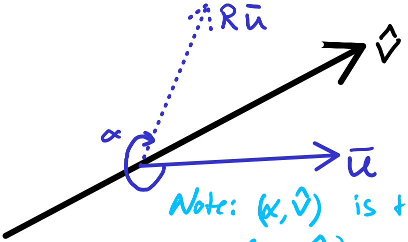
Rotations in 3D



Note: $(\alpha, \hat{v})$ is the rotation

$(-\alpha, -\hat{v})$ same rotation!

$(\alpha + 2k\pi, \hat{v})$ —〃—

- In 2D: $R_{2D} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

- In 3D: Not so simple!

- Orthogonal matrices with determinant = 1

  those wit $\det R = -1$ are reflections

- 3 DoF

- Many different parametrisations, useful in different situations

Tait-Bryan Angles

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$ rotation around the x-axis

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$ y-axis

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ z-axis

General: $R = R(\alpha,\beta,\gamma) = R_x(\alpha)R_y(\beta)R_z(\gamma)$

Euler angles:

$R_x(\alpha)R_y(\beta)R_x(\gamma)$

$R_z(\alpha)R_x(\beta)R_z(\gamma)$

Rodrigues' Formula

Rotating a single vector: axis $\hat{v}$, angle α

$$y' = y + \sin\alpha(\hat{v}\times y) + (1-\cos\alpha)(\hat{v}\times(\hat{v}\times y))$$

$$\left(I + \sin\alpha[\hat{v}]_\times + \underbrace{(1-\cos\alpha)}_{\sin^2(\frac{\alpha}{2})}[\hat{v}]_\times^2\right)y$$

Reading off the rotation matrix:

$$R = I + \sin\alpha\,[\hat{v}]_\times + (1-\cos\alpha)\,[\hat{v}]_\times^2$$

$$[\hat{v}]_\times^3 = -[\hat{v}]_\times$$

$$[\hat{v}]_\times^2 = \hat{v}\hat{v}^T - I$$

Rodrigues' Formula

$$\hat{v} \times x = \underbrace{\hat{v} \times x_{\parallel}}_{=0} + \hat{v} \times x_{\perp}$$

$$Rx = R(x_{\parallel} + x_{\perp} = Rx_{\parallel} + Rx_{\perp}$$

But

$$Rx_{\perp} = \cos\varphi\, x_{\perp} + \sin\varphi\,(\hat{v} \times x_{\perp}) =$$

$$= -\cos\varphi\,(\hat{v} \times (\hat{v} \times x_{\perp})) + \sin\varphi\,(\hat{v} \times x_{\perp}) =$$

$$= -\cos\varphi\,(\hat{v} \times (\hat{v} \times x)) + \sin\varphi\,(\hat{v} \times x).$$

Now,

$$Rx = Rx_{\parallel} + Rx_{\perp} = x_{\parallel} + Rx_{\perp} =$$

$$= \underbrace{x_{\parallel} + x_{\perp}}_{x} + \sin\varphi\,[\hat{v}]_{\times} x + [\hat{v}]_{\times}^{2} x - \cos\varphi\,[\hat{v}]_{\times}^{2} x =$$

$$= x + \sin\varphi\,[\hat{v}]_{\times} x + (1 - \cos\varphi)[\hat{v}]_{\times}^{2} x.$$

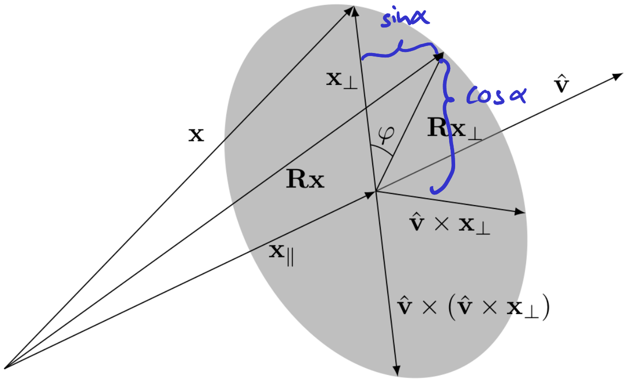


Figure: Geometric interpretation of Rodrigues' formula.

Matrix Exponentials

$$(S\Lambda S^{-1})^k = S\Lambda \cancel{S^{-1}S}\Lambda \cancel{S^{-1}S}\cdots S^{-1} = S\Lambda^k S^{-1}$$

For $A \in \mathbb{R}^{n\times n}$, define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

If A is diagonalisable: $A = S\Lambda S^{-1}$, $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Rightarrow \Lambda^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}$

$$e^A = \sum_{k=0}^{\infty} \frac{(S\Lambda S^{-1})^k}{k!} = \sum_{k=0}^{\infty} S\frac{\Lambda^k}{k!}S^{-1} = S\left(\sum_{k=0}^{\infty}\frac{\Lambda^k}{k!}\right)S^{-1} = S e^{\Lambda} S^{-1}$$

$$= S\begin{pmatrix} \sum_{k=0}^{\infty}\frac{\lambda_1^k}{k!} & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty}\frac{\lambda_n^k}{k!} \end{pmatrix} S^{-1} = S\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} S^{-1}$$

$$AS = S\Lambda$$

$$A(s_1 \cdots s_n) = (\lambda_1 s_1 \cdots \lambda_n s_n)$$

Rotations as Exponentials

Let $R(\alpha)$ be an orthogonal 3×3 matrix:

$$RR^T = I \Rightarrow \frac{d}{d\alpha}(RR^T) = \frac{dI}{d\alpha} = 0$$

$$\frac{dR}{d\alpha}R^T + R\frac{dR^T}{d\alpha} = 0 \Rightarrow \frac{dR}{d\alpha}R^T = -\left(\frac{dR}{d\alpha}R^T\right)^T = /\text{anti-symmetric}/ = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

$$= [v]_\times$$

$$\frac{dR}{d\alpha}R^T = [v]_\times \Leftrightarrow \frac{dR}{d\alpha} = [v]_\times R = R(\alpha) = e^{[v]_\times \alpha}\underbrace{R(0)}_{=I} = e^{[v]_\times \alpha}$$

eigenvalues: $e^{\pm i\alpha}, 1$

if $\|v\| = 1$

Recovering Axis-Angle from R

$$[\hat{v}]_\times^2 = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -v_3^2-v_2^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & -v_3^2-v_1^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & -v_2^2-v_1^2 \end{pmatrix}$$

Rodrigues': $R = \underline{I} + \underline{\sin\alpha\,[\hat{v}]_\times} + \underline{(1-\cos\alpha)}\underline{[\hat{v}]_\times^2}$

consider $\operatorname{tr} R = 3 + 0 + (1-\cos\alpha)(-2\underbrace{(v_1^2+v_2^2+v_3^2)}_{=\|\hat{v}\|^2=1})$

$= 3 - 2 + 2\cos\alpha = 1 + 2\cos\alpha \iff \cos\alpha = \dfrac{\operatorname{tr} R - 1}{2}$

and $R - R^T = 0 + 2\sin\alpha[\hat{v}]_\times + 0 \implies \sin\alpha[\hat{v}]_\times = \dfrac{R-R^T}{2} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$

$(\sin\alpha)\hat{v} = n \implies \|n\| = \sin\alpha$

$\hat{v} = \dfrac{n}{\|n\|}$

Use <u>both</u> $\sin\alpha$ and $\cos\alpha$ to determine the angle!