of its vector space structure.

IMPORTANT NOTE:

> Points can be represented in many ways, and sometimes we can do meaningful things to one representation, which, geometrically, are not meaningful at all!

The Real Projective Plane

The real projective plane consists of ordinary points (x_1, x_2) together with 'ideal points' (points at infinity).



Homogeneous Coordinates

$u \sim v \Leftrightarrow u = \lambda v$

$\lambda \neq 0$

An ordinary point (x_1, x_2) is represented by its homogeneous coordinates $(x_1, x_2, 1) \sim (\lambda x_1, \lambda x_2, \lambda)$ for any non-zero λ.

P-normalisation: $(x_1, x_2, x_3) \sim \underbrace{\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)}_{\text{standard form}}$

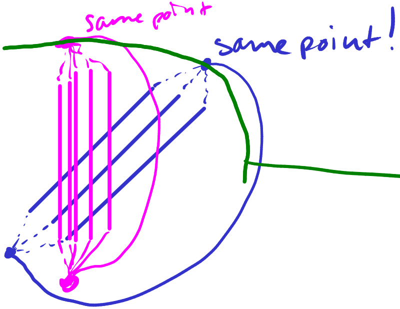
$\text{norm}_p \, \mathbf{x}$

Ideal points?

$(\lambda x_1, \lambda x_2, 1) \sim (x_1, x_2, \frac{1}{\lambda}) \longrightarrow (x_1, x_2, 0)$ as $\lambda \rightarrow \pm\infty$

The Real Projective Plane (contd.)

The line at infinity ("horizon"):

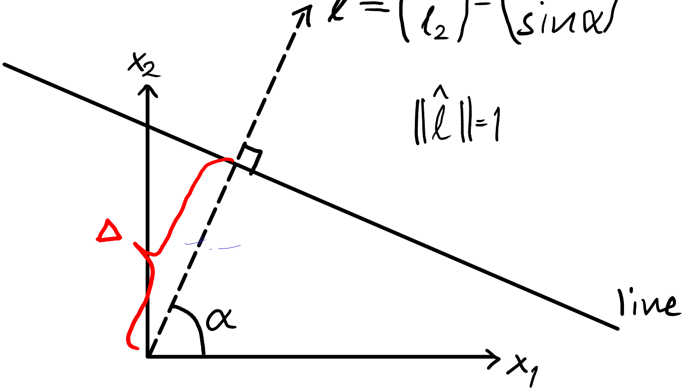


horizon line

contains all points at infinity

Lines in 2D

$$\hat{\ell} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$



$$\|\hat{\ell}\| = 1$$

Equation (normal form):

$$\ell_1 x_1 + \ell_2 x_2 = \Delta$$

$\left(2\ell_1 x_1 + 2\ell_2 x_2 = 2\Delta \right.$ no direct interpretation $\left. \right)$

$$\ell_1 x_1 + \ell_2 x_2 - \Delta \cdot 1 = 0$$

Lines in 2D (contd.)

Dual homogeneous coordinates $\ell = \begin{pmatrix} \hat{\ell} \\ -\Delta \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \\ -\Delta \end{pmatrix} \sim \begin{pmatrix} \lambda \ell_1 \\ \lambda \ell_2 \\ -\lambda\Delta \end{pmatrix}$

$x = (x_1, x_2, 1)$

line eq: $\ell^T x = 0$

$$(\ell_1 \ \ \ell_2 \ \ -\Delta) \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = 0$$

D-normalisation:

length $= 1$

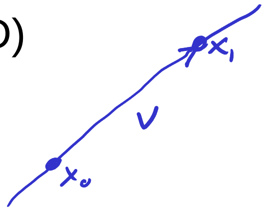
$$\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} \sim \frac{-\operatorname{sign} \ell_3}{\sqrt{\ell_1^2 + \ell_2^2}} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}$$

$\mathrm{norm}_D\, \ell$

$$\Longrightarrow \Delta = \frac{\operatorname{sign} \ell_3}{\sqrt{\ell_1^2 + \ell_2^2}} \cdot \ell_3$$

Lines in 2D (contd.)

Line through two points (parametric form, 2D and 3D)



- Non-homogeneous: ℓ: $X(\lambda) = x_0 + \lambda v =$

$$= x_0 + \lambda(x_1 - x_0) =$$

$$= \underbrace{(1-\lambda)}_{\lambda_0} x_0 + \underbrace{\lambda}_{\lambda_1} x_1 = x(\lambda_0, \lambda_1) \quad \text{with } \lambda_0 + \lambda_1 = 1$$

- Homogeneous:

$$X(\lambda_0, \lambda_1) = \lambda_0 x_0 + \lambda_1 x_1 \sim \underbrace{\mu\lambda_0}_{\tilde{\lambda}_0} x_0 + \underbrace{\mu\lambda_1}_{\tilde{\lambda}_1} x_1 \qquad \cancel{\text{with } \lambda_0 + \lambda_1 = 1}$$

Lines in 2D (contd.)

Line through two points (homogeneous, 2D only!):

x_1 and x_2 points on ℓ $\Rightarrow$ $\begin{cases} \ell^T x_1 = 0 \\ \ell^T x_2 = 0 \end{cases}$ $\Rightarrow$ $\ell \sim x_1 \times x_2$

Intersection of two lines (homogeneous, 2D only!):

ℓ_1 and ℓ_2 dual homogeneous

$$\begin{cases} \ell_1^T x = 0 \\ \ell_2^T x = 0 \end{cases} \Leftrightarrow \begin{cases} x^T \ell_1 = 0 \\ x^T \ell_2 = 0 \end{cases} \Rightarrow x \sim \ell_1 \times \ell_2$$

Examples

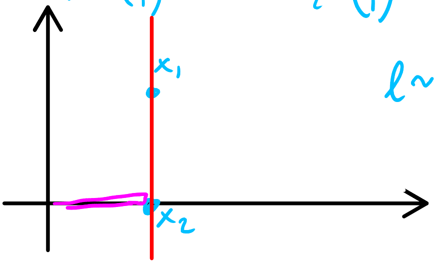
Let $x_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$ and $x_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ (two ideal points)

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

The line through x_1 and x_2 is $l_\infty \sim x_1 \times x_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$l_\infty^T \begin{pmatrix}x\\y\\0\end{pmatrix} = 0$$

Let $x_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $x_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$



$l \sim x_1 \times x_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ normal

distance Δ

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

Plücker coordinates (more interesting in 3D)

Define $L = x_1 x_2^T - x_2 x_1^T \in \mathbb{R}^{3\times3}$ where $x_1, x_2 \in \mathbb{R}^3$ are homogeneous coordinates

This is anti-symmetric: $L^T = x_2 x_1^T - x_1 x_2^T = -L$

Ideal point on L: $x_1 = \begin{pmatrix} u_1 \\ u_2 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix} \Rightarrow L = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 \\ u_2 v_1 & u_2 v_2 & u_2 \\ v_1 & v_2 & 1 \end{pmatrix} - \begin{pmatrix} u_1 v_1 & u_2 v_1 & v_1 \\ u_1 v_2 & u_2 v_2 & v_2 \\ u_1 & u_2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & & u_1 - v_1 \\ & 0 & u_2 - v_2 \\ & & 0 \end{pmatrix}$$

ideal point on L