Homogeneous Coordinates for Points in 3D

Let $\bar{\mathbf{x}} = (x_1, x_2, x_3)$ be Cartesian coordinates for a point in $\mathbb{E}^3$.

- Then $\mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda)$ represents the same point in homogeneous coordinates (as long as $\lambda \neq 0$).
- P-normalisation: $\mathrm{norm}_\mathrm{P}\big((x_1, x_2, x_3, x_4)\big) = (\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}, 1)$, as long as $x_4 \neq 0$ (*proper points*).
- *Ideal points* (points at infinity) are of the form $\mathbf{x} = (x_1, x_2, x_3, 0)$.

$X(\lambda) = (\lambda x_1, \lambda x_2, \lambda x_3, 1) \sim (x_1, x_2, x_3, \frac{1}{\lambda}) \rightarrow (x_1, x_2, x_3, 0)$ as $\lambda \rightarrow \pm\infty$

Not a line!

$0 = p^T x = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 = 0$

$(ax + by + cz + d = 0)$

Dual Homogeneous Coordinates for Planes in 3D

Let $\mathbf{p} = (\beta_1, \beta_2, \beta_3, \beta_4)$ be dual homogeneous coordinates for a plane.

- D-normalisation:

length $= 1$

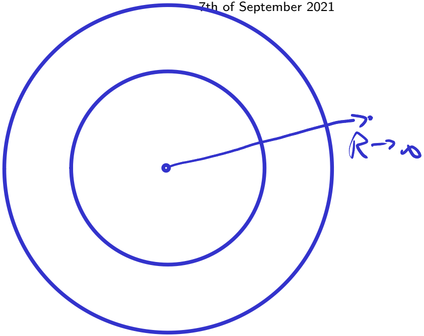
$$\mathrm{norm}_{\mathrm{D}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \frac{-\operatorname{sign} \beta_4}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{p}} \\ -\Delta \end{pmatrix}$$

- $(\mathrm{norm}_{\mathrm{P}}\, \mathbf{x})^\top (\mathrm{norm}_{\mathrm{D}}\, \mathbf{p})$ gives the *signed distance* from $\mathbf{x}$ to $\mathbf{p}$.

In particular: $= 0$ if on the plane

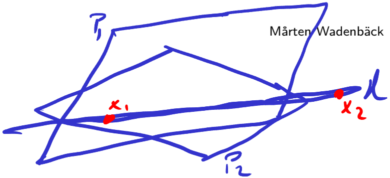
The Ideal Plane (Plane at Infinity)

Ideal points: $(x_1, x_2, x_3, 0)$



- The ideal plane consists of all ideal points.
- Dual homogeneous coordinates $\mathbf{p}_\infty = (0, 0, 0, 1)$.
- We cannot D-normalise $\mathbf{p}_\infty$.
- We may think of $\mathbf{p}_\infty$ as a sphere with infinite radius.

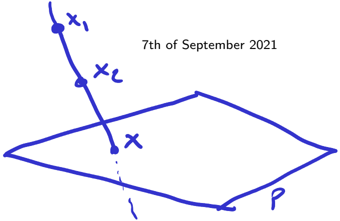
Lines in 3D



Lines in $3D$ *cannot* be written as $\mathbf{l}^\top \mathbf{x} = 0$ (this is a *plane*).
Alternative representations:

- Parameter form: $\bar{\mathbf{x}}(t) = (1-t)\bar{\mathbf{x}}_1 + t\bar{\mathbf{x}}_2$. $= \bar{x}_1 + t(\bar{x}_2 - \bar{x}_1) = \bar{x}_1 + tv$
- In homogeneous coordinates: $\mathbf{x}(\lambda) = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$, where $\lambda_1^2 + \lambda_2^2 \neq 0$.
- *Plücker coordinates*: $\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top$ (anti-symmetric, rank two).
- *Dual Plücker coordinates*: $\tilde{\mathbf{L}} = \mathbf{p}_1\mathbf{p}_2^\top - \mathbf{p}_2\mathbf{p}_1^\top$ (also anti-symmetric, rank two).

Example: Intersection between Plane and Line



Suppose $\mathbf{L}$ represents a line (Plücker coordinates) in 3D, and $\mathbf{p}$ represents a plane. Find the intersection!

$$L = x_1 x_2^T - x_2 x_1^T \sim x_1 x^T - x x_1^T$$ (show as exercise!)

Compute Lp: $$Lp \sim x_1 \underbrace{x^T p}_{\substack{=0 \\ \text{on plane!}}} - x \underbrace{x_1^T p}_{\text{scalar}} = -(x_1^T p)\, x \sim x$$

Spatial Configuration of Points

$$x_j = \begin{pmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \\ x_{4j} \end{pmatrix}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be homogeneous coordinates of points in the extended Euclidean space. The rank of the matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^\top \\ \vdots \\ \mathbf{x}_n^\top \end{pmatrix} \in \mathbb{R}^{n \times 4}$$

"data matrix"

tells us what the spatial configuration of the points is!

Spatial Configuration of Points (contd.)

rank $X = 0$: (entire matrix is zero) does not happen!

rank $X = 1$: all x_k are the same! (up to scale)

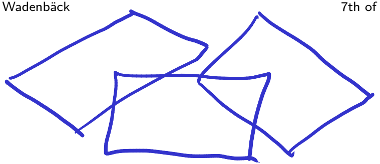
rank $X = 2$: all points on a line!

rank $X = 3$: all points on a plane p

$$Xp = 0 \iff \begin{cases} x_1^T p = 0 \\ x_2^T p = 0 \\ \vdots \\ x_n^T p = 0 \end{cases}$$

rank $X = 4$: general position (configuration)

Spatial Configuration of Planes

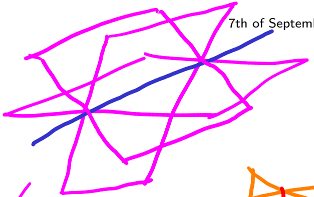


Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be dual homogeneous coordinates of planes in the extended Euclidean space. The rank of the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1^\top \\ \vdots \\ \mathbf{p}_n^\top \end{pmatrix} \in \mathbb{R}^{n \times 4}$$

tells us what the spatial configuration of the planes is!

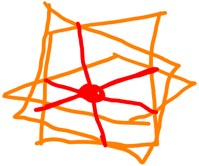
Spatial Configuration of Planes (contd.)



rank $P=0$: does not happen!

rank $P=1$: all planes are the same!

rank $P=2$: all planes contain the same line!



rank $P=3$: the system $Px=0$ will have a 1D solution $x(t) = tx_0 \sim x_0$

rank $P=4$: planes are in general position (configuration) no common point!

Algebraic vs Geometric Error (Estimation Sneak Peek)

Algebraic error: $\varepsilon_A(p) = \|Xp\|^2 = \left\|\begin{pmatrix} x_1^T p \\ \vdots \\ x_n^T p \end{pmatrix}\right\|^2 = (x_1^T p)^2 + \dots + (x_n^T p)^2 = \sum_{k=1}^{n} (x_k^T p)^2$

Bad idea? Scales with the points (or p)!

Does not measure "actual" distances

Geometric distance: $\varepsilon_G(p) = \sum_{k=1}^{n} \left((\mathrm{norm}_p x_k)^T (\mathrm{norm}_D p)\right)^2$

Measures the "actual" distances

Messier dependence on p $\Rightarrow$ harder to minimise

Geometric Transformations in 2D and 3D

Table: Geometric transformations in 2D and 3D. Each type includes, as subgroups, the types listed below it.

Type	Matrix	Constraints	DoF (2D)	DoF (3D)
Affine	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$	$\det \mathbf{A} \neq 0$	6	12
Similarity	$\begin{bmatrix} s\mathbf{Q} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$	$\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}$ and $s \neq 0$	4	7
Rigid	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$	3	6

most important in 3D!

Dual Transformations in 3D

Assume we apply a transformation $\mathbf{T}$ to 3D space as $\mathbf{x}' = \mathbf{T}\mathbf{x}$.

- What happens to planes in dual homogeneous coordinates?

Let p be a plane: $p^T x = 0$ How do we get p' such that $(p')^T x' = 0$?

$(p')^T T x = 0 \Leftrightarrow (Sp)^T T x = 0 \Leftrightarrow p^T \underbrace{S^T T}_{I} x = 0$ $\Rightarrow \tilde{T} = (T^{-1})^T = T^{-T}$

I gives $p^T x = 0$

- What happens to planes in Plücker coordinates?

$L = x_1 x_2^T - x_2 x_1^T$

$L' = T x_1 (T x_2)^T - T x_2 (T x_1)^T = T \underbrace{(x_1 x_2^T - x_2 x_1^T)}_{L} T^T = T L T^T$