Guide to Exercises for Introduction to Representations and Estimation in Geometry

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Version: 0.40– September 19, 2018

1 Basic geometric objects

1.1. **Alternative 1:** The point $\bar{\mathbf{y}}_0$ is on the line since

$$(\Delta\, \mathbf{l}_1)\, l_1 + (\Delta\, \mathbf{l}_2)\, l_2 = \Delta\, (l_1^2 + l_2^2) = \Delta.$$

We can set $\hat{\mathbf{t}} = (l_2, -l_1)$ as a direction vector or tangent vector of the line. This works since $\bar{\mathbf{y}}_0 + s\, t = (\Delta\, l_1 + s\, l_2, \Delta\, l_2 - s\, l_1)$ is a point on the line:

$$(\Delta\, l_1 + s\, l_2)\, l_1 + (\Delta\, l_2 - s\, l_1)\, l_2 = \Delta\, (l_1^2 + l_2^2) + s\, (l_2 l_1 - l_1 l_2) = \Delta.$$

Finally we investigate the distance from $\bar{\mathbf{y}}_0 + s\, \hat{\mathbf{t}}$ to the origin, by looking at the square of this distance:

$$\|\bar{\mathbf{y}}_0 + s\, \hat{\mathbf{t}}\|^2 = (\bar{\mathbf{y}}_0 + s\, \hat{\mathbf{t}}) \cdot (\bar{\mathbf{y}}_0 + s\, \hat{\mathbf{t}}) = \bar{\mathbf{y}}_0 \cdot \bar{\mathbf{y}}_0 + 2\, s\, \underbrace{\bar{\mathbf{y}}_0 \cdot \hat{\mathbf{t}}}_{=0} + s^2\, \hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = \|\bar{\mathbf{y}}_0\|^2 + s^2.$$

This expression shows that the distance is minimized for $s = 0$, from which follows that $\bar{\mathbf{y}}_0$ is the closest point to the origin.

Alternative 2: The vector $\hat{\mathbf{l}} = (l_1, l_2)$ is a normal vector of the line, i.e., it is orthogonal (perpendicular) to the line. Starting at the origin and moving along the direction of $\hat{\mathbf{l}}$, the distance s gives the point $s\, \bar{\mathbf{l}}$. The path of this point intersects the line when

$$\Delta = (s\, l_1)\, l_1^2 + (s\, l_2)\, l_2^2 = s(l_1^2 + l_2^2) = s.$$

The intersection point is $(0, 0) + s\, \hat{\mathbf{l}} = \Delta\, (l_1, l_2) = \bar{\mathbf{y}}_0$.

1.2. **Alternative 3:** Change coordinates from (u, v) to $(a, b) = (u, v) - (u_0, v_0)$, a translation by subtracting (u_0, v_0). In these new coordinates, the point with coordinates (u_0, v_0) in the original system lies at the origin. In the new coordinates, the equation of the line (same line as before) is given as

$$\Delta = u\, l_1 + v\, l_2 = (a + u_0)\, l_1 + (b + v_0)\, l_2 = a\, l_1 + b\, l_2 + (u_0 l_1 + v_0 l_2),$$

or

$$a\, l_1 + b\, l_2 = \Delta - (u_0 l_1 + v_0 l_2).$$

This means that after the coordinate transformation, the line as the same normal vector (l_1, l_2) as before the transformation, but the distance to the new origin in the direction of (l_1, l_2) is given as $\Delta - (u_0 l_1 + v_0 l_2)$ (can become negative, which simply means that it is $-(l_1, l_2)$ that points from the new origin to the line).

Using the result from the previous exercise, the point closest to the origin is given as

$$(\Delta - (u_0 l_1 + v_0 l_2))\, (l_1, l_2),$$

These coordinates are expressed in the new system, and to return to the original system, we need to translate by adding (u_0, y_0). The result is

$$(\Delta - (u_0 l_1 + v_0 l_2))\, (l_1, l_2) + (u_0, y_0).$$

The resulting point can also be expressed as:

$$\begin{pmatrix} 1 - l_1^2 & -l_1 l_2 \\ -l_1 l_2 & 1 - l_2^2 \end{pmatrix} + \Delta \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = /l_1^2 + l_2^2 = 1/ = \begin{pmatrix} l_2^2 & -l_1 l_2 \\ -l_1 l_2 & l_1^2 \end{pmatrix} + \Delta \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

1.3. The results are independent of the handedness of the coordinate system: no assumption about the handedness is made in the derivations.

1.4. For example, using Alternative 1 in exercise 1.1, we find the point in the plane that is closest to $\bar{\mathbf{x}}_1$ by moving $\bar{\mathbf{x}}_1$ along a path in the direction of the normal vector of the plane until the path intersects the plane. This is the same as finding s such that $\bar{\mathbf{x}}_1 + s\,\bar{\mathbf{p}}$ is a point in the plane, where $\bar{\mathbf{p}}$ is a normal vector of the plane, for example: $\bar{\mathbf{p}} = (2, 4, -1)$. Inserted into the equation of the plane, we get:

$$(-1 + 2\,s) \cdot 2 + (2 + 4\,s) \cdot 4 - (5 - s) = 3,$$

or

$$21\,s + 1 = 3, \quad \Rightarrow \quad s = \frac{2}{21}.$$

Hence, the point in the plane that is closest to $\bar{\mathbf{x}}_1$ has Cartesian coordinates

$$(-1, 2, 5) + s\,(2, 4, -1) = (-1, 2, 5) + \frac{2}{21}\,(2, 4, -1) = \frac{1}{21}(-17, 50, 103).$$

The distance from this point to $\bar{\mathbf{x}}_1$ is given as

$$\left\| \frac{1}{21}(-17, 50, 103) - (-1, 2, 5) \right\| = \left\| \frac{1}{21}(4, 8, -2) \right\| = \frac{\sqrt{84}}{21} = \frac{2}{\sqrt{21}}.$$

1.5. We can represent the line in parametric form: $\bar{\mathbf{x}}(s) = \bar{\mathbf{x}}_1 + s\,\bar{\mathbf{t}}$, where $\bar{\mathbf{t}}$ is a tangent vector pointing in the direction of the line. We can choose

$$\bar{\mathbf{t}} = \bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 = (-3, 3, 0), \quad \Rightarrow \quad \bar{\mathbf{x}}(s) = (-1 - 3\,s, 2 + 3\,s, 5).$$

We want to find the point on this line that also lie in *plane 1*, i.e, find s such that $\bar{\mathbf{x}}(s)$ lies in the plane:

$$3 = 2 \cdot (-1 - 3\,s) + 4 \cdot (2 + 3\,s) - 5 = 1 + 6\,s, \quad \Rightarrow \quad s = \frac{1}{3}.$$

This gives the point where the line intersects the plane as

$$\bar{\mathbf{x}}\left(\frac{1}{3}\right) = (-2, 3, 5)$$

1.6. We seek parameters p_1, p_2, p_3, Δ of the plane such that

$$x_1 p_1 + x_2 p_2 + x_3 p_3 = \Delta$$

for Cartesian coordinates (x_1, x_2, x_3) of any point in the plane. In particular, the previous equation should be valid for the three points $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3$, which leads to three linear equations in the unknown parameters:

$$\begin{aligned} -p_1 + 2\,p_2 + 5\,p_3 &= \Delta \\ 2\,p_1 - p_2 + 5\,p_3 &= \Delta \\ 4\,p_1 + 3\,p_2 + p_3 &= \Delta \end{aligned}$$

We have three equations and four unknowns, so it is not possible to solve all the unknown parameters. For example, we can express p_1, p_2, p_3 in terms of Δ, which can be done using Gaussian elimination:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{\Delta}{51} \begin{pmatrix} 6 \\ 6 \\ 9 \end{pmatrix}.$$

Consequently, the equation of the plane that includes $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3$ is given as

$$\frac{6}{51}\,\Delta\, x_1 + \frac{6}{51}\,\Delta\, x_2 + \frac{9}{51}\,\Delta\, x_3 = \Delta,$$

or, after dividing by $\frac{\Delta}{51}$ on both sides,

$$6\,x_1 + 6\,x_2 + 9\,x_3 = 51.$$

1.7. We have two equations in the Cartesian coordinates (x_1, x_2, x_3) of a point that lies on the line:

$$\begin{aligned} 2\,x_1 + 4\,x_2 - x_3 &= 3, \\ 6\,x_1 + 6\,x_2 + 9\,x_3 &= 51. \end{aligned}$$

We can then eliminate (for example) either x_1 from the first equation or x_3 from the second equation:

$$\begin{aligned} -6\,x_2 + 12\,x_3 &= 42, \\ 24\,x_1 + 42\,x_2 &= 78. \end{aligned}$$

The coordinates of points on the intersecting line can be represented in parametric form as (for example):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{13}{4} \\ 0 \\ \frac{7}{2} \end{pmatrix} + s \begin{pmatrix} -\frac{7}{4} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

1.8. **Alternative 1:** Find the parameter value s that gives the shortest distance to $\bar{\mathbf{x}}_1$ from $\bar{\mathbf{x}}(s)$. This leads to the minimization of a second order expression in s, which is done by computing the derivative of the distance squared and setting it to zero.

Alternative 2: Find a point on the line such that the difference $\bar{\mathbf{x}}(s) - \bar{\mathbf{x}}_1$ is perpendicular to the direction of the line. A direction vector is given as $\bar{\mathbf{t}} = (-3, 2, 1)$, i.e., we want to find s such that

$$\begin{aligned} 0 = \bar{\mathbf{t}} \cdot (\bar{\mathbf{x}}(s) - \bar{\mathbf{x}}_1) &= (-3, 2, 1) \cdot \big((1, 5, -7) + s\,(-3, 2, 1) - (-1, 2, 5)\big) = \\ &= (-3, 2, 1) \cdot \big((2, 3, -12) + s\,(-3, 2, 1)\big) = -12 + 14\,s \end{aligned}$$

This gives $s = \frac{6}{7}$, and the point on the line closest to $\bar{\mathbf{x}}_1$ is

$$\bar{\mathbf{x}}\left(\frac{6}{7}\right) = (1, 5, -7) + \frac{6}{7}\,(-3, 2, 1) = \frac{1}{7}\,(-11, 47, -43).$$

The distance from this point to $\bar{\mathbf{x}}_1$ is

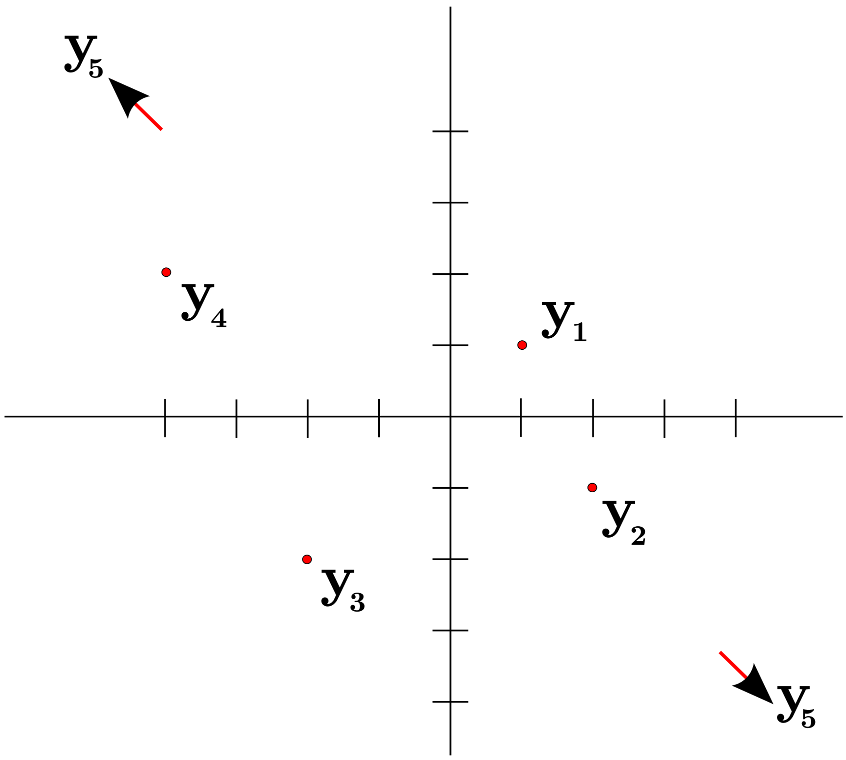
$$\left\| \bar{\mathbf{x}}\left(\frac{6}{7}\right) - \bar{\mathbf{x}}_1 \right\| = \left\| \frac{1}{7}\,(-11, 47, -43) - (-1, 2, 5) \right\| = \left\| \frac{1}{7}\,(-4, 33, -78) \right\| = \frac{\sqrt{7189}}{7} = \sqrt{\frac{1027}{7}} \approx 12.1$$

2 Homogeneous representations in 2D

2.1. We normalize the homogeneous coordinates of each point such that the third element = 1. The first two element in the resulting vectors are the coordinates of the corresponding 2D point. The homogeneous coordinates of the fifth point $\mathbf{y}_5$ cannot be P-normalized since it is a point at infinity. It lies in the orientation given by $\pm(1, -1)$.

$$\mathbf{y}_1 \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{y}_2 \sim \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{y}_3 \sim \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{y}_4 \sim \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$$

The four proper points together with the fifth point at infinity are plotted in the figure below:



2.2. We normalize the homogeneous coordinates of each line such that the sum of squares of the first two elements =1, and change the sign of the result if the third element becomes positive. The first two element in the resulting vectors are $(\cos\alpha, \sin\alpha)$ of a normal vector of the line that points away from the origin. The third element is $-L$, with $L =$ the shortest distance between the origin and the line.

$$\mathbf{l}_1 \sim \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \qquad \mathbf{l}_2 \sim \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \end{pmatrix} \qquad \mathbf{l}_3 \sim \begin{pmatrix} 0.447 \\ -0.894 \\ -0.894 \end{pmatrix} \qquad \mathbf{l}_4 \sim \begin{pmatrix} -0.707 \\ 0.707 \\ -0.707 \end{pmatrix}$$

Alternatively, for each line we can determine the homogeneous coordinates of two points that lie on the line, and then draw a line through these pairs of points. These pairs of points are determined from the relation $\mathbf{y} \cdot \mathbf{l} = 0$ when $\mathbf{l}$ and $\mathbf{y}$ are the homogeneous coordinates of a line and of a point that lie on the line. For line $\mathbf{l}_1$ all points with homogeneous coordinates

$$\mathbf{y} = \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$$

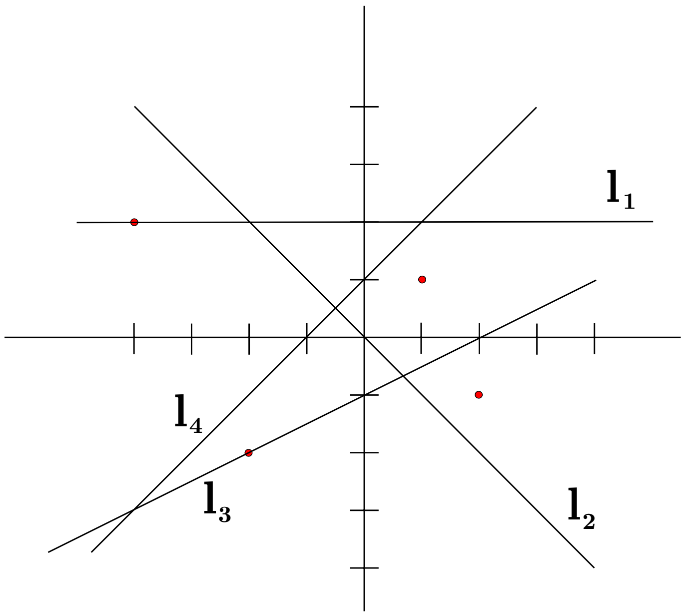
lie on the line since $\mathbf{y} \cdot \mathbf{l}_1 = 0$. For line $\mathbf{l}_2$ the two points with homogeneous coordinates given by

$$\mathbf{y}_2' = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{y}_2'' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

both lie on this line for the same reason. Similarly, for lines $\mathbf{l}_3$ and $\mathbf{l}_4$, these pairs of points

$$\mathbf{y}_3' = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{y}_3'' = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{y}_4' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{y}_4'' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

lie on the respective line. The four lines are plotted in the figure below:



2.3. For example: take lines $\mathbf{l}_1$ and $\mathbf{l}_2$. The homogeneous coordinates of the point of intersection, $\mathbf{y}$, are given by

$$\mathbf{y} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \sim \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

This are the homogeneous coordinates of the 2D point $(-2, 2)$.

2.4. For example: take points $\mathbf{y}_1$ and $\mathbf{y}_2$. The dual homogeneous coordinates of the intersecting line, $\mathbf{l}$, are given by

$$\mathbf{l} = \mathbf{y}_1 \times \mathbf{y}_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 0.894 \\ 0.447 \\ -1.342 \end{pmatrix}$$

This are the dual homogeneous coordinates of the 2D line that has a normal vector $(0.894, 0.447)$ located at distance 1.342 from the origin.

As an example of a line that passes through $\mathbf{y}_5$, a point at infinity, consider the line that also intersects with $\mathbf{y}_3$:

$$\mathbf{l} = \mathbf{y}_5 \times \mathbf{y}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} \sim \begin{pmatrix} -0.707 \\ -0.707 \\ -2.828 \end{pmatrix}$$

2.5. For example, consider the line $\mathbf{l}_3$ and the three points $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$. In order to determine the distances, the homogeneous coordinates of both the line and the points must be properly normalized. With these normalized homogeneous coordinates at hand, the signed distances from the line to the points are given by a simple scalar product of the homogeneous vectors:

$$d_{13} = \mathbf{y}_1 \cdot \mathbf{l}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0.447 \\ -0.894 \\ -0.894 \end{pmatrix} = -1.342$$

$$d_{23} = \mathbf{y}_2 \cdot \mathbf{l}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0.447 \\ -0.894 \\ -0.894 \end{pmatrix} = 0.894$$

$$d_{33} = \mathbf{y}_3 \cdot \mathbf{l}_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0.447 \\ -0.894 \\ -0.894 \end{pmatrix} = 0$$

The 2D point $\mathbf{y}_1$ lies on the same side of the line as the origin, therefore the distance is negative. The 2D point $\mathbf{y}_2$ lies on the opposite side of the line relative the origin, therefore the distance is positive. In the third case, the 2D point $\mathbf{y}_3$ lies on the line.

2.6. The basic result that is used here is

$\mathbf{y} \cdot \mathbf{l} = 0 \quad \Leftrightarrow$ $\mathbf{y}$ is the homogeneous coordinates of a 2D point that lies on the line with dual homogeneous coordinates $\mathbf{l}$.

Since the two points with homogeneous coordinates $\mathbf{y}_1$ and $\mathbf{y}_2$ both lie on the line, we have

$$\mathbf{y}_1 \cdot \mathbf{l} = \mathbf{y}_2 \cdot \mathbf{l} = 0$$

Let $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$, be any linear combination of the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$. From the above results follows then

$$\mathbf{y} \cdot \mathbf{l} = (c_1\mathbf{y}_1 + c_2\mathbf{y}_2) \cdot \mathbf{l} = c_1(\mathbf{y}_1 \cdot \mathbf{l}) + c_2(\mathbf{y}_2 \cdot \mathbf{l}) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

All vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{l}$ lie in $\mathbb{R}^3$ and since $\mathbf{y}_1$ and $\mathbf{y}_2$ represent distinct points on the line, it must be the case that the two vectors span the orthogonal space to the vector $\mathbf{l}$. Any other point on the line, with homogeneous coordinates $\mathbf{y}_3$, lies in the orthogonal complement of $\mathbf{l}$, and therefore can be written as

$$\mathbf{y}_3 = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$$

for some scalars c_1, c_2.

2.7. Set

$$\mathbf{l}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad \mathbf{l}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

We apply a D-normalization on these dual homogeneous coordinates to get the canonical forms:

$$\mathbf{l}_1 \sim \frac{1}{\sqrt{a_1^2 + b_1^2}} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad \mathbf{l}_2 \sim \frac{1}{\sqrt{a_2^2 + b_2^2}} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

Still, the signs of teach canonical form is not properly set, but the orientations of the two normal vectors, one for each line, are given by the two normal vectors

$$\hat{\mathbf{l}}_1 = \frac{1}{\sqrt{a_1^2 + b_1^2}} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \hat{\mathbf{l}}_2 = \frac{1}{\sqrt{a_2^2 + b_2^2}} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

The smallest angle between them, α must satisfy

$$\cos\alpha = |\hat{\mathbf{l}}_1 \cdot \hat{\mathbf{l}}_2| = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

2.8. The matrix $\mathbf{Y}$ has columns that are linearly dependent exactly when the three points lie on a line. This is also exactly the case when $\mathbf{Y}$ is singular, i.e., $\mathbf{Y}$ has a well-defined inverse exactly when the three points do not lie on a line. In this case, the first column in $\mathbf{Y}^{-\top}$ is orthogonal to the second and third column of $\mathbf{Y}$, i.e., it represents the dual homogeneous coordinates of a line that passes through points $\mathbf{y}_2$ and $\mathbf{y}_3$. Similarly, the second column of $\mathbf{Y}^{-\top}$ is a line that passes through $\mathbf{y}_1$ and $\mathbf{y}_3$, and the third column is a line that passes through $\mathbf{y}_1$ and $\mathbf{y}_2$.

2.9. Using the same arguments as in the previous exercise, $\tilde{\mathbf{Y}}^{-\top}$ is well-defined exactly when the three lines do not intersect at a single point. In this case, the first column of $\tilde{\mathbf{Y}}^{-\top}$ represents the homogeneous coordinates of the intersecting point of lines $\mathbf{l}_2$ and $\mathbf{l}_3$, the second column of $\tilde{\mathbf{Y}}^{-\top}$ is the intersecting point of $\mathbf{y}_1$ and $\mathbf{y}_3$, and the third column is the intersecting point of $\mathbf{y}_1$ and $\mathbf{y}_2$.

3 Transformations in 2D

3.1. We can decompose $\mathbf{M}_1$ as follows:

$$\mathbf{M}_1 = \begin{pmatrix} 3 & 4 & 2 \\ -4 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{translation}} \begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{translation}} \underbrace{\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation}} \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{scaling}}$$

The translation can be determined directly from the upper right 2×1 sub-matrix of $\mathbf{M}_1$, but only because we want to extract the translation as a factor to the left in the resulting product. The scaling that is determined from the remaining factor by assuring that it produces a 2×2 rotation matrix in the upper left sub-matrix.

3.2. We also can decompose $\mathbf{M}_1$ as follows:

$$\mathbf{M}_1 = \begin{pmatrix} 3 & 4 & 2 \\ -4 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{scaling}} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & \frac{2}{5} \\ -\frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{scaling}} \underbrace{\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation}} \underbrace{\begin{pmatrix} 1 & 0 & \frac{14}{25} \\ 0 & 1 & \frac{2}{25} \\ 0 & 0 & 1 \end{pmatrix}}_{\text{translation}}$$

This is the same scaling and rotation as in the previous exercise, since rotation and scaling are commuting transformations.

3.3. We start by normalising the transformation matrix such that element (3,3) is =1, and then decompose it into a translation and a pure linear transformation onto the 2D coordinates:

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0.5 & -1 & 1 \\ 1.5 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & -1 & 0 \\ 1.5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This means that the affine transformation $\mathbf{M}$ is given by a linear transformation of the 2D coordinates defined by the matrix:

$$\begin{pmatrix} 0.5 & -1 \\ 1.5 & 1 \end{pmatrix}$$

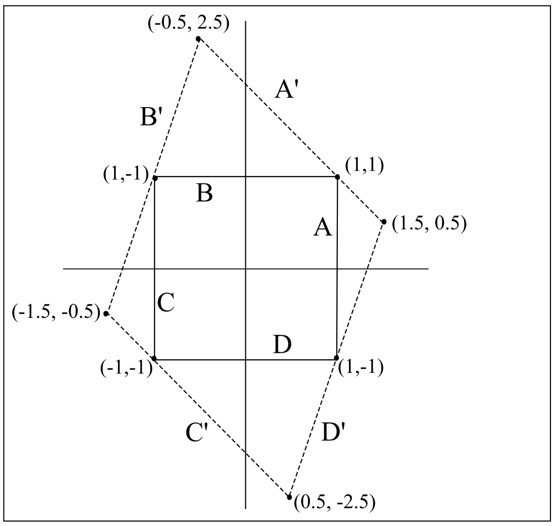
followed by a translation by $(1, -1)$. The linear transformation can be illustrated by mapping a square centered at the origin, e.g., with corner points:

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is mapped to $\begin{pmatrix} -0.5 \\ 2.5 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is mapped to $\begin{pmatrix} -1.5 \\ -0.5 \end{pmatrix}$

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ is mapped to $\begin{pmatrix} 0.5 \\ -2.5 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is mapped to $\begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$



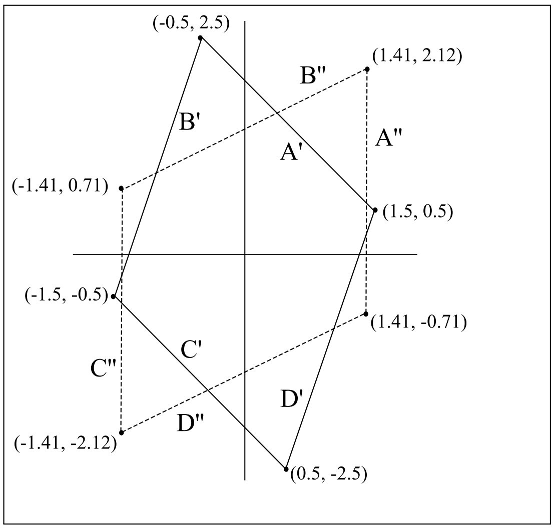
To better understand this transformation we can rotate the result such that the two parallel sides A' and C' become aligned with sides A and C. The corresponding rotation is 45° clockwise. This gives new points:

$$\begin{pmatrix}-0.5\\2.5\end{pmatrix} \text{ is mapped to } \begin{pmatrix}1.41\\2.12\end{pmatrix}$$

$$\begin{pmatrix}-1.5\\-0.5\end{pmatrix} \text{ is mapped to } \begin{pmatrix}-1.41\\0.71\end{pmatrix}$$

$$\begin{pmatrix}0.5\\-2.5\end{pmatrix} \text{ is mapped to } \begin{pmatrix}-1.41\\-2.12\end{pmatrix}$$

$$\begin{pmatrix}1.5\\0.5\end{pmatrix} \text{ is mapped to } \begin{pmatrix}1.41\\-0.71\end{pmatrix}$$



The remaining transformation on these points is now given by

$$\begin{pmatrix}\cos 45° & \sin 45°\\ -\sin 45° & \cos 45°\end{pmatrix}\begin{pmatrix}0.5 & -1\\ 1.5 & 1\end{pmatrix} = \begin{pmatrix}1.41 & 0\\ 0.71 & 1.41\end{pmatrix} = 1.41\begin{pmatrix}1 & 0\\ 0.5 & 1\end{pmatrix}$$

This corresponds to a scaling by $\sqrt{2}$ and a shearing transformation along the second coordinate axis: the corner points the original square will be displaced 0.5 units, up on the right side and down on the left side. In summary the affine transformation can be decomposed as a sequence of transformations applied to the points:

$$\mathbf{y}' = \underbrace{\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}}_{\text{translation}}\underbrace{\begin{pmatrix}\cos 45° & -\sin 45° & 0\\ \sin 45° & \cos 45° & 0\\ 0 & 0 & 1\end{pmatrix}}_{45° \text{ anti-clockwise rotation}}\underbrace{\begin{pmatrix}1.41 & 0 & 0\\ 0 & 1.41 & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{uniform scaling}}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0.5 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{shearing}}\mathbf{y}$$

3.4. For example: uniform scaling commutes with both rotation and shearing transformations, but rotation and shearing do not commute in general. Translations do not commute with either of scalings, rotations, and shearing transformations.

3.5. For example, take 2D point $\mathbf{y}_4$ and 2D line $\mathbf{l}_1$, previously defined. They intersect since $\mathbf{y}_4 \cdot \mathbf{l}_1 = 0$. Transform the point to a new point $\mathbf{y}'_4$:

$$\mathbf{y}'_4 = \mathbf{M}\,\mathbf{y}_4 = \begin{pmatrix}1 & -2 & 2\\ 3 & 2 & -2\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}-4\\ 2\\ 1\end{pmatrix} = \begin{pmatrix}-6\\ -10\\ 2\end{pmatrix} \sim \begin{pmatrix}-3\\ -5\\ 1\end{pmatrix}$$

i.e., the result is the 2D point $(-3, -5)$. The line is transformed by $\mathbf{M}^{-\top}$:

$$\mathbf{M}^{-\top} = \begin{pmatrix}1/4 & -3/8 & 0\\ 1/4 & 1/8 & 0\\ 0 & 1/2 & 1/2\end{pmatrix} \sim \begin{pmatrix}2 & -3 & 0\\ 2 & 1 & 0\\ 0 & 4 & 4\end{pmatrix}$$

The new line $\mathbf{l}_1'$ is then given by

$$\mathbf{l}_1' = \mathbf{M}^{-\top}\,\mathbf{l}_1 = \begin{pmatrix} 2 & -3 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix}$$

The homogeneous coordinates of the transformed point and line still satisfy $\mathbf{y}_4' \cdot \mathbf{l}_1' = 0$, i.e., the line still intersects the point

3.6. Let $\mathbf{l}$ be the homogeneous coordinates of a 2D line. This $\mathbf{l}$ is represented by a vector in $\mathbb{R}^2$. Any vector $\mathbf{y} \in \mathbb{R}^3$ that is perpendicular to $\mathbf{l}$ can then be interpreted as the homogeneous coordinates of a point on the line, in fact any point on the line has a homogeneous representation $\mathbf{y}$ of this type. Consequently, the 2-dimensional subspace V of $\mathbb{R}^3$ that is perpendicular to $\mathbf{l}$ represents all 2D points on the line. Apply an affine transformation to these 2D points by applying a corresponding transformation matrix $\mathbf{M}$ to their homogeneous coordinates, i.e., to V. As a result we get, $\mathbf{M}(V)$, another 2-dimensional subspace of $\mathbb{R}^3$, representing the homogeneous coordinates of the mapped points on the line. This means that there exists a 1-dimensional subspace perpendicular to $\mathbf{M}(V)$, representing the homogeneous coordinates of some line $\mathbf{l}'$ that must intersects all these points. Since all the mapped points lie on a line, it follows that the image of the original line under the affine transformation must be a line.

3.7. We need to show that $[\mathbf{l}']_\times \sim \mathbf{L}'$, i.e., that

$$[\mathbf{M}^{-\top}\mathbf{l}]_\times \sim \mathbf{M}\,\mathbf{L}\,\mathbf{M}^\top \sim \mathbf{M}\,[\mathbf{l}]_\times\,\mathbf{M}^\top$$

for any vector $\mathbf{l} \in \mathbb{R}^3$ and invertible 3×3 matrix $\mathbf{M}$. Both left and right hand sides of this relation are a 3×3 matrix, and both matrices are anti-symmetric. Any 3×3 anti-symmetric matrix is the cross product operator of some vector in $\mathbb{R}^3$, i.e., it must be the case that

$$\mathbf{M}\,[\mathbf{l}]_\times\,\mathbf{M}^\top = [\mathbf{a}]_\times$$

for some $\mathbf{a} \in \mathbb{R}^3$. As a projective element, this $\mathbf{a}$ is unique in the sense that only this $\mathbf{a}$ will satisfy $[\mathbf{a}]_\times \mathbf{a} = \mathbf{0}$, i.e.,

$$\mathbf{M}\,[\mathbf{l}]_\times\,\mathbf{M}^\top\mathbf{a} = \mathbf{0} \quad \Leftrightarrow \quad [\mathbf{l}]_\times\,\mathbf{M}^\top\mathbf{a} = \mathbf{0}$$

This means that $\mathbf{l} \sim \mathbf{M}^\top\mathbf{a}, \Leftrightarrow \mathbf{a} \sim \mathbf{M}^{-\top}\mathbf{l} \Rightarrow$

$$\mathbf{M}\,[\mathbf{l}]_\times\,\mathbf{M}^\top = [\mathbf{M}^{-\top}\mathbf{l}]_\times$$

3.8. Using the procedure described in Toolbox Section 8.1.4, and since $\mathbf{C} = \mathbf{0}$, we get the Schur complement $\mathbf{S} = \mathbf{R}$, from which follows:

$$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{t} \cdot 1 \\ \mathbf{0} & 1^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$$

We verify:

$$\mathbf{T}^{-1}\mathbf{T} = \begin{pmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}^\top\mathbf{R} & \mathbf{R}^\top\mathbf{t} - \mathbf{R}^\top\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{T}\,\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}\,\mathbf{R}^\top & -\mathbf{R}\,\mathbf{R}^\top\mathbf{t} + \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \mathbf{I}$$

3.9. The product of $\mathbf{T}_1$ and $\mathbf{T}_2$ is given as

$$\mathbf{T}_1\mathbf{T}_2 = \begin{pmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \\ \mathbf{0} & 1 \end{pmatrix}.$$

This is a rigid transformation: it first rotates by $\mathbf{R}_1\mathbf{R}_2 \in SO(2)$, and then translates according to the vector $\mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \in \mathbb{R}^2$.

If we do the same calculation for $\mathbf{T}_2\mathbf{T}_1$, we get

$$\mathbf{T}_2\mathbf{T}_1 = \begin{pmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_2\mathbf{R}_1 & \mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix}.$$

This is a rigid transformation: it first rotates by $\mathbf{R}_2\mathbf{R}_1 \in SO(2)$, and then translates according to the vector $\mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2 \in \mathbb{R}^2$.

Rotations in 2D, $SO(2)$, commute: $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2\mathbf{R}_1$. But in general, $\mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \neq \mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2$.

3.10. 1) The group operation must be closed, this is shown in exercise 3.9.

2) The group operation must be associative. Concatenation corresponds to matrix multiplication, and this operation is associative.

3) There must be a neutral element: it is given by the rigid transformation where $\mathbf{R} = \mathbf{I}$ and $\bar{\mathbf{t}} = 0$ (no rotation and no translation).

4) Each transformation must have an inverse, this is shown in exercise 3.8.

3.11. The requested transformation is obtained by first translating to make $\bar{\mathbf{t}}$ end up at the origin, the a rotation by $\mathbf{R}$ about the origin, and finally a translation that moves the origin back to $\bar{\mathbf{t}}$:

$$\begin{pmatrix} \mathbf{I} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\,\bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix}.$$

We verify that this transformation has $\bar{\mathbf{t}}$ as a fix-point:

$$\begin{pmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\,\bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{t}} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}\,\bar{\mathbf{t}} + (\mathbf{I} - \mathbf{R})\,\bar{\mathbf{t}} \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{t}} \\ 1 \end{pmatrix} \quad !$$

3.12. From the previous exercise, we get

$$(\mathbf{I} - \mathbf{R})\,\bar{\mathbf{t}}_0 = \bar{\mathbf{t}} \quad \Rightarrow \quad \bar{\mathbf{t}}_0 = (\mathbf{I} - \mathbf{R})^{-1}\bar{\mathbf{t}}.$$

Notice that this requires that $\mathbf{R} \neq \mathbf{I}$, but in the case that $\mathbf{R} = \mathbf{I}$ then $\mathbf{T}$ is a pure translation that cannot be related to any rotation.

3.13. We are looking for conditions that make the two transformation commute, i.e., $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$. From the solution of exercise 3.9 we see that the two rotations commute if and only if: $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2\mathbf{R}_1$ and $\mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 = \mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2$. The first requirement is already satisfied, since all 2D rotations commute. The second requirement can be reformulated as

$$(\mathbf{R}_1 - \mathbf{I})\,\mathbf{t}_2 = (\mathbf{R}_2 - \mathbf{I})\,\mathbf{t}_1.$$

This condition is satisfied for all $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^2$ when $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{I}$. If only $\mathbf{R}_2 = \mathbf{I}$, it must then be the case that $\mathbf{t}_2 = \mathbf{0}$, and if only $\mathbf{R}_1 = \mathbf{I}$ we must set $\mathbf{t}_1 = \mathbf{0}$. In the general case, neither $\mathbf{R}_1$ nor $\mathbf{R}_2$ equals $\mathbf{I}$, and we require that

$$\mathbf{t}_2 = (\mathbf{R}_1 - \mathbf{I})^{-1}\,(\mathbf{R}_2 - \mathbf{I})\,\mathbf{t}_1.$$

3.14. We can write

$$\bar{\mathbf{y}}_1 = \bar{\mathbf{y}}_0 + \Delta\bar{\mathbf{y}}, \quad \bar{\mathbf{y}}_2 = \bar{\mathbf{y}}_0 - \Delta\bar{\mathbf{y}} \quad \text{where} \quad \bar{\mathbf{y}}_0 = \frac{1}{2}(\bar{\mathbf{y}}_1 + \bar{\mathbf{y}}_2), \quad \Delta\bar{\mathbf{y}} = \frac{1}{2}(\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2).$$

Similarly for the transformed points:

$$\bar{\mathbf{y}}'_1 = \bar{\mathbf{y}}'_0 + \Delta\bar{\mathbf{y}}', \quad \bar{\mathbf{y}}'_2 = \bar{\mathbf{y}}'_0 - \Delta\bar{\mathbf{y}}' \quad \text{where} \quad \bar{\mathbf{y}}'_0 = \frac{1}{2}(\bar{\mathbf{y}}'_1 + \bar{\mathbf{y}}'_2), \quad \Delta\bar{\mathbf{y}}' = \frac{1}{2}(\bar{\mathbf{y}}'_1 - \bar{\mathbf{y}}'_2).$$

The two sets of points are related by a rigid transformation:

$$\bar{\mathbf{y}}'_1 = \mathbf{R}\,\mathbf{y}_1 + \bar{\mathbf{t}} \quad \text{and} \quad \bar{\mathbf{y}}'_2 = \mathbf{R}\,\mathbf{y}_2 + \bar{\mathbf{t}}.$$

We can insert these expressions into $\Delta\bar{\mathbf{y}}'$ to get

$$\Delta\,\bar{\mathbf{y}}' = \frac{1}{2}\mathbf{R}(\mathbf{y}_1 - \mathbf{y}_2) = \mathbf{R}\,\Delta\bar{\mathbf{y}}.$$

Consequently, $\mathbf{R} \in SO(2)$ rotates $\Delta\bar{\mathbf{y}} = (-1, -2)$ to $\Delta\bar{\mathbf{y}}' = (1, -2)$. The rotation angle α can, e.g., be computed from this relation between complex numbers: $(1 - 2i) = e^{i\alpha}(-1 - 2i)$, which gives $\alpha \approx 0.927$ rad. With this result at hand, we can then determine $\mathbf{t}$, e.g., from

$$\mathbf{t} = \bar{\mathbf{y}}'_1 - \mathbf{R}\,\bar{\mathbf{y}}_1, \quad \text{where} \quad \mathbf{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \Rightarrow \quad \mathbf{t} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.2 \\ -0.4 \end{pmatrix}$$

We can easily verify that this unique choice of $\mathbf{R}$ and $\mathbf{t}$ satisfies $\bar{\mathbf{y}}'_k = \mathbf{R}\,\bar{\mathbf{y}}_k + \bar{\mathbf{t}}$ for $k = 1, 2$.

3.15. No, they have to represent a set of points before and after a rigid transformation, i.e., the distance between $\bar{\mathbf{y}}_1$ and $\bar{\mathbf{y}}_2$ must be the same as the distance between $\bar{\mathbf{y}}'_1$ and $\bar{\mathbf{y}}'_2$.

3.16. The requested transformation can be decomposed into a sequence of transformations: first a translation that moves point $\bar{\mathbf{y}}_0$ to the origin:

$$\mathbf{T}_{\text{transl}} = \begin{pmatrix} \mathbf{I} & -\bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix}.$$

Then a uniform scaling by the factor s:

$$\mathbf{T}_{\text{scale}} = \begin{pmatrix} s\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}.$$

Finally, we move the origin back to the point $\bar{\mathbf{y}}_0$:

$$\mathbf{T}^{-1}_{\text{transl}} = \begin{pmatrix} \mathbf{I} & \bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix}.$$

Consequently, the total sequence of transformations is given as

$$\mathbf{T}^{-1}_{\text{transl}}\mathbf{T}_{\text{scale}}\mathbf{T}_{\text{transl}} = \begin{pmatrix} \mathbf{I} & \bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} s\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} s\,\mathbf{I} & (1-s)\,\bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix}.$$

This transformation has the property that

$$\begin{pmatrix} s\,\mathbf{I} & (1-s)\,\bar{\mathbf{y}}_0 \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{y}}_0 \\ 1 \end{pmatrix}.$$

4 Homogeneous representations in 3D

4.1. We normalise the homogeneous coordinates of each point such that the fourth element $= 1$. The first three element in the resulting vectors are the coordinates of the corresponding 3D point.

$$\mathbf{x}_1 \sim \begin{pmatrix} 1/4 \\ 1/2 \\ 3/4 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 \sim \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

The 3D coordinates of the two points are: $(1/4, 1/2, 3/4)$ and $(1, 2, 3)$.

4.2. We normalise the homogeneous coordinates of the plane such that the sum of squares of the first three elements $=1$, and change the sign of the result if the fourth element becomes positive. The first three element in the resulting vectors are a normal vector of the plane that points away from the origin. The fourth element is $-L$, with $L =$ the shortest distance between the origin and the plane. This normalisation gives

$$\mathbf{p}_1 \sim \begin{pmatrix} 0.408 \\ -0.408 \\ 0.816 \\ -0.408 \end{pmatrix}$$

and the normal vector is $(0.408, -0.408, 8.16)$ and the distance from the origin to the plane is 0.408.

4.3. In order to determine the distance, the homogeneous coordinates of both the plane and the point must be properly normalised. With these normalised homogeneous coordinates at hand, the signed distances from the plane to the point is given by a simple scalar product of the homogeneous vectors:

$$d_1 = \mathbf{x}_1 \cdot \mathbf{l}_3 = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.75 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0.408 \\ -0.408 \\ 0.816 \\ -0.408 \end{pmatrix} = 0.1021$$

$$d_2 = \mathbf{x}_1 \cdot \mathbf{l}_3 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0.408 \\ -0.408 \\ 0.816 \\ -0.408 \end{pmatrix} = 1.6330$$

4.4. If the points lie on the same side of the line, their signed distances must have the equal signs. In this case, both distances are positive: the points both lie on the opposite side of the plane as the origin.

4.5. The Plücker coordinates of the line that intersects $\mathbf{x}_1$ and $\mathbf{x}_2$ is given by

$$\mathbf{L}_1 = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top = \begin{pmatrix} 0 & -5 & -10 & -15 \\ 5 & 0 & -5 & -10 \\ 10 & 5 & 0 & -5 \\ 15 & 10 & 50 & \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

4.6. The duality mapping of $\mathbf{L}_1$ gives the corresponding dual Plücker coordinates as

$$\tilde{\mathbf{L}}_1 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.$$

4.7. For example, use the two points with Cartesian coordinates $\bar{\mathbf{x}}_3 = (1, 0, 0)$ and $\bar{\mathbf{x}}_4 = (0, 1, 0)$. The corresponding planes, $\mathbf{p}_3$ and $\mathbf{p}_4$, are then given as

$$\mathbf{p}_3 = \tilde{\mathbf{L}}_1\, \mathbf{x}_3 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix},$$

$$\mathbf{p}_4 = \tilde{\mathbf{L}}_1\, \mathbf{x}_3 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ -2 \end{pmatrix}.$$

4.8. The dual Plücker coordinates of the line where the two planes intersect is given as

$$\mathbf{p}_3\mathbf{p}_4^\top - \mathbf{p}_4\mathbf{p}_3^\top = \begin{pmatrix} 0 & 4 & -8 & 4 \\ -4 & 0 & 12 & -8 \\ 8 & -12 & 0 & 4 \\ -4 & 8 & -4 & 0 \end{pmatrix} \sim \tilde{\mathbf{L}}.$$

4.9. The point $\mathbf{x}$ lies on the line $\mathbf{L}$ if and only if $\tilde{\mathbf{L}}\, \mathbf{x} = \mathbf{0}$.

4.10. The point of intersection, $\mathbf{x}_0$, between the line $\mathbf{L}_1$ and the plane $\mathbf{p}$ is given by

$$\mathbf{x}_0 = \mathbf{L}\, \mathbf{p} = \begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1/3 \\ 2/2 \\ 1 \end{pmatrix}$$

This are the homogeneous coordinates of the 3D point $(0, 1/3, 2/3)$. The point $\mathbf{x}_0$ lies on the plane $\mathbf{p}$ since

$$\mathbf{x}_0 \cdot \mathbf{p} = (0, 1, 2, 3) \cdot (1, -1, 2, -1) = 0$$

It also lies in the plane $\mathbf{L}_1$ since

$$\tilde{\mathbf{L}}_1\, \mathbf{x}_0 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{0}$$

4.11. A necessary condition on $\mathbf{L}_2$ to represent the Plücker coordinates of a 3D line is $\det(\mathbf{L}_2) =$. In this case this is satisfied.

4.12. In order to L-normalize $\mathbf{L}_2$ we divide it with the norm of the first three elements in the fourth column:

$$\operatorname{norm}_L(\mathbf{L}_2) = \frac{1}{\|(3,6,6)\|}\begin{pmatrix} 0 & 1 & 3 & 3 \\ -1 & 0 & 4 & 6 \\ -3 & -4 & 0 & 6 \\ -3 & -6 & -6 & 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 0 & 1 & 3 & 3 \\ -1 & 0 & 4 & 6 \\ -3 & -4 & 0 & 6 \\ -3 & -6 & -6 & 0 \end{pmatrix}$$

The first three elements in the fourth column is then a unit vector that points in the direction of the line: $\hat{\mathbf{t}} = -(1/3, 2/3, 2/3)$. The point on the line that lies closest to the origin is given as

$$\bar{\mathbf{x}}' = \mathbf{A}\,\hat{\mathbf{t}} = \frac{1}{9}\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}\frac{1}{3} = \frac{1}{27}\begin{pmatrix} -8 \\ -7 \\ 11 \end{pmatrix}$$

To verify this result we first notice that

$$\tilde{\mathbf{L}}_2\,\mathbf{x}' = \begin{pmatrix} 0 & 6 & -6 & 4 \\ -6 & 0 & 3 & -3 \\ 6 & -3 & 0 & 1 \\ -4 & 3 & -1 & 0 \end{pmatrix}\begin{pmatrix} -8 \\ -7 \\ 11 \\ 27 \end{pmatrix} = \mathbf{0}$$

which means that $\mathbf{x}'$ indeed lies on the line $\mathbf{L}_2$. Furthermore, we can find a second point on the line, for example, by moving in the direction of $\hat{\mathbf{t}}$ from $\bar{\mathbf{x}}'$:

$$\bar{\mathbf{x}}'' = \bar{\mathbf{x}}' + \hat{\mathbf{t}} = \frac{1}{27}\begin{pmatrix} -8 \\ -7 \\ 11 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -\frac{1}{27}\begin{pmatrix} 17 \\ 25 \\ 7 \end{pmatrix}$$

The Plücker coordinates of the line that passes through $\mathbf{x}'$ and $\mathbf{x}''$ is given as

$$\mathbf{x}'(\mathbf{x}'')^\top - \mathbf{x}''(\mathbf{x}')^\top \sim \mathbf{L}$$

4.13. The two lines intersect if and only if $\tilde{\mathbf{L}}_1 \cdot \mathbf{L}_2 = 0$ (or vice versa). Here, we use the Frobenius scalar product! In the case of the specific lines we have here:

$$\tilde{\mathbf{L}}_1 \cdot \mathbf{L}_2 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 3 & 3 \\ -1 & 0 & 4 & 6 \\ -3 & -4 & 0 & 6 \\ -3 & -6 & -6 & 0 \end{pmatrix} = -8$$

This means that the two lines do not intersect.

4.14. Set

$$\mathbf{L}_1 = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top, \quad \text{and} \quad \tilde{\mathbf{L}}_2 = \mathbf{p}_1\mathbf{p}_2^\top - \mathbf{p}_2\mathbf{p}_1^\top.$$

Gives

$$\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{x}_1\left((\mathbf{x}_2\cdot\mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_2\cdot\mathbf{p}_2)\mathbf{p}_1^\top\right) - \mathbf{x}_2\left((\mathbf{x}_1\cdot\mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_1\cdot\mathbf{p}_2)\mathbf{p}_1^\top\right) \tag{1}$$

We show first that $\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{0}$ requires the two lines to be identical. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ represent two distinct points, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent as vectors in $\mathbb{R}^4$. Therefore, $\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{0}$ implies

$$\mathbf{0} = \mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{x}_1\underbrace{\left((\mathbf{x}_2\cdot\mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_2\cdot\mathbf{p}_2)\mathbf{p}_1^\top\right)}_{=\mathbf{0}} - \mathbf{x}_2\underbrace{\left((\mathbf{x}_1\cdot\mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_1\cdot\mathbf{p}_2)\mathbf{p}_1^\top\right)}_{=\mathbf{0}}$$

or

$$(\mathbf{x}_2 \cdot \mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_2 \cdot \mathbf{p}_2)\mathbf{p}_1^\top = \mathbf{0},$$
$$(\mathbf{x}_1 \cdot \mathbf{p}_1)\mathbf{p}_2^\top - (\mathbf{x}_1 \cdot \mathbf{p}_2)\mathbf{p}_1^\top = \mathbf{0}.$$

Applying the same arguments on $\mathbf{p}_1, \mathbf{p}_2$, as two distinct planes, leads to the following relations:

$$\mathbf{x}_1 \cdot \mathbf{p}_1 = \mathbf{x}_1 \cdot \mathbf{p}_2 = \mathbf{x}_2 \cdot \mathbf{p}_1 = \mathbf{x}_2 \cdot \mathbf{p}_2 = 0. \tag{2}$$

This means that both points, $\mathbf{x}_1$ and $\mathbf{x}_2$, lie in both planes, $\mathbf{p}_1$ and $\mathbf{p}_2$. Consequently, the first line $\mathbf{L}_1$ must be identical to the second line $\mathbf{L}_2$.

If the two lines are identical, then Equation (2) is true. From Equation (1) follows then immediately that $\mathbf{L}_1\tilde{\mathbf{L}}_2 = \mathbf{0}$.

4.15. We can write the two matrices as

$$\mathbf{L} \sim \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}, \quad \tilde{\mathbf{L}} \sim \begin{pmatrix} 0 & \tilde{a} & \tilde{b} & \tilde{c} \\ -\tilde{a} & 0 & \tilde{d} & \tilde{e} \\ -\tilde{b} & -\tilde{d} & 0 & \tilde{f} \\ -\tilde{c} & -\tilde{e} & -\tilde{f} & 0 \end{pmatrix}.$$

The equations suggested in the hint are $[\mathbf{L}\,\tilde{\mathbf{L}}]_{kl} = 0$ where $kl = \{11, 12, 13, 22, 23\}$:

$$\begin{aligned} -a\,\tilde{a} - b\,\tilde{b} - c\,\tilde{c} &= 0, \\ -b\,\tilde{d} - c\,\tilde{e} &= 0, \\ -a\,\tilde{d} - c\,\tilde{f} &= 0, \\ -a\,\tilde{a} - d\,\tilde{d} - e\,\tilde{e} &= 0, \\ -a\,\tilde{b} - e\,\tilde{f} &= 0. \end{aligned}$$

From these equations we solve for the variables $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}$:

$$\tilde{a} = -\frac{(cd - be)\tilde{f}}{a^2} = /\text{IREG Equation (5.29)}/ = \frac{f\tilde{f}}{a},$$
$$\tilde{b} = -\frac{e\tilde{f}}{a}, \quad \tilde{c} = \frac{d\tilde{f}}{a}, \quad \tilde{d} = \frac{\tilde{f}}{a}, \quad \tilde{e} = -\frac{b\tilde{f}}{a}.$$

Assuming $a \neq 0$, we can always find λ such that $\tilde{f} = \lambda\, a$. Inserted into $\tilde{\mathbf{L}}$ in Equation (4), we get the final expression for $\tilde{\mathbf{L}}$:

$$\tilde{\mathbf{L}} = \lambda \begin{pmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{pmatrix}. \tag{3}$$

If $a = 0$, we can solve for a different set of variables from a different set of equations, and still get the same result.

4.16. With

$$\mathbf{L} \sim \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top$$

we get

$$\mathbf{L}^2 \sim (\mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top)(\mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top) = (\mathbf{x}_1 \cdot \mathbf{x}_2)(\mathbf{x}_1\mathbf{x}_2^\top + \mathbf{x}_2\mathbf{x}_1^\top) - \|\mathbf{x}_2\|^2\mathbf{x}_1\mathbf{x}_1^\top - \|\mathbf{x}_1\|^2\mathbf{x}_2\mathbf{x}_2^\top$$

As a projective element, $\mathbf{L}$, and therefore also $\mathbf{L}^2$, is independent of which two points $\mathbf{x}_1$ and $\mathbf{x}_2$ are chosen as long as both lie on the line and they are distinct. Let $\mathbf{x}_0$ and $\mathbf{x}_1$ be the homogeneous coordinates of two such points. A third point $\mathbf{x}_2$ is then given as

$$\mathbf{x}_2 \sim t\,\mathbf{x}_1 + (1-t)\,\mathbf{x}_0$$

for some real number t. Consider the scalar product between homogeneous coordinates of points $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1 \cdot (t\,\mathbf{x}_1 + (1-t)\,\mathbf{x}_0) = t\,\mathbf{x}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_0) + \mathbf{x}_0)$$

This means that we can (almost) always determine t such that $\mathbf{x}_1 \dot{\mathbf{x}}_2 = 0$, and since $\mathbf{L}$ is independent of this choice, we can assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal:

$$\mathbf{L}^2 \sim -\|\mathbf{x}_2\|^2 \mathbf{x}_1 \mathbf{x}_1^\top - \|\mathbf{x}_1\|^2 \mathbf{x}_2 \mathbf{x}_2^\top$$

Furthermore, as a projective element, $\mathbf{L}$ is independent of the norms of $\mathbf{x}_1$ and $\mathbf{x}_2$, so we can replace both with unit vectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$:

$$\mathbf{L}^2 \sim \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_1^\top + \hat{\mathbf{x}}_2 \hat{\mathbf{x}}_2^\top$$

This is a projection operator that projects any vector in $\mathbb{R}^4$ onto the 2D space in $\mathbb{R}^4$ spanned by the vectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$.

4.17. With reference to the previous exercise: a point $\mathbf{x}$ lies on the 3D line if and only if its homogeneous coordinates can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. This, in turn, is equivalent to $\mathbf{L}^2\mathbf{x} \sim \mathbf{x}$ since $\mathbf{L}_2$ is a projection operator onto the 2D space in $\mathbb{R}^4$ spanned by the vectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$. In summary: the point $\mathbf{x}$ lies on the line $\mathbf{L}$ if and only if $\mathbf{L}^2\mathbf{x} \sim \mathbf{x}$.

5 Transformations in 3D

5.1. Two planes with dual homogeneous coordinates

$$\mathbf{p}_1 = \begin{pmatrix} \hat{\mathbf{p}}_1 \\ -\Delta_1 \end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix} \hat{\mathbf{p}}_1 \\ -\Delta_1 \end{pmatrix}$$

are parallel if and only if the normal vectors are parallel: $\hat{\mathbf{p}}_1 = \pm\hat{\mathbf{p}}_2$. The transformation matrix that transform homogeneous coordinates of points in accordance with an affine transformation looks like:

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix}$$

The corresponding dual transformation, transforming the dual homogeneous coordinates of planes, is then given as $\tilde{\mathbf{T}}^{-1} = (\mathbf{T}^{-1})^\top$.

$$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \quad \Rightarrow \quad \tilde{\mathbf{T}} = \mathbf{T}^{-\top} = \begin{pmatrix} \mathbf{A}^{-\top} & \mathbf{0} \\ -\mathbf{A}^{-\top}\bar{\mathbf{t}} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}' & \mathbf{0} \\ \mathbf{t}' & 1 \end{pmatrix}$$

We apply this transformation to the two planes:

$$\mathbf{p}_1' = \tilde{\mathbf{T}}\,\mathbf{p}_1 = \begin{pmatrix} \mathbf{A}'\,\hat{\mathbf{p}}_1 \\ \mathbf{t}' \cdot \hat{\mathbf{p}}_1 - \Delta_1 \end{pmatrix}, \quad \mathbf{p}_2' = \tilde{\mathbf{T}}\,\mathbf{p}_2 = \begin{pmatrix} \mathbf{A}'\,\hat{\mathbf{p}}_2 \\ \mathbf{t}' \cdot \hat{\mathbf{p}}_2 - \Delta_2 \end{pmatrix}.$$

This means that the normal vectors of the resulting planes are, too, parallel: $\mathbf{A}'\,\hat{\mathbf{p}}_1 = \pm\mathbf{A}'\,\hat{\mathbf{p}}_2$. Since the normal vectors are parallel, the two planes $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are parallel.

5.2. If $\mathbf{x}$ is a proper point, the canonical form of its homogeneous coordinates is $\mathbf{x} = (\,\bar{\mathbf{x}}\,1\,)$, where $\bar{\mathbf{x}}$ are the Cartesian coordinates. We apply the affine transformation $\mathbf{T}$, defined in the previous exercise, to $\mathbf{x}$:

$$\mathbf{x}' = \mathbf{T}\,\mathbf{x} = \begin{pmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\,\bar{\mathbf{x}} + \bar{\mathbf{t}} \\ 1 \end{pmatrix}$$

This is, again, a proper point since the third element of $\mathbf{x}'$ is not zero.

A proper line in 3D must intersect with a proper point, in fact with infinitely many proper points. If we transform the line in accordance with the affine transformation $\mathbf{T}$ (using the dual transformation $\tilde{\mathbf{T}}$) then these proper points remain proper. The transformed line must intersect these proper points and is, therefore, a proper line.

The same argument applies to a proper plane, which must include at least one proper point (or, in fact, infinitely many proper points).

5.3. If $\mathbf{x}$ is a point at infinity, its homogeneous coordinates can be written as $\mathbf{x} = (\bar{\mathbf{x}}0)$, where $\bar{\mathbf{x}}$ represents an orientation in 3D space. We apply the affine transformation $\mathbf{T}$, defined in the previous exercise, to $\mathbf{x}$:

$$\mathbf{x}' = \mathbf{T}\,\mathbf{x} = \begin{pmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\,\bar{\mathbf{x}} + \bar{\mathbf{t}} \\ 0 \end{pmatrix}$$

This is, again, a point at infinity since the third element of $\mathbf{x}'$ is zero.

A line at infinity in 3D includes only points at infinity. If we transform the line in accordance with the affine transformation $\mathbf{T}$, then these points at infinity remain at infinity. The transformed line includes only these points at infinity and is, therefore, a line at infinity.

The same argument applies to the plane at infinity, which include exactly all points at infinity. They are transformed by $\mathbf{T}$ to the set of all points at infinity and intersect with the plane at infinity.

5.4. An affine 3D transformation can be described in terms of the Cartesian coordinates of a point before, (u_k, v_k, w_k), and after, (u'_k, v'_k, w'_k), the transformation as

$$\begin{pmatrix} u'_k \\ v'_k \\ w'_k \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

If the coordinates (u_k, v_k, w_k) and $(u'_k, v'_k . w'_k)$ are known, this last equation represents three linear equations in the parameters of the affine transformation:

$$\begin{pmatrix} u'_k \\ v'_k \\ w'_k \end{pmatrix} = \begin{pmatrix} u_k & v_k & w_k & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u_k & v_k & w_k & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_k & v_k & w_k & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

Given four pairs of points, before and after the transformation, for $k = 1, \ldots, 4$, this leads to

$$\underbrace{\begin{pmatrix} u'_1 \\ v'_1 \\ w'_1 \\ u'_2 \\ v'_2 \\ w'_2 \\ u'_3 \\ v'_3 \\ w'_3 \\ u'_4 \\ v'_4 \\ w'_4 \end{pmatrix}}_{=\mathbf{b}} = \underbrace{\begin{pmatrix} u_1 & v_1 & w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & v_1 & w_1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 & v_1 & w_1 & 0 & 0 & 1 \\ u_2 & v_2 & w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & v_2 & w_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_2 & v_2 & w_2 & 0 & 0 & 1 \\ u_3 & v_3 & w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u_3 & v_3 & w_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3 & v_3 & w_3 & 0 & 0 & 1 \\ u_4 & v_4 & w_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u_4 & v_4 & w_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_4 & v_4 & w_4 & 0 & 0 & 1 \end{pmatrix}}_{=\mathbf{A}} \underbrace{\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}}_{=\mathbf{z}},$$

$$\Rightarrow \quad \mathbf{A}\,\mathbf{z} = \mathbf{b}, \quad \Rightarrow \quad \mathbf{z} = \mathbf{A}^{-1}\mathbf{b}$$

5.5. XXX

5.6. Define a new coordinate system with origin at $\Delta\,\hat{\mathbf{p}}$, the point on the plane closest to the origin. An ON-basis is given by $\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \hat{\mathbf{p}}$, where $\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2$ are tangent to the plane. The reflection implies that if (c_1, c_2, c_3) are the Cartesian coordinates of a point relative to this coordinate system, then the reflected point has coordinates $(c_1, c_2, -c_3)$. Let $\bar{\mathbf{x}}$ be the Cartesian coordinates of some point relative to the original coordinate system. Its coordinates in the new coordinate system are then:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{t}}_1 \cdot (\bar{\mathbf{x}} - \Delta\,\hat{\mathbf{p}}) \\ \hat{\mathbf{t}}_2 \cdot (\bar{\mathbf{x}} - \Delta\,\hat{\mathbf{p}}) \\ \hat{\mathbf{p}} \cdot (\bar{\mathbf{x}} - \Delta\,\hat{\mathbf{p}}) \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{t}}_1^\top \bar{\mathbf{x}} \\ \hat{\mathbf{t}}_2^\top \bar{\mathbf{x}} \\ \hat{\mathbf{p}}^\top \bar{\mathbf{x}} - \Delta \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \\ -c_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{t}}_1^\top \bar{\mathbf{x}} \\ \hat{\mathbf{t}}_2^\top \bar{\mathbf{x}} \\ \Delta - \hat{\mathbf{p}}^\top \bar{\mathbf{x}} \end{pmatrix}$$

We reconstruct the Cartesian coordinates of the transformed point relative to the original coordinate system:

$$\bar{\mathbf{x}}' = \hat{\mathbf{t}}_1 c_1 + \hat{\mathbf{t}}_2 c_2 - \hat{\mathbf{p}} c_3 + \Delta\,\hat{\mathbf{p}} = \hat{\mathbf{t}}_1 \hat{\mathbf{t}}_1^\top \bar{\mathbf{x}} + \hat{\mathbf{t}}_2 \hat{\mathbf{t}}_2^\top \bar{\mathbf{x}} - \hat{\mathbf{p}} \hat{\mathbf{p}}^\top \bar{\mathbf{x}} + 2\,\Delta\,\hat{\mathbf{p}}$$

The same expression in homogeneous coordinates read

$$\mathbf{x}' = \begin{pmatrix} \bar{\mathbf{x}}' \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{t}}_1 \hat{\mathbf{t}}_1^\top \bar{\mathbf{x}} + \hat{\mathbf{t}}_2 \hat{\mathbf{t}}_2^\top \bar{\mathbf{x}} - \hat{\mathbf{p}} \hat{\mathbf{p}}^\top \bar{\mathbf{x}} + 2\,\Delta\,\hat{\mathbf{p}} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{\mathbf{t}}_1 \hat{\mathbf{t}}_1^\top + \hat{\mathbf{t}}_2 \hat{\mathbf{t}}_2^\top - \hat{\mathbf{p}} \hat{\mathbf{p}}^\top & 2\,\Delta\,\hat{\mathbf{p}} \\ \mathbf{0} & 1 \end{pmatrix}}_{=\mathbf{T} \text{ in IREG Equation (6.6)}} \underbrace{\begin{pmatrix} \bar{\mathbf{x}} \\ 1 \end{pmatrix}}_{=\mathbf{x}}$$

5.7. XXX

6 Introduction to estimation

6.1. We use the computations outlined in IREG, Section 12.1:

$$(s_1, s_2) = (0,0), \quad s_{11} = 1, s_2 2 = 2, s_{12} = 0$$

The line that minimizes ϵ_V has parameters given as

$$k_V = \frac{s_{12}}{s_{11}} = \frac{0}{1} = 0, \quad l_V = \frac{s_{11}s_2 - s_{12}s_1}{s_{11}} = \frac{0}{1} = 0$$

This is a horizontal line that intersects the origin. The line that minimizes ϵ_H has parameters given as

$$k_H = \frac{s_{22}}{s_{12}} = \frac{2}{0} = \infty, \quad l_H = \frac{s_{12}s_2 - s_{22}s_1}{s_{12}} = \frac{0}{0}$$

This is a vertical line. From the symmetry of the data, it too must intersect the origin. In summary, we get two lines estimated from the same data that are perpendicular.

6.2. We use the parametric representation of the line, as suggested in the exercise:

$$\bar{\mathbf{x}}(t) = \bar{\mathbf{x}}_0 + t\,\hat{\mathbf{n}},$$

where $\bar{\mathbf{x}}_0$ is some point on the line, and $\hat{\mathbf{n}}$ is a normalized tangent direction of the line. Using the same approach as in exercise 1.8, the point $\bar{\mathbf{x}}(t)$ on the line that is closest to some point $\bar{\mathbf{x}}$ is given by $t = \hat{\mathbf{n}} \cdot (\bar{\mathbf{x}} - \bar{\mathbf{x}}_0)$. The distance between $\bar{\mathbf{x}}$ and the line is then

$$d(\bar{\mathbf{x}}) = \|(\bar{\mathbf{x}} - \bar{\mathbf{x}}_0) - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top(\bar{\mathbf{x}} - \bar{\mathbf{x}}_0)\| = \|(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top)\,(\bar{\mathbf{x}} - \bar{\mathbf{x}}_0)\| = \|\mathbf{P}\,(\bar{\mathbf{x}} - \bar{\mathbf{x}}_0)\|.$$

Notice that $\mathbf{P} = \mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T$ is a projection operator onto the subspace of $\mathbb{R}^3$ that is orthogonal to $\hat{\mathbf{n}}$. Therefore: $\mathbf{P}^\top\mathbf{P} = \mathbf{P}\,\mathbf{P} = \mathbf{P}$.

The cost function that we want to minimize is

$$\epsilon = \sum_{k=1}^{N} d(\bar{\mathbf{x}}_k)^2 = \sum_{k=1}^{N} \|\mathbf{P}\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_0)\|^2 = \sum_{k=1}^{N} (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_0)^\top \mathbf{P}\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_0).$$

We want to minimize ϵ over $\bar{\mathbf{x}}_0 \in \mathbb{R}^3$ and $\hat{\mathbf{n}} \in S^2$. Starting with $\bar{\mathbf{x}}_0$, it should satisfy

$$\mathbf{0} = \nabla_{\mathbf{x}_0}\epsilon = \frac{d\epsilon}{d\bar{\mathbf{x}}_0} = 2\sum_{k=1}^{N} \mathbf{P}\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_0).$$

or,

$$N\,\mathbf{P}\,\bar{\mathbf{x}}_0 = \mathbf{P}\sum_{k=1}^{N} \bar{\mathbf{x}}_k, \quad \Rightarrow \quad \mathbf{P}\,\bar{\mathbf{x}}_0 = \mathbf{P}\,\frac{1}{N}\sum_{k=1}^{N} \bar{\mathbf{x}}_k.$$

A straight-forward choice for $\bar{\mathbf{x}}_0$ is

$$\bar{\mathbf{x}}_0 = \frac{1}{N}\sum_{k=1}^{N} \bar{\mathbf{x}}_k = \bar{\mathbf{x}}_c,$$

the center of gravity, or mean position, of the 3D points. We can also add an arbitrary component of $\hat{\mathbf{n}}$ to this $\bar{\mathbf{x}}_0$, which is then canceled by $\mathbf{P}$, and this still gives a point on the line.

By choosing $\bar{\mathbf{x}}_0 = \bar{\mathbf{x}}_c$, we can rewrite ϵ as

$$\epsilon = \sum_{k=1}^{N} (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_c)^\top \mathbf{P}\, (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_0) = \overbrace{\sum_{k=1}^{N} \|\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_c\|^2}^{=\epsilon_0} - \overbrace{\hat{\mathbf{n}}^\top \underbrace{\left(\sum_{k=1}^{N} (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_c)\, (\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_c)^\top \right)}_{=\mathbf{A}\,\mathbf{A}^\top} \hat{\mathbf{n}}}^{=\epsilon_1},$$

where $\mathbf{A}$ is a $3 \times N$ matrix that holds the coordinates of the 3D points relative to the center of gravity in its columns:

$$\mathbf{A} = \big(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_c, \bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_c, \ldots, \bar{\mathbf{x}}_N - \bar{\mathbf{x}}_c\big)\,. \tag{4}$$

This ϵ should now be minimized over unit vectors $\hat{\mathbf{n}}$. Notice that the first term, ϵ_0, does not depend on $\hat{\mathbf{n}}$, and that minimizing ϵ then means to *maximize* the second term, ϵ_1. Since $\hat{\mathbf{n}}$ is constrained by $c = \|\hat{\mathbf{n}}\|^2 = \hat{\mathbf{n}}^\top \hat{\mathbf{n}} = 1$, we need to use Lagrange's method and choose $\hat{\mathbf{n}}$ such that

$$\nabla_{\hat{\mathbf{n}}} \epsilon_1 = \lambda\, \nabla_{\hat{\mathbf{n}}} c \quad \Rightarrow \quad \mathbf{A}\,\mathbf{A}^\top \hat{\mathbf{n}} = \lambda\, \hat{\mathbf{n}}. \tag{5}$$

Here, λ is the Lagrange multiplier of the optimization problem. Notice that $\mathbf{A}\,\mathbf{A}^\top$ is a 3×3 symmetric and positive indefinite matrix, i.e., all eigenvalues of $\mathbf{A}\,\mathbf{A}^\top$ are non-negative, and we can always find an ON-basis of corresponding eigenvectors. The last result in Equation (5) means that $\hat{\mathbf{n}}$ is a normalized eigenvector of $\mathbf{A}\,\mathbf{A}^\top$, with eigenvalue λ, which leads to

$$\epsilon = \epsilon_0 - \hat{\mathbf{n}}^\top \mathbf{A}\,\mathbf{A}^\top \hat{\mathbf{n}} = \epsilon_0 - \lambda.$$

To minimize ϵ, λ should be the largest eigenvalue of $\mathbf{A}\,\mathbf{A}^\top$. We can now summarize all this to: $\bar{\mathbf{x}}_0$ can be chosen as the mean of the 3D points and $\hat{\mathbf{n}}$ should be a normalized eigenvector corresponding to the largest eigenvalue of $\mathbf{A}\,\mathbf{A}^\top$, where $\mathbf{A}$ is described in Equation (4).

Notice that this formulation is the same as was derived in IREG Section 12.2 for the case of a 2D line. An alternative formulation of $\hat{\mathbf{n}}$ is as a right singular vector of $\mathbf{A}$ corresponding to the largest singular value.

6.3. To determine the Plücker coordinates, we need two distinct points on the line. From exercise 6.2, we know that one point can be chosen as $\bar{\mathbf{x}}_c$, the center of gravity of the point set. Another point on the line is the ideal point (at infinity) in the direction of $\pm\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a normalized eigenvector corresponding to the largest eigenvalue of $\mathbf{A}\,\mathbf{A}^\top$. These two points have homogeneous coordinates given as

$$\mathbf{x}_1 \sim \begin{pmatrix} \bar{\mathbf{x}}_c \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 \sim \begin{pmatrix} \hat{\mathbf{n}} \\ 0 \end{pmatrix}.$$

The Plücker coordinates of the line that passes through these two points is given as

$$\mathbf{L} \sim \mathbf{x}_1 \mathbf{x}_2^\top - \mathbf{x}_2 \mathbf{x}_1^\top = \begin{pmatrix} \bar{\mathbf{x}}_c \hat{\mathbf{n}}^\top - \hat{\mathbf{n}}\, \bar{\mathbf{x}}_c^\top & -\hat{\mathbf{n}} \\ \hat{\mathbf{n}}^\top & 0 \end{pmatrix}.$$

6.4. XXX

6.5. XXX

6.6. XXX

6.7. XXX

6.8. The relation $\tilde{\mathbf{L}}_k \mathbf{x} = \mathbf{0}$ for $k = 1, \ldots, N$ can also be written as

$$\underbrace{\begin{pmatrix} \tilde{\mathbf{L}}_1 \\ \tilde{\mathbf{L}}_2 \\ \vdots \\ \tilde{\mathbf{L}}_N \end{pmatrix}}_{=\mathbf{A}} \mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{A}\,\mathbf{x} = \mathbf{0}$$

6.9. Using the result from the previous exercise:

$$\mathbf{A}^\top \mathbf{A} = \begin{pmatrix} \tilde{\mathbf{L}}_1^\top & \tilde{\mathbf{L}}_2^\top & \ldots & \tilde{\mathbf{L}}_N^\top \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{L}}_1 \\ \tilde{\mathbf{L}}_2 \\ \vdots \\ \tilde{\mathbf{L}}_N \end{pmatrix} = \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{L}}_1 + \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{L}}_2 + \ldots + \tilde{\mathbf{L}}_N^\top \tilde{\mathbf{L}}_N$$

7 Homographies

7.1. **The unsophisticated approach (or: don't think, just do the math)**.

A 3D line has a parametric representation as

$$\bar{\mathbf{x}}(s) = \bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}} \tag{6}$$

where $\bar{\mathbf{x}}_0$ is the Cartesian coordinates of some point on the line and $\hat{\mathbf{t}}$ is a tangent vector of the line. We want to show that these points are mapped to points that also corresponds to a parametric representation of a 3D line. The homography transformation can be expressed in homogeneous coordinates:

$$\mathbf{x}'(s) = \begin{pmatrix} \bar{\mathbf{x}}'(s) \\ 1 \end{pmatrix} \sim \mathbf{H}\,\mathbf{x} = \underbrace{\begin{pmatrix} \mathbf{H}_{11} & \bar{\mathbf{h}}_{12} \\ \bar{\mathbf{h}}_{21}^\top & h_{33} \end{pmatrix}}_{4\times 4} \begin{pmatrix} \bar{\mathbf{x}}(s) \\ 1 \end{pmatrix}$$

Here, 4×4 matrix $\mathbf{H}$ is partitioned into a 3×3 matrix $\mathbf{H}_{11}$, two 3-dim vectors $\bar{\mathbf{h}}_{12}$ and $\bar{\mathbf{h}}_{21}$, and a scalar h_{33}.

The image of the line under the homography transformation is a curve given by the Cartesian coordinates

$$\bar{\mathbf{x}}'(s) = \frac{\mathbf{H}_{11}\bar{\mathbf{x}}(s) + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top\bar{\mathbf{x}}(s) + h_{33}}$$

We want to show that this curve is a line. We expand this expression using Equation (6):

$$\bar{\mathbf{x}}'(s) = \frac{\mathbf{H}_{11}(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + h_{33}} = \underbrace{\frac{\mathbf{H}_{11}\bar{\mathbf{x}}_0 + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top\bar{\mathbf{x}}_0 + h_{33}}}_{=\bar{\mathbf{x}}_0'} + \underbrace{\frac{\mathbf{H}_{11}(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + h_{33}} - \frac{\mathbf{H}_{11}\bar{\mathbf{x}}_0 + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top\bar{\mathbf{x}}_0 + h_{33}}}_{=\bar{\mathbf{x}}_1'(s)}$$

Note that $\bar{\mathbf{x}}_0'$ is the image of $\bar{\mathbf{x}}_0$ under the homography mapping, and that $\bar{\mathbf{x}}_1'(s)$ describes the curve relative to $\bar{\mathbf{x}}_0'$. We simplify $\bar{\mathbf{x}}_1'(s)$:

$$\begin{aligned} \bar{\mathbf{x}}_1'(s) &= \frac{\mathbf{H}_{11}(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + h_{33}} - \frac{\mathbf{H}_{11}\bar{\mathbf{x}}_0 + \bar{\mathbf{h}}_{12}}{\bar{\mathbf{h}}_{21}^\top\bar{\mathbf{x}}_0 + h_{33}} = \\ &= \underbrace{\frac{t}{(\bar{\mathbf{h}}_{21}^\top(\bar{\mathbf{x}}_0 + s\,\hat{\mathbf{t}}) + h_{33})(\bar{\mathbf{h}}_{21}^\top\bar{\mathbf{x}}_0 + h_{33})}}_{\text{a scalar function } \sigma(s)} \underbrace{\left((\mathbf{h}_{21}^\top\bar{\mathbf{x}}_0 + h_{33})\mathbf{H}_{11}\hat{\mathbf{t}} - (\mathbf{h}_{21}^\top\hat{\mathbf{t}}(\mathbf{H}_{11}\bar{\mathbf{x}}_0 + \mathbf{h}_{12})\right)}_{\text{a vector } \bar{\mathbf{t}}',\text{ independent of } s} \end{aligned}$$

In summary:

$$\bar{\mathbf{x}}'(s) = \bar{\mathbf{x}}_0' + \sigma(s)\,\bar{\mathbf{t}}'$$

This is the parametric representation of a 3D line that passes through the point $\bar{\mathbf{x}}_0'$ with $\bar{\mathbf{t}}'$ as tangent vector.

The sophisticated approach (or: think first, then do the math)

The homogeneous coordinates of all points on a 3D line form a 2D subspace $S \subset \mathbb{R}^4$. Since a homography is represented as a non-singular linear transformation on $\mathbb{R}^4$, the image of S is, again, a 2D subspace S'. Also this S' contains exactly the homogeneous coordinates of all points on a 3D line.

7.2. The homography transforms a 3D point x to a 3D point x' in accordance with $\mathbf{y}' \sim \mathbf{H}\,\mathbf{x}$. The Plücker coordinates of a 3D line is formed from the homogeneous coordinates of two points on the line, say $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^\top - \mathbf{x}_2\mathbf{x}_1^\top.$$

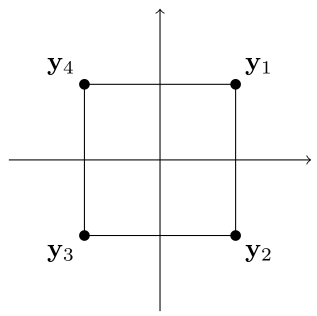
We get the transformed Plücker coordinates by first transforming the two points and then forming the Plücker coordinates:

$$\begin{aligned}\mathbf{L}' \sim \mathbf{x}_1'\mathbf{x}_2'^{\top} - \mathbf{x}_2'\mathbf{x}_1'^{\top} &= \mathbf{H}\,\mathbf{x}_1\big(\mathbf{H}\,\mathbf{x}_2\big)^{\top} - \mathbf{H}\,\mathbf{x}_2\big(\mathbf{H}\,\mathbf{x}_1\big)^{\top} = \\ &= \mathbf{H}\,\mathbf{x}_1\mathbf{x}_2^{\top}\mathbf{H}^{\top} - \mathbf{H}\,\mathbf{x}_2\mathbf{x}_1\mathbf{H}^{\top} = \mathbf{H}\,\mathbf{L}\,\mathbf{H}^{\top}.\end{aligned}$$

See also IREG Section 6.6.

7.3. The four points to be transformed have Cartesian coordinates given as

$$\bar{\mathbf{y}}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \bar{\mathbf{y}}_2 = \begin{pmatrix}1\\-1\end{pmatrix}, \quad \bar{\mathbf{y}}_3 = \begin{pmatrix}-1\\-1\end{pmatrix}, \quad \bar{\mathbf{y}}_4 = \begin{pmatrix}-1\\1\end{pmatrix}.$$



The corresponding homogeneous coordinates are inserted into the columns of a matrix $\mathbf{Y}$:

$$\mathbf{Y} = \begin{pmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}.$$

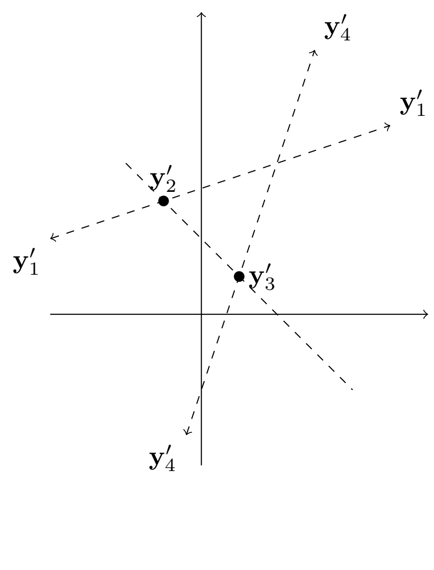
The homogeneous coordinates of the transformed points are obtained by multiplying $\mathbf{H}$ onto $\mathbf{Y}$, from left:

$$\mathbf{Y}' = \mathbf{H}\,\mathbf{Y} = \begin{pmatrix}1 & 1 & 1\\ -1 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}3 & 1 & -1 & 1\\ 1 & -3 & -1 & 3\\ 0 & -2 & -2 & 0\end{pmatrix}.$$

This means that the first and fourth points are both transformed to points at infinity. $\mathbf{y}_1'$ lies in the direction $\pm(3,1)$ and $\mathbf{y}_4'$ lies in the direction $\pm(1,3)$. The Cartesian coordinates of the transformation of points two and three are

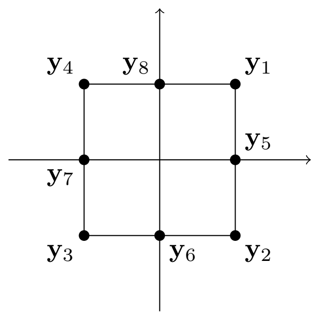
$$\bar{\mathbf{y}}_2' = \frac{1}{(-2)}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{3}{2}\end{pmatrix}, \quad \bar{\mathbf{y}}_3' = \frac{1}{(-2)}\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}.$$

We draw these transformed points in a figure. Missing in this figure is the line between points $\mathbf{y}_1'$ and $\mathbf{y}_4'$. Since both these points are at infinity, this line is *the line at infinity.*



It remains to determine how the interior of the square is mapped to the transformed space. A simple approach is to transform a set of points that lie between the four corners of the square. For example, we can look at these points:

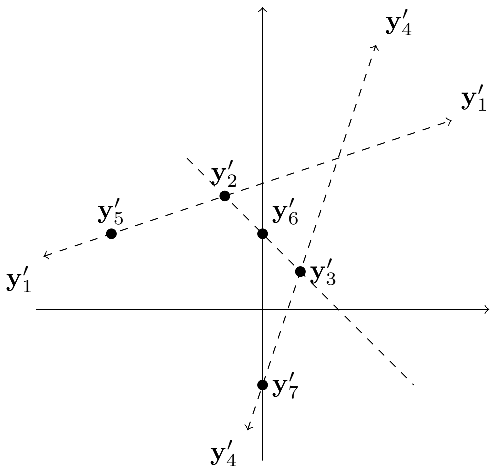
$$\bar{\mathbf{y}}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\mathbf{y}}_6 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \bar{\mathbf{y}}_7 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \bar{\mathbf{y}}_8 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



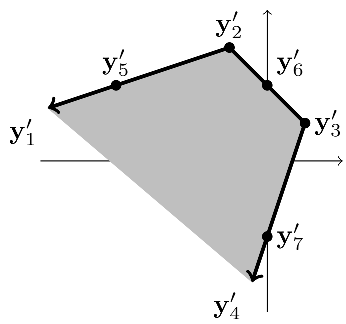
If we apply the homography transformation also to these points, we get the corresponding Cartesian coordinates:

$$\bar{\mathbf{y}}'_5 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \bar{\mathbf{y}}'_6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{\mathbf{y}}'_7 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathbf{y}'_8 \text{ lies a infinity.}$$

We plot these new points in the figure:



As a result, the square is transformed to the infinitely large shape depicted in the figure below. Notice that $\mathbf{y}'_1$ and $\mathbf{y}'_4$ lie at infinity.



7.4. From the solution of exercise 7.3 follows that both $\bar{\mathbf{y}}_1 = (1, 1)$ and $\bar{\mathbf{y}}_4 = (-1, 1)$ are mapped to infinity.

7.5. The dual homogeneous coordinates of the line that passes through $\mathbf{y}_1$ and $\mathbf{y}_4$ is given as

$$\mathbf{l}_{14} \sim \mathbf{y}_1 \times \mathbf{y}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

The dual transformation of $\mathbf{H}$ is given as

$$\tilde{\mathbf{H}} = \mathbf{H}^{-\top} \sim \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 1 & -3 \end{pmatrix}.$$

We apply this transformation to the line $\mathbf{l}_{14}$:

$$\mathbf{l}'_{14} = \tilde{\mathbf{H}}\,\mathbf{l} = \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}.$$

This is the line at infinity, which is the line that passes through every point at infinity

8 Pinhole cameras

8.1. XXX

8.2. Any line that passes through the camera center is mapped to a point in the image instead of a line. The point is at infinity if the line is parallel to the image plane.

8.3. For example: the three plane with dual homogeneous coordinates given as the three rows of the camera matrix $\mathbf{C}$. Each such row must be orthogonal to the homogeneous coordinates of the camera center, $\mathbf{n}$, since $\mathbf{C}\,\mathbf{n} = \mathbf{0}$.

8.4. XXX

8.5. The homogeneous coordinates of the image center, $\mathbf{n}$, in the world coordinate system related to the camera mapping, must satisfy

$$\mathbf{C}\,\mathbf{n} = \mathbf{0}$$

In this case, this this leads to the equation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -2 \end{pmatrix}$$

where (n_1, n_2, n_3) are the 3D coordinates of the camera center in the world coordinate system. Solving this equation (e.g., using Matlab) gives $(n_1, n_2, n_3) = (5/3, -2, -5/6)$.

8.6. The images of the two points are given by

$$\mathbf{y}_1 \sim \mathbf{C}\,\mathbf{x}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 17 \end{pmatrix} \sim \begin{pmatrix} 30/17 \\ 20/17 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 30/17 \\ 20/17 \end{pmatrix}$$

$$\mathbf{y}_2 \sim \mathbf{C}\,\mathbf{x}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \\ 13 \end{pmatrix} \sim \begin{pmatrix} 20/13 \\ 30/13 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 20/13 \\ 30/13 \end{pmatrix}$$

8.7. The dual homogeneous coordinates of the intersecting line is given by

$$\mathbf{l} = \mathbf{y}_1 \times \mathbf{y}_2 = \begin{pmatrix} -250 \\ -45 \\ 450 \end{pmatrix} \sim \begin{pmatrix} -50 \\ -9 \\ 90 \end{pmatrix}$$

8.8. XXX

8.9. The camera center can be solved from the camera matrix in the same as in the previous exercise, giving its 3D coordinates in the world coordinate system as

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 474.1 \\ 1116.7 \\ 653.5 \end{pmatrix} \quad [\mathrm{mm}]$$

Alternatively, we can solve it from the matrix decomposition provided in the exercise. In the camera centered 3D coordinate system, the camera center is $(0, 0, 0)$. This point is the result of first rotating (n_1, n_2, n_3) by $\mathbf{R}$ and then translating by $\mathbf{t}$, or, vice versa, if $-\mathbf{t}$ is rotated by $\mathbf{R}^{-1} = \mathbf{R}^\top$, the result is (n_1, n_2, n_3):

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = -\mathbf{R}^\top \mathbf{t} = \begin{pmatrix} 474.1 \\ 1116.7 \\ 653.5 \end{pmatrix} \quad [\text{mm}]$$

8.10. In the camera centered 3D coordinate system, the optical axis has direction $(0, 0, 1)$. This direction is obtained by taking the direction in the world coordinate system and rotate it with $\mathbf{R}$, or, vice versa, if we rotate $(0, 0, 1)$ by $\mathbf{R}^{-1} = \mathbf{R}^\top$ we get the direction of the optical axis in the world coordinate system. The result of this operation is the third column of $\mathbf{R}^\top$ = the third row of $\mathbf{R}$:

$$\text{optical axis in the world coordinate system} = \begin{pmatrix} -0.2591 \\ -0.8330 \\ -0.4888 \end{pmatrix}$$

8.11. In camera centered image coordinates, the homogeneous coordinates of the principal point is $(0, 0, 1)$, and these camera centered image coordinates are transformed by $\mathbf{A}$, i.e, the homogeneous coordinates of the principal point is given by the third column of $\mathbf{A}$:

$$\text{homogeneous coordinates of the principal point} = \begin{pmatrix} 948.3 \\ 1298.2 \\ 1 \end{pmatrix}$$

and the corresponding image coordinates are $(948.3, 1298.2)$. These are the pixel coordinates where the first gives the vertical position and the second the horizontal position relative an origin at the top left corner. Assuming that the image is 1944 pixels high and 2592 pixels wide, the image center is at $(972, 1296)$. This is relatively close in the horizontal direction, but more than 20 pixels off in the vertical direction.

8.12. The square will become slightly wider than high (a rectangle) and on top of that it will be slightly sheared, i.e., the rectangle will become slightly rhombic.

8.13. It is a point on the projection line generated by the image point $\mathbf{y}$. This follows from the fact that $\mathbf{C}\,\mathbf{C}^+\mathbf{y} = \mathbf{I}\,\mathbf{y} = \mathbf{y}$, it the 3D point given by $\mathbf{C}^+\mathbf{y}$ is projected back to the image point $\mathbf{y}$. Notice that the 3D point $\mathbf{C}^+\mathbf{y}$ must be distinct from the camera center $\mathbf{n}$, otherwise $\mathbf{C}\,\mathbf{C}^+\mathbf{y} = \mathbf{0}$.

8.15. XXX

8.16. XXX

8.17. XXX

9 Estimation of transformations

9.1. The data matrix $\mathbf{A}$, by definition, should be formulated such that $\mathbf{A}\,\mathbf{z} = \mathbf{0}$ in the ideal (noise free) case, where $\mathbf{z}$ is the model to be determined. In this case $\mathbf{z} = \mathbf{l}$, the dual homogeneous coordinates of the 2D line, and the rows should therefore be the homogeneous coordinates of the points:

$$\mathbf{A} = \begin{pmatrix} -4.89 & 105.4 & 1 \\ 0.88 & 100.2 & 1 \\ 5.55 & 95.68 & 1 \end{pmatrix} \quad \text{has singular values} \quad (\sigma_1, \sigma_2, \sigma_3) = (174.09, 7.43, 0.0014)$$

This profile is ambiguous since there is indeed one very small singular value, σ_3, but also σ_2 is several orders of magnitude smaller than σ_1. Therefore it is difficult to state if one or both of σ_3 and σ_2 should be considered as approximately zero.

9.2. The right singular vector of $\mathbf{A}$ corresponding to the smallest singular value, σ_3 is

$$\mathbf{l}_1 = \begin{pmatrix} -0.0092 \\ -0.0099 \\ 0.9999 \end{pmatrix} \sim \begin{pmatrix} 0.6810 \\ 0.7323 \\ -73.8935 \end{pmatrix} = \mathbf{l}_{10}$$

The vector $\mathbf{l}_1$ has norm $= 1$, the vector $\mathbf{l}_{10}$ is D-normalized to proper dual homogeneous coordinates.

9.3. As a geometric error we can use $\|\mathbf{A}\mathbf{l}_{10}\|$ where $\mathbf{A}$ is the data matrix with properly normalized homogeneous coordinates of the 2D points and $\mathbf{l}_{10}$ is the properly normalized dual homogeneous coordinates of the estimated line. This is equal to the sum of squares of distances between the points and the line. We get $\|\mathbf{A}\mathbf{l}_{10}\| = 0.102696$.

9.4. The centroid of the points is given by

$$\frac{1}{3}\left[\begin{pmatrix} -4.89 \\ 105.4 \end{pmatrix} + \begin{pmatrix} 0.88 \\ 100.2 \end{pmatrix} + \begin{pmatrix} 5.55 \\ 95.68 \end{pmatrix}\right] = \begin{pmatrix} 0.5133 \\ 100.4267 \end{pmatrix}$$

Translating the points by $(-0.5133, -100.4267)$ then produces a new set of points with centroid at $(0, 0)$:

$$\begin{pmatrix} -5.4033 \\ 4.9733 \end{pmatrix} \quad \begin{pmatrix} 0.3667 \\ -0.2267 \end{pmatrix} \quad \begin{pmatrix} 5.0367 \\ -4.7467 \end{pmatrix}$$

The mean distance from $(0, 0)$ for these three points is 4.8986. By multiplying the coordinates of the last three points with $\sqrt{2}/4.8986 = 0.2887$, we get Hartley-transformed points. They have their centroid at $(0, 0)$ and the mean distance to $(0, 0)$ is $= \sqrt{2}$:

$$\begin{pmatrix} -1.5599 \\ 1.4358 \end{pmatrix} \quad \begin{pmatrix} 0.1059 \\ -0.0654 \end{pmatrix} \quad \begin{pmatrix} 1.4541 \\ -1.3704 \end{pmatrix}$$

The transformation matrix that produces the corresponding homogeneous coordinates is given by

$$\mathbf{M} = \underbrace{\begin{pmatrix} 0.2887 & 0 & 0 \\ 0 & 0.2887 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{uniform scaling}} \underbrace{\begin{pmatrix} 1 & 0 & -0.5133 \\ 0 & 1 & -100.4267 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{translation}} = \begin{pmatrix} 0.2887 & 0 & -0.1482 \\ 0 & 0.2887 & -28.9932 \\ 0 & 0 & 1 \end{pmatrix}$$

9.5.
$$\mathbf{A}' = \begin{pmatrix} -1.5599 & 1.4358 & 1 \\ 0.1059 & -0.0654 & 1 \\ 1.4541 & -1.3704 & 1 \end{pmatrix} \quad \text{has singular values} \quad (2.9158, 1.7321, 0.0296)$$

This profile is less ambiguous than before: the first two singular values are of the same order of magnitude, and the third is several orders of magnitude less and can be assumed to be approximately zero. It is then more reasonable to assume that this last singular value corresponds to a unique solution, given by the corresponding right singular vector.

9.6. We start with the right singular vector of $\mathbf{A}'$ corresponding to the smallest singular value. This gives us the estimated line in the Hartley-transformed coordinate system. In order to transform point coordinates back to the original system, we multiply their homogeneous representation by $\mathbf{M}^{-1}$. The corresponding transformation for lines is given by $[\mathbf{M}^{-1}]^{-\top} = \mathbf{M}^{\top}$. Consequently, the new estimate of the line, $\mathbf{l}_2$ is given as the right singular vector of $\mathbf{A}'$ corresponding to the smallest singular value, multiplied by $\mathbf{M}^{\top}$:

$$\mathbf{l}_2 = \begin{pmatrix} 0.1966 \\ 0.2114 \\ -21.3312 \end{pmatrix} \sim \begin{pmatrix} 0.6810 \\ 0.7322 \\ -73.8870 \end{pmatrix}$$

The second numerical vector above, $\mathbf{l}_{10}$, is normalized to proper dual homogeneous coordinates. This line is differs only slightly from the previous estimate.

9.7. Using the same geometric error as above gives us $\|\mathbf{A}\mathbf{l}_{10}\| = 0.102691$. This is only a very small decrease in the geometric error compared to the first estimate. NOTE: here we are computing the error in the original coordinate system.

9.8. Based on the relation $\mathbf{y}'_k \sim \mathbf{H}\,\mathbf{y}_k$ for the ideal, noise free, case, several alternatives are possible:

(1): The simplest one is a unsymmetric total L_2 error:

$$\epsilon_1 = \sum_k \|\mathrm{norm}_{\mathrm{P}}(\mathbf{y}'_k) - \mathrm{norm}_{\mathrm{P}}(\mathbf{H}\,\mathbf{y}_k)\|^2$$

This error quantifies the discrepancy between the transformed points only in one image, where the primed coordinates are located.

(2): An alternative is to use a symmetric error and measure the discrepancy in both images:

$$\epsilon_2 = \sum_k \left[\|\mathrm{norm}_{\mathrm{P}}(\mathbf{y}'_k) - \mathrm{norm}_{\mathrm{P}}(\mathbf{H}\,\mathbf{y}_k)\|^2 + \|\mathrm{norm}_{\mathrm{P}}(\mathbf{y}_k) - \mathrm{norm}_{\mathrm{P}}(\mathbf{H}^{-1}\,\mathbf{y}'_k)\|^2\right]$$

IMPORTANT: With $\mathbf{y}_k = (u_k, v_k)$ and $\mathbf{y}'_k = (u'_k, v_k)$, each term in the "forward error" ϵ_1 can be express explicitly as

$$\left\|(u'_k, v'_k) - \left(\frac{h_{11}u_k + h_{12}v_k + h_{13}}{h_{31}u_k + h_{32}v_k + h_{33}}, \frac{h_{21}u_k + h_{22}v_k + h_{23}}{h_{31}u_k + h_{32}v_k + h_{33}}\right)\right\|^2$$

The corresponding terms for the "backward error" are given as

$$\left\|(u_k, v_k) - \left(\frac{h'_{11}u'_k + h'_{12}v'_k + h'_{13}}{h'_{31}u'_k + h'_{32}v'_k + h'_{33}}, \frac{h'_{21}u'_k + h'_{22}v'_k + h'_{23}}{h'_{31}u'_k + h'_{32}v'_k + h'_{33}}\right)\right\|^2$$

where h'_{ij} are the elements of $\mathbf{H}^{-1}$. These terms are, in general, different than those of the "forward error", and they do not add to the same value. Consequently, the "forward" and "backward" errors are not the same in general, and should be added to get a proper symmetric error.

(3): Another alternative is to use an L_1 error, for example:

$$\epsilon_1 = \sum_k \|\mathrm{norm_P}(\mathbf{y}'_k) - \mathrm{norm_P}(\mathbf{H}\,\mathbf{y}_k)\|$$

Here, the distances are not squared. This type of error, however, has singularities in the gradient with respect to $\mathbf{H}$ for any $\mathbf{H}$ that makes one of the terms $= 0$, since the sign of the corresponding derivative changes sign in a discontinuous way. Therefore, this type of error is not suitable for iterative optimization, unless we can sure that we can avoid these singularities.

9.9. None of the alternatives for a geometric error can be minimized by solving some simple (linear) equation. Instead iterative methods must be used.

9.10. The lines as related as

$$\mathbf{l}'_k \sim \tilde{\mathbf{H}}\,\mathbf{l}_k \tag{7}$$

where $\tilde{\mathbf{H}}$ is the dual transformation of $\mathbf{H}$, i.e., $\tilde{\mathbf{H}} = \mathbf{H}^{-\top}$. This leaves, at least, two options for the estimation of $\mathbf{H}$.

(1): Rewrite Equation (7) using DLT into

$$\mathbf{0} = [\,\mathbf{l}'_k\,]_\times \tilde{\mathbf{H}}\,\mathbf{l}_k$$

This generates two linear homogeneous equations in $\tilde{\mathbf{H}}$ for each pair of corresponding lines, and we can then estimate $\tilde{\mathbf{H}}$ in the same way as $\mathbf{H}$ is estimated from points. Once $\tilde{\mathbf{H}}$ is determined, we get $\mathbf{H} = \tilde{\mathbf{H}}^{-\top}$.

(2): rewrite Equation (7) by applying the transformation $\mathbf{H}^\top$ to both sides:

$$\mathbf{H}^\top \mathbf{l}'_k \sim \mathbf{H}^\top \tilde{\mathbf{H}}\,\mathbf{l}_k = \mathbf{H}^\top \mathbf{H}^{-\top}\,\mathbf{l}_k = \mathbf{l}_k$$

Now, use DLT to get

$$[\,\mathbf{l}_k\,]_\times \mathbf{H}^\top \mathbf{l}'_k = \mathbf{0}$$

This generates two linear homogeneous equations in $\mathbf{H}$ (or in $\mathbf{H}^\top$) for each pair of corresponding lines, and we can then estimate $\mathbf{H}$ directly.

In general, the two strategies gives slightly different estimates since the corresponding error functions are not identical. The minimal number of corresponding lines is 4, which follows for the same reason as for estimation of $\mathbf{H}$ from point pairs.

9.11. XXX

9.12. XXX

9.13. An idea (RANSAC): Select a minimal set of hypothetical correspondences (4 pairs!), estimate a homography from these hypothetical correspondences, check how many of the other points that can be brought into correspondence by means of the estimated homography. If the number of correspondences that emerges is sufficiently high: use them, otherwise select a new minimal set and start all over again. The selection of hypothetical points can either be made systematically or randomly. The second option is useful when the number of correct correspondences in relation to the number of possible selections is small.

9.14. XXX

10 Representations of 3D rotations

10.1. Set

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

As parameters we can choose, for example, r_{11}, r_{12}, and r_{21}. Notice that, except for the sign, r_{31} is given directly from r_{11} and r_{21}:

$$r_{31}^2 = 1 - r_{11}^2 - r_{21}^2, \tag{8}$$

which is derived from one of the constraints on $\mathbf{R}$: $r_{11}^2 + r_{21}^2 + r_{31}^2 = 1$. This means that we cannot use r_{31} as a free parameter together with r_{11} and r_{21}. Similarly, r_{32} is given from r_{12} and r_{22}:

$$r_{32}^2 = 1 - r_{12}^2 - r_{22}^2.$$

To determine also r_{22} we can use the constraint

$$r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0,$$

which together with the two constraints above leads to

$$(1 - r_{11}^2 - r_{21}^2)(1 - r_{12}^2 - r_{22}^2) - (r_{11}r_{12} + r_{21}r_{22})^2 = 0.$$

The left hand side is a second order polynomial in the unknown r_{22}, that can be solved by standard methods.

Once r_{11}, r_{12}, and r_{21} are determined, then r_{31}, r_{22} and r_{32} can be determined, although with multiple choices. In order to satisfy Equation (8) we must require that $r_{11}^2 + r_{21}^2 \leq 1$. There is also a constraint

$$r_{13}^2 = 1 - r_{11}^2 - r_{12}^2$$

which can be satisfied only if $r_{11}^2 + r_{12}^2 \leq 1$.

For each values of (r_{12}, r_{12}, r_{21}) within these ranges, there are two possible values for r_{31} and for r_{22}. For r_{31} these two values only differ by a sign, while for r_{22} the absolute values of the two possibilities are distinct, in general. This means that r_{32} can have 4 different values for each set of (r_{12}, r_{12}, r_{21}).

Once the values of the first two columns in $\mathbf{R}$ have been determined from (r_{12}, r_{12}, r_{21}), with 4 different outcomes, the third columns if given uniquely as the cross product of the first and second column.

In summary, this representation of $\mathbf{R} \in SO(3)$ is ambiguous (one-to-many) since the remaining elements cannot be uniquely determined from only three of them. There are no singularities, however. But because of the ambiguities, the representation is not very useful.

10.2. Given a rotation axis $\hat{\mathbf{n}}$ and rotation angle α, Rodrigues formula gives the rotation matrix $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{I} + (1 - \cos\alpha)[\,\hat{\mathbf{n}}\,]_\times^2 + \sin\alpha[\,\hat{\mathbf{n}}\,]_\times$$

For small angles: $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$, leading to

$$\mathbf{R} \approx \mathbf{I} + \alpha[\,\hat{\mathbf{n}}\,]_\times = \mathbf{I} + [\,\alpha\,\hat{\mathbf{n}}\,]_\times$$

Assuming that this approximation is valid, we get

$$[\,\alpha\,\hat{\mathbf{n}}\,]_\times = \mathbf{R} - \mathbf{I}, \quad \Rightarrow \quad \alpha\,\hat{\mathbf{n}} = [\,\mathbf{R} - \mathbf{I}\,]^\times$$

This allows us to uniquely determine both $\hat{\mathbf{n}}$ and α, at least if the right hand side is $\neq 0$.

10.3. XXX

10.4. By definition $\mathbf{R}_2$ and $\mathbf{R}_1$ form a twisted pair if $\mathbf{Q}_1 = \mathbf{R}_2^\top \mathbf{R}_1$ is a rotation by 180° about some axis $\hat{\mathbf{n}}$, so the first combination $\mathbf{Q}_1$ is already checked. The inverse of this rotation must also be a rotation by 180°, about the same axis $\hat{\mathbf{n}}$, and it is given as $(\mathbf{R}_2^\top \mathbf{R}_1)^{-1} = (\mathbf{R}_2^\top \mathbf{R}_1)^\top = \mathbf{R}_1^\top \mathbf{R}_2 = \mathbf{Q}_2$. This $\mathbf{Q}_2$ is the second combination in the list. Notice that $\mathbf{Q}_1 = \mathbf{Q}_2$ since rotating by 180° twice gives an identity operation.

The third combination is $\mathbf{R}_2 \mathbf{R}_1^\top$, which can be rewritten as

$$\mathbf{Q}_3 = \mathbf{R}_2 \mathbf{R}_1^\top = \mathbf{R}_2 \mathbf{R}_1^\top \underbrace{\mathbf{R}_2 \mathbf{R}_2^\top}_{=\mathbf{I}} = \mathbf{R}_2 \underbrace{\mathbf{R}_1^\top \mathbf{R}_2}_{=\mathbf{Q}_2} \mathbf{R}_2^\top = \mathbf{R}_2 \mathbf{Q}_2 \mathbf{R}_2^\top$$

We can then interpret $\mathbf{Q}_3$ as: it does the same thing as $\mathbf{Q}_2$ does, i.e., rotates by 180°, but in a coordinate system that is rotated by $\mathbf{R}_2^\top$. Consequently, instead of rotating about $\hat{\mathbf{n}}$, $\mathbf{Q}_3$ rotates about $\hat{\mathbf{n}}'$ that is related to $\hat{\mathbf{n}}$ by $\hat{\mathbf{n}} = \mathbf{R}_2^\top \hat{\mathbf{n}}'$, or $\hat{\mathbf{n}}' = \mathbf{R}_2 \hat{\mathbf{n}}$.

For the same reason that $\mathbf{Q}_1 = \mathbf{Q}_2$, it follows that $\mathbf{Q}_3 = \mathbf{Q}_4$ where $\mathbf{Q}_4 = \mathbf{R}_1 \mathbf{R}_2^\top$. Consequently, $\mathbf{Q}_4$ too is rotation by 180° about $\hat{\mathbf{n}}'$. But, in a similar way as for $\mathbf{Q}_3$ it can also be rewritten as $\mathbf{Q}_4 = \mathbf{R}_1 \mathbf{Q}_1 \mathbf{R}_1^\top$. We can interpret this as: $\mathbf{Q}_4$ does the same thing as $\mathbf{Q}_1$ does, i.e., rotates by 180°, but in a coordinate system that is rotated by $\mathbf{R}_1^\top$. The rotation axis for $\mathbf{Q}_1$ is $\hat{\mathbf{n}}$ and for $\mathbf{Q}_4$ it is $\hat{\mathbf{n}}'$. They must be related as $\hat{\mathbf{n}} = \mathbf{R}_1^\top \hat{\mathbf{n}}'$, or $\hat{\mathbf{n}}' = \mathbf{R}_1 \hat{\mathbf{n}}$.

10.5. See the previous exercise.

10.6. Let q_1 and q_2 be two quaternions, each with its four elements given as:

$$\mathbf{q}_1 = \begin{pmatrix} s_1 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} s_2 \\ a_2 \\ b_2 \\ c_2 \end{pmatrix}.$$

We can also represent the two quaternions in terms of a combination of a scalar (the real part) and a vector (the imaginary part):

$$q_1 : \left[s_1, \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \right], \quad q_2 : \left[s_2, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right], \quad \Rightarrow \quad q_2^* : \left[s_2, \begin{pmatrix} -a_2 \\ -b_2 \\ -c_2 \end{pmatrix} \right].$$

The scalar component of the quaternion product $q_2^* \circ q_1$ is given as (see IREG Equation (7.10)):

$$s_1 s_2 - \begin{pmatrix} -a_2 \\ -b_2 \\ -c_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = s_1 s_2 + a_1 a_2 + b_1 b_2 + c_1 c_2 = \mathbf{q}_1 \cdot \mathbf{q}_2$$

This means that the scalar component vanishes, corresponding to a rotation by 180°, exactly when the vectors representing the quaternions are orthogonal: $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$.

11 Estimation involving transformations

11.1. XXX