12 Dual bases

12.1. The metric $\mathbf{G}$ is defined as

$$\mathbf{G} = \begin{pmatrix} \langle\, \mathbf{b}_1 \mid \mathbf{b}_1 \,\rangle & \langle\, \mathbf{b}_2 \mid \mathbf{b}_1 \,\rangle \\ \langle\, \mathbf{b}_1 \mid \mathbf{b}_2 \,\rangle & \langle\, \mathbf{b}_2 \mid \mathbf{b}_2 \,\rangle \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Its inverse it then

$$\mathbf{G}^{-1} = \frac{1}{\det \mathbf{G}} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

12.2. The dual basis vectors are given as

$$\tilde{\mathbf{b}}_1 = \mathbf{b}_1\, G^{-1}_{11} + \mathbf{b}_2\, G^{-1}_{21} = 2\,\mathbf{b}_1 - \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\tilde{\mathbf{b}}_1 = \mathbf{b}_1\, G^{-1}_{12} + \mathbf{b}_2\, G^{-1}_{22} = -\mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

12.3. The dual basis vectors should satisfy

$$\begin{pmatrix} \langle\, \tilde{\mathbf{b}}_1 \mid \mathbf{b}_1 \,\rangle & \langle\, \tilde{\mathbf{b}}_2 \mid \mathbf{b}_1 \,\rangle \\ \langle\, \tilde{\mathbf{b}}_1 \mid \mathbf{b}_2 \,\rangle & \langle\, \tilde{\mathbf{b}}_2 \mid \mathbf{b}_2 \,\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inserting the numerical values of the original and the dual basis vectors shows that this is correct.

12.4. The coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ are given as

$$c_1 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_1 \,\rangle = -3$$
$$c_2 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_2 \,\rangle = 2$$

12.5. The dual coordinates of $\mathbf{v}$ are given as

$$\tilde{c}_1 = \langle\, \mathbf{v} \mid \mathbf{b}_1 \,\rangle = -1$$
$$\tilde{c}_2 = \langle\, \mathbf{v} \mid \mathbf{b}_2 \,\rangle = 1$$

12.6. The last two results can be verified by using the coordinates in a linear combination with the basis vectors, or the dual coordinates in a linear combination with the dual basis vectors. In both cases the result should be $\mathbf{v}$:

$$c_1\, \mathbf{b}_1 + c_2\, \mathbf{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{v}$$
$$\tilde{c}_1\, \tilde{\mathbf{b}}_1 + \tilde{c}_2\, \tilde{\mathbf{b}}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{v}$$

12.7. The metric is

$$\mathbf{G} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and its inverse is

$$\mathbf{G}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

12.8. The dual basis matrix $\tilde{\mathbf{B}}$ is

$$\tilde{\mathbf{B}} = \mathbf{B}\, \mathbf{G}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

This means that the two dual basis vectors are

$$\tilde{\mathbf{b}}_1 = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \tilde{\mathbf{b}}_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We can verify this result by looking at the pairwise scalar products between basis vectors and dual basis vectors:

$$\mathbf{B}^\top \mathbf{G}_0 \tilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

This means that the basis and the dual basis are in a dual relation to each other.

12.9. The coordinates of $\mathbf{v}$ are computed as

$$c_1 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_1 \,\rangle = \tilde{\mathbf{b}}_1^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -3$$

$$c_2 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_2 \,\rangle = \tilde{\mathbf{b}}_2^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{3} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2$$

Notice that these coordinates are the same as we computed in Exercise 12.4. This is reasonable since we are still determining the coordinates of the same vector relative to the same basis.

12.10. The dual coordinates of $\mathbf{v}$ are given as

$$\tilde{c}_1 = \langle\, \mathbf{v} \mid \mathbf{b}_1 \,\rangle = \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{3} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -4$$

$$\tilde{c}_2 = \langle\, \mathbf{v} \mid \mathbf{b}_2 \,\rangle = \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{3} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1$$

The dual coordinates have changed relative to Exercise 12.6 since the dual basis has changed due to the new scalar product. We verify this result by computing the linear combination of the dual coordinates and the dual basis vectors:

$$\tilde{c}_1 \tilde{\mathbf{b}}_1 + \tilde{c}_2 \tilde{\mathbf{b}}_2 = -4 \cdot \frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{v}$$

12.11. All properties of the scalar product are trivially satisfied except that $\langle\, \mathbf{v} \mid \mathbf{v} \,\rangle > 0$ for all $\mathbf{v} \neq \mathbf{0}$. To check this property we need to check that $\mathbf{G}_0$ is positive definite, i.e., that its eigenvalues are positive. They are the roots of the characteristic polynomial of $\mathbf{G}_0$:

$$\det\left(\mathbf{G}_0 - \lambda\, \mathbf{I}\right) = \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0, \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 3$$

12.12. The basis matrix is

$$\mathbf{B} = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix},$$

which gives the Gram matrix as

$$\mathbf{G} = \mathbf{B}^\star \mathbf{G}_0 \mathbf{B} = \begin{pmatrix} -i & 0 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 3 & -2-2i \\ -2+2i & 3 \end{pmatrix}.$$

12.13. To determine the dual basis, we need the inverse of $\mathbf{G}$:

$$\mathbf{G}^{-1} = \frac{1}{\det \mathbf{G}} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 3 & 2+2i \\ 2-2i & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2+2i \\ 2-2i & 3 \end{pmatrix}.$$

The dual basis matrix is then given as

$$\tilde{\mathbf{B}} = \mathbf{B}\,\mathbf{G}^{-1} = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix} \begin{pmatrix} 3 & 2+2i \\ 2-2i & 3 \end{pmatrix} = \begin{pmatrix} 2+i & 1+2i \\ 2+2i & 3i \end{pmatrix}$$

The corresponding dual basis vectors are found in the columns of $\tilde{\mathbf{B}}$:

$$\tilde{\mathbf{b}}_1 = \begin{pmatrix} 2+i \\ 2+2i \end{pmatrix}, \quad \tilde{\mathbf{b}}_2 = \begin{pmatrix} 1+2i \\ 3i \end{pmatrix}$$

We can verify this result by checking the scalar products between basis vectors and dual basis vectors:

$$\langle\, \mathbf{b}_1 \mid \tilde{\mathbf{b}}_1 \,\rangle = \tilde{\mathbf{b}}_1^\star \mathbf{G}_0 \mathbf{b}_1 = \begin{pmatrix} 2-i & 2-2i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} = 1,$$

$$\langle\, \mathbf{b}_1 \mid \tilde{\mathbf{b}}_2 \,\rangle = \tilde{\mathbf{b}}_2^\star \mathbf{G}_0 \mathbf{b}_1 = \begin{pmatrix} 1-2i & -3i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} = 0,$$

$$\langle\, \mathbf{b}_2 \mid \tilde{\mathbf{b}}_1 \,\rangle = \tilde{\mathbf{b}}_1^\star \mathbf{G}_0 \mathbf{b}_2 = \begin{pmatrix} 2-i & 2-2i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0,$$

$$\langle\, \mathbf{b}_2 \mid \tilde{\mathbf{b}}_2 \,\rangle = \tilde{\mathbf{b}}_2^\star \mathbf{G}_0 \mathbf{b}_2 = \begin{pmatrix} 1-2i & -3i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 1.$$

This means that $\langle\, \mathbf{b}_i \mid \tilde{\mathbf{b}}_j \,\rangle = \delta_{ij}$, which is the defining relation between a basis and its dual basis.

12.14. The coordinates of $\mathbf{v}$ relative to the basis $\mathbf{b}_1, \mathbf{b}_2$ are computed as

$$c_1 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_1 \,\rangle = \tilde{\mathbf{b}}_1^\star \mathbf{G}_0 \mathbf{v} = \begin{pmatrix} 2-i & 2-2i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2+i,$$

$$c_2 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_2 \,\rangle = \tilde{\mathbf{b}}_2^\star \mathbf{G}_0 \mathbf{v} = \begin{pmatrix} 1-2i & -3i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -2i,$$

We can verify this result:

$$c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 = (2+i) \begin{pmatrix} i \\ 0 \end{pmatrix} - 2i \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{v}.$$

12.15. The dual coordinates are given as scalar products between the vector $\mathbf{v}$ and the basis vectors:

$$\tilde{c}_1 = \langle\, \mathbf{v} \mid \mathbf{b}_1 \,\rangle = \mathbf{b}_1^\star \mathbf{G}_0 \mathbf{v} = \begin{pmatrix} -i & 0 \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2 + 7i,$$

$$\tilde{c}_2 = \langle\, \mathbf{v} \mid \mathbf{b}_2 \,\rangle = \mathbf{b}_2^\star \mathbf{G}_0 \mathbf{v} = \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 3 & -2+i \\ -2-i & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -6 - 4i,$$

We can verify this result:

$$\tilde{c}_1 \tilde{\mathbf{b}}_1 + \tilde{c}_2 \tilde{\mathbf{b}}_2 = (2+7i) \begin{pmatrix} 2+i \\ 2+2i \end{pmatrix} + (-6-4i) \begin{pmatrix} 1+2i \\ 3i \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{v}.$$

12.16. $\mathbf{G}_0$ is Hermitian, i.e., $\mathbf{G}_0^\star = \mathbf{G}_0$ (where $\mathbf{G}_0^\star$ denotes transpose and complex conjugation). As a consequence, all eigenvalues of $\mathbf{G}_0$ are real, but they should also be positive to assure that $\mathbf{G}_0$ represents a scalar product. We check this by determining the eigenvalues of $\mathbf{G}_0$, for example as roots of the characteristic polynomial of $\mathbf{G}_0$:

$$\lambda_1 \approx 4.79, \qquad \lambda_2 \approx 0.21.$$

Since both eigenvalues are positive and $\mathbf{G}_0$ is Hermitian, we conclude that it represents a scalar product.

12.17. XXX

13 Subspaces

13.1. The metric is computed as

$$\mathbf{G} = \begin{pmatrix} \langle\, \mathbf{b}_1 \mid \mathbf{b}_1 \,\rangle & \langle\, \mathbf{b}_2 \mid \mathbf{b}_1 \,\rangle \\ \langle\, \mathbf{b}_1 \mid \mathbf{b}_2 \,\rangle & \langle\, \mathbf{b}_2 \mid \mathbf{b}_1 \,\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_2 \\ \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The inverse metric is

$$\mathbf{G}^{-1} = \frac{1}{\det \mathbf{G}} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

13.2. The basis matrix $\mathbf{B}$ holds the basis vectors in its columns:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The dual basis matrix is computed as

$$\tilde{\mathbf{B}} = \mathbf{B}\,\mathbf{G}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

The columns of $\tilde{\mathbf{B}}$ contain the dual basis vectors:

$$\tilde{\mathbf{b}}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad \tilde{\mathbf{b}}_2 = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

13.3. The basis vectors and the dual basis vectors should be related as $\langle\, \mathbf{b}_i \mid \tilde{\mathbf{b}}_j \,\rangle = \delta_{ij}$. This can also be expressed in terms of the basis matrices as $\tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{B} = \mathbf{I}$. We check this last expression:

$$\tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{B} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}, \qquad \text{OK!}$$

13.4. The coordinates of $\mathbf{v}_1$ are given as the scalar products between the dual basis vectors and the vector $\mathbf{v}$:

$$\mathbf{c} = \tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

13.5. To obtain the vector $\mathbf{v}_1$ we form a linear combination of the basis vectors in $\mathbf{B}$ and the coordinates in $\mathbf{c}$:

$$\mathbf{v}_1 = \mathbf{B}\,\mathbf{c} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}.$$

13.6. The component of $\mathbf{v}$ that lies in the orthogonal complement of the subspace is determined as

$$\mathbf{v}_0 = \mathbf{v} - \mathbf{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\4\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

We can verify that $\mathbf{v}_0$ lies in the orthogonal complement of the subspace by checking that $\mathbf{v}_0$ is orthogonal to the subspace basis. To do this we compute the scalar products between the subspace basis and $\mathbf{v}_0$:

$$\mathbf{B}^\top \mathbf{G}_0 \mathbf{v}_0 = \frac{1}{3}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

13.7. The metric is computed in the same way in exercise 13.1, but now with a different scalar product:

$$\mathbf{G} = \begin{pmatrix}\langle\, \mathbf{b}_1 \,|\, \mathbf{b}_1 \,\rangle & \langle\, \mathbf{b}_2 \,|\, \mathbf{b}_1 \,\rangle\\ \langle\, \mathbf{b}_1 \,|\, \mathbf{b}_2 \,\rangle & \langle\, \mathbf{b}_2 \,|\, \mathbf{b}_1 \,\rangle\end{pmatrix} = \begin{pmatrix}\mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_2\\ \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_2\end{pmatrix} = \begin{pmatrix}3 & 2\\2 & 3\end{pmatrix}.$$

The inverse metric is

$$\mathbf{G}^{-1} = \frac{1}{\det \mathbf{G}}\begin{pmatrix}G_{22} & -G_{12}\\ -G_{21} & G_{11}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3 & -2\\-2 & 3\end{pmatrix}.$$

13.8. The dual basis vectors are computed in the same way in as exercise 13.2, but now with a different metric:

$$\tilde{\mathbf{B}} = \mathbf{B}\,\mathbf{G}^{-1} = \begin{pmatrix}1 & 0\\1 & 1\\0 & 1\end{pmatrix}\frac{1}{5}\begin{pmatrix}3 & -2\\-2 & 3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3 & -2\\1 & 1\\-2 & 3\end{pmatrix}.$$

The columns of $\tilde{\mathbf{B}}$ contain the dual basis vectors:

$$\tilde{\mathbf{b}}_1 = \frac{1}{5}\begin{pmatrix}3\\1\\-2\end{pmatrix}, \qquad \tilde{\mathbf{b}}_2 = \frac{1}{5}\begin{pmatrix}-2\\1\\3\end{pmatrix}.$$

13.9. The vector $\mathbf{v}_1$ is computed in the same way as in exercises 13.4 and 13.5, but now with a different dual basis. First, we determine the coordinates of $\mathbf{v}_1$:

$$\mathbf{c} = \tilde{\mathbf{B}}^\top \mathbf{G}_0 \mathbf{v} = \frac{1}{5}\begin{pmatrix}3 & 1 & -2\\-2 & 1 & 3\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3\\3\end{pmatrix}.$$

Then, these coordinates are used in a linear combination with the basis vectors to form $\mathbf{v}_1$:

$$\mathbf{v}_1 = \mathbf{B}\,\mathbf{c} = \begin{pmatrix}1 & 0\\1 & 1\\0 & 1\end{pmatrix}\frac{1}{5}\begin{pmatrix}3\\3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3\\6\\3\end{pmatrix}.$$

13.10. The vector $\mathbf{v}_1$ is still the orthogonal projection of $\mathbf{v}$ into the subspace spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$. But since the scalar product has changed, the notion of orthogonality has changed, and therefore how to orthogonally project a vector into the subspace has changed. As a consequence, $\mathbf{v}_1$ changes when $\mathbf{G}_0$ changes.

13.11. XXX

13.12. We get

$$\mathbf{P}^2 = \mathbf{P}\,\mathbf{P} = \mathbf{B}\,\underbrace{(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0\,\mathbf{B}}_{=\mathbf{I}}(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0 = \mathbf{B}\,(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0 = \mathbf{P}$$

This shows that $\mathbf{P}$ is a projection operator. We can decompose $\mathbf{P}$ as follows:

$$\mathbf{P} = \underbrace{\mathbf{B}}_{\text{forms a linear combination with the subspace basis}}\ \underbrace{(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}}_{\text{transforms dual coord. to standard coord.}}\ \underbrace{\mathbf{B}^\star\mathbf{G}_0}_{\text{forms scalar products with subspace basis = dual coord.}}$$

Consequently, $\mathbf{P}$ performs an orthogonal projection onto the subspace spanned by the basis $\mathbf{B}$.

13.13. We get

$$\begin{aligned}
\|\mathbf{v}_1\|^2 &= \langle\,\mathbf{v}_1 \mid \mathbf{v}_1\,\rangle = \mathbf{v}_1^*\mathbf{G}_0\mathbf{v}_1 = \left(\mathbf{B}\,(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0\mathbf{v}\right)^*\mathbf{G}_0\mathbf{B}\,(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0\mathbf{v} = \\
&= \mathbf{v}^*\mathbf{G}_0\mathbf{B}\,\underbrace{(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^*\mathbf{G}_0\mathbf{B}}_{=\mathbf{I}}(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0\mathbf{v} = \\
&= \mathbf{v}^*\mathbf{G}_0\,\underbrace{\mathbf{B}\,(\mathbf{B}^\star\mathbf{G}_0\mathbf{B})^{-1}\mathbf{B}^\star\mathbf{G}_0\mathbf{v}}_{=\mathbf{v}_1} = \mathbf{v}^*\mathbf{G}_0\mathbf{v}_1 = \langle\,\mathbf{v}_1 \mid \mathbf{v}\,\rangle
\end{aligned}$$

The identity $\langle\,\mathbf{v}_1 \mid \mathbf{v}_1\,\rangle = \langle\,\mathbf{v} \mid \mathbf{v}_1\,\rangle$ follows from the symmetry of the scalar product.

14 Normalized convolution, 1D signals

14.1. The basis vectors represented as columns vectors which are also functions of a discrete variable k, with the origin of the variable at the center element of the vector. The two basis vectors are then

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{They are collected into a basis matrix:} \quad \mathbf{B} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

14.2. Formally, the filter functions are given from the basis functions as

$$g_l[k] = a[-k]\,\overline{b_l[-k]} = a[k]\,b_l[-k] \tag{1}$$

The complex conjugation can be skipped since we assume a real signal space. The applicability a is symmetric: $a[-k] = a[k]$. In this case we get

$$g_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} \text{ point-wise multiplied with } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} \text{ point-wise multiplied with } \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \\ -2 \end{pmatrix}$$

14.3. In this case, with signal certainty $= 1$, the scalar product $\mathbf{G}_0$ is only defined by the applicability function. $\mathbf{G}_0$ is a diagonal matrix with the applicability function in its diagonal elements:

$$\mathbf{G}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

14.4. The metric $\mathbf{G}$ is the scalar product between all pairs of basis vectors:

$$\mathbf{G} = \begin{pmatrix} \langle\, \mathbf{b}_1 \,|\, \mathbf{b}_1 \,\rangle & \langle\, \mathbf{b}_2 \,|\, \mathbf{b}_1 \,\rangle \\ \langle\, \mathbf{b}_1 \,|\, \mathbf{b}_2 \,\rangle & \langle\, \mathbf{b}_2 \,|\, \mathbf{b}_2 \,\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_1 & \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_1 \\ \mathbf{b}_1^\top \mathbf{G}_0 \mathbf{b}_2 & \mathbf{b}_2^\top \mathbf{G}_0 \mathbf{b}_2 \end{pmatrix} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{B} = \begin{pmatrix} 9 & 0 \\ 0 & 12 \end{pmatrix}$$

Its inverse is then

$$\mathbf{G}^{-1} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}$$

14.5. The filter results computed at the center of the signal is given as

$$\tilde{c}_1 = \begin{pmatrix} 11 \\ 13 \\ 14 \\ 20 \\ 17 \end{pmatrix} \text{ dot multiplied with } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 136$$

$$\tilde{c}_2 = \begin{pmatrix} 11 \\ 13 \\ 14 \\ 20 \\ 17 \end{pmatrix} \text{ dot multiplied with } \begin{pmatrix} -2 \\ -2 \\ 0 \\ 2 \\ 2 \end{pmatrix} = 26$$

These filter responses are dual coordinates of the local signal relative to the basis. They can be transformed into proper coordinates by means of $\mathbf{G}^{-1}$:

$$c_1 = G_{11}^{-1}\,\tilde{c}_1 + G_{12}^{-1}\,\tilde{c}_2 \approx 15.11$$
$$c_2 = G_{21}^{-1}\,\tilde{c}_1 + G_{22}^{-1}\,\tilde{c}_2 \approx 2.17$$

If we approximate the Taylor expansion of the local signal with a first order polynomial, these two coordinates correspond to the mean and the first order derivative of the local signal. These numerical values appear reasonable given the values of the signal.

14.6. XXX

14.7. In this case the scalar product matric $\mathbf{G}_0$ contains in the diagonal the product of the applicability function and the signal certainty, i.e., it is a function of the position along the signal. At the center point it becomes

$$\mathbf{G}_0 = \begin{pmatrix} 1\cdot 1 & 0 & 0 & 0 & 0 \\ 0 & 2\cdot 0 & 0 & 0 & 0 \\ 0 & 0 & 3\cdot 1 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot 1 & 0 \\ 0 & 0 & 0 & 0 & 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

14.8. The metric is also a function of position along the signal, and at the center point it becomes:

$$\mathbf{G} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{B} = \begin{pmatrix} 7 & 2 \\ 2 & 10 \end{pmatrix}$$

Its inverse is then

$$\mathbf{G}^{-1} = \frac{1}{66}\begin{pmatrix} 10 & -2 \\ -2 & 7 \end{pmatrix}$$

14.9. Multiplying the certainty function c onto the signal and then convolving the result with the two filters gives the following filter responses at the center of the signal:

$$\tilde{c}_1 = \begin{pmatrix} 11 \\ 0 \\ 14 \\ 20 \\ 17 \end{pmatrix} \text{ dot multiplied with } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 110$$

$$\tilde{c}_2 = \begin{pmatrix} 11 \\ 0 \\ 14 \\ 20 \\ 17 \end{pmatrix} \text{ dot multiplied with } \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \\ -2 \end{pmatrix} = 52$$

They can be transformed into proper coordinates:

$$c_1 = G^{-1}_{11}\,\tilde{c}_1 + G^{-1}_{12}\,\tilde{c}_2 \approx 15.09$$
$$c_2 = G^{-1}_{21}\,\tilde{c}_1 + G^{-1}_{22}\,\tilde{c}_2 \approx 2.18$$

14.10. Not taking the signal certainty into account, just multiplying setting unknown signal elements $= 0$, gives the same filter responses as above $(\tilde{c}_1, \tilde{c}_2) = (110, 52)$. They will, however, be transformed to the proper coordinates using the $\mathbf{G}$ in exercise 14.4:

$$c_1 = G^{-1}_{11}\,\tilde{c}_1 + G^{-1}_{12}\,\tilde{c}_2 \approx 12.22$$
$$c_2 = G^{-1}_{21}\,\tilde{c}_1 + G^{-1}_{22}\,\tilde{c}_2 \approx 4.33$$

This deviates quite a bit from the estimates in exercises 14.5 and 14.9, and also does not reflect the behavior of the local signal with the unknown sample disregarded.

14.11. The dual coordinates are functions of signal position given as

$$\tilde{c}_1[k] = (f * g_1)[k]$$
$$\tilde{c}_2[k] = (f * g_2)[k]$$

They are then transformed into proper coordinates:

$$c_1[k] = G^{-1}_{11}\,\tilde{c}_1[k] + G^{-1}_{12}\,\tilde{c}_2[k] = \left(f * (G^{-1}_{11}\,g_1 + G^{-1}_{12}\,g_2)\right)[k]$$
$$c_2[k] = G^{-1}_{21}\,\tilde{c}_1[k] + G^{-1}_{22}\,\tilde{c}_2[k] = \left(f * (G^{-1}_{21}\,g_1 + G^{-1}_{22}\,g_2)\right)[k]$$

14.12. This means that the proper coordinates can be obtained directly by convolving the signal by the dual filters

$$\tilde{g}_1 = G^{-1}_{11}\,g_1 + G^{-1}_{12}\,g_2 \approx \begin{pmatrix} 0.11 \\ 0.22 \\ 0.33 \\ 0.22 \\ 0.11 \end{pmatrix} \qquad \tilde{g}_2 = G^{-1}_{21}\,g_1 + G^{-1}_{22}\,g_2 \approx \begin{pmatrix} 0.17 \\ 0.17 \\ 0 \\ -0.17 \\ -0.17 \end{pmatrix}$$

Notice, that these filters are dependent on the applicability function a, so changing a also changes the dual basis filters.

14.13. In this case the single basis function is $b[k] = 1$, and the corresponding filter function is

$$g[k] = a[k]\, b[-k] = a[k]$$

The signal is multiplied by the signal certainty function c before being convolved by the filter function, and the result is the dual coordinate relative to the basis

$$\tilde{c}[k] = ((f \cdot c) * g)[k] = ((f \cdot c) * a)[k]$$

The scalar product $\mathbf{G}_0[k]$ is a position dependent diagonal matrix given as

$$\mathbf{G}_0[k] = \mathrm{diag}(a \cdot c[k])$$

where the "·" in this case denotes point-wise multiplication by the fixed applicability and the position dependent certainty function, the latter as it appears within the local signal window. The metric $\mathbf{G}[k]$, also a position dependent entity, is given as

$$\mathbf{G}[k] = \mathbf{b}^\top \mathbf{G}_0[k]\, \mathbf{b} =$$

With the basis function $\mathbf{b}$ as a vector of only elements $= 1$, this leads to

$$\mathbf{G}[k] = \sum_l c[k+l]a[l] = /a \text{ is symmetric}/ = \sum_l c[k-l]a[l] = (c * a)[k]$$

Finally, to get the proper coordinate relative to the basis, we transform the dual coordinate with the inverse of $\mathbf{G}$, leading to

$$c = \frac{(f \cdot c) * a}{c * a} \quad \text{(point-wise division)}$$

15 Normalized convolution, 2D signals

15.1. The basis functions can initially be represented as three 2D neighborhoods of size 3×3:

$$\text{basis}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{basis}_2 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad \text{basis}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

These functions can be vectorized, e.g., by stacking each columns of the neighborhood on top of each other. In that case, the basis matrix becomes

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

15.2. The filters are obtained by multiplying (point-wise) each basis function with the applicability function, followed by a mirroring operation, $f_m[k_1, k_2] = \text{basis}_m[-k_1, -k_2]\, a[-k_1, -k_2]$. Since the filters are 2D, they are represented as 3×3 matrices:

$$\text{filter}_1 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{filter}_2 = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}, \quad \text{filter}_3 = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

15.3. The scalar product matrix $\mathbf{G}_0$ is in this case (certainty = 1) defined as a diagonal matrix that contains the applicability. The applicability needs to be vectorized before it is put in the diagonal of $\mathbf{G}_0$:

$$\mathbf{G}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

15.4. The metric $\mathbf{G}$ is defines as the scalar products between all different basis vectors:

$$\mathbf{G} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{B} = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \qquad \mathbf{G}^{-1} = \begin{pmatrix} \frac{1}{15} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}.$$

15.5. The mean and the horizontal and vertical derivatives are the coordinates of the local signal relative the basis $\{1, x, y\}$. The corresponding dual coordinates are given as the filter responses from the three filters, generated at the center of the image neighborhood. The dual coordinates are equivalently given as the scalar products between the local signal vector $\mathbf{v}$ and the basis vectors:

$$\mathbf{v} = \begin{pmatrix} 10 \\ 11 \\ 12 \\ 11 \\ 12 \\ 14 \\ 12 \\ 14 \\ 17 \end{pmatrix}, \qquad \tilde{\mathbf{c}} = \mathbf{B}^\top \mathbf{G}_0 \mathbf{v} = \begin{pmatrix} 187 \\ 13 \\ -13 \end{pmatrix}.$$

To obtain the proper coordinates relative to the basis, the dual coordinates need to be transformed by means of the inverse metric:

$$\mathbf{c} = \mathbf{G}^{-1}\, \tilde{\mathbf{c}} = \begin{pmatrix} \frac{1}{15} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 187 \\ 13 \\ -13 \end{pmatrix} \approx \begin{pmatrix} 12.5 \\ 1.6 \\ -1.6 \end{pmatrix}$$

The (weighted) mean is approximately 12.5, the horizontal derivative is 1.6 (increasing to the right) and the vertical derivative is -1.6 (decreasing when going up). These figures are consistent with the numerical values of the pixels in the image neighborhood.

15.6. XXX

15.7. XXX

16 Filter optimization

16.1. A general result that is used in this exercise is Parseval's theorem for discrete signals:

$$\sum_{k=-\infty}^{\infty} a[k]\, b^*[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(u)\, B^*(u)\, du,$$

where a and b are two sequences with Fourier transforms A and B. By setting $a = b$ we get

$$\sum_{k=-\infty}^{\infty} |a[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(u)|^2\, du,$$

Applied to the terms in ϵ, this gives

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |h[k] - h_{ideal}[k]|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(u) - H_{ideal}(u)|^2\, du = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(u)\, F(u) - G(u)\, F_{ideal}(u)|^2\, du = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(u)|^2\, |F(u) - F_{ideal}(u)|^2\, du\end{aligned}$$

Applying E , as the expectation value operator over all signals, then gives

$$\begin{aligned}\epsilon = \mathsf{E} \sum_{k=-\infty}^{\infty} |h[k] - h_{ideal}[k]|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\mathsf{E}\ |G(u)|^2\right) |F(u) - F_{ideal}(u)|^2\, du = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_g(u)\ |F(u) - F_{ideal}(u)|^2\, du\end{aligned}$$

Minimizing this expression over F is then a weighted least squares problem, the same as in standard filter optimization, where the weighting in the frequency domain is defined by S_g.

16.2. For the different dimensions, we get the following volumes of the hyper-cube (V_c) and volume of the enclosed hyper-sphere (V_s).

- **n = 1** :$V_c = P$, $V_s = P$, ratio $1/1 = 1$.
- **n = 2** :$V_s = \pi\, (P/2)^2$, ratio $\pi/4 \approx 0.78$.
- **n = 3** :$V_s = 4\pi/3\, (P/2)^3$, ratio $\pi/6 \approx 0.52$.
- **n = 4** :$V_s = \pi^2/2\, (P/2)^4$, ratio $\pi^2/32 \approx 0.31$.

16.3. We want to minimize

$$\epsilon = \| \sum_k f[k] e^{-iuk} - F_{\text{ideal}}(u) \|_W^2,$$

where W is the frequency weighting function, over the filter coefficients in f. In vector/matrix notation the problem can be formulated as follows. The frequency function of the resulting filter is a vector $\hat{\mathbf{f}}$:

$$\hat{\mathbf{f}} = \mathbf{B}\, \mathbf{f}$$

where $\mathbf{f}$ is the vector of filter coefficients and $\mathbf{B}$ is a basis matrix that hold the functions e^{-iuk} in its columns. The ideal frequency function is denoted $\hat{\mathbf{f}}_{\text{ideal}}$. The error function can be expressed as

$$\epsilon = \langle\, \hat{\mathbf{f}} - \hat{\mathbf{f}}_{\text{ideal}} \mid \hat{\mathbf{f}} - \hat{\mathbf{f}}_{\text{ideal}} \,\rangle = \langle\, \hat{\mathbf{f}} \mid \hat{\mathbf{f}} \,\rangle - \langle\, \hat{\mathbf{f}} \mid \hat{\mathbf{f}}_{\text{ideal}} \,\rangle - \langle\, \hat{\mathbf{f}}_{\text{ideal}} \mid \hat{\mathbf{f}} \,\rangle + \langle\, \hat{\mathbf{f}}_{\text{ideal}} \mid \hat{\mathbf{f}}_{\text{ideal}} \,\rangle$$

where the scalar product is defined in terms of the diagonal matrix $\mathbf{W}$ that holds the frequency weighting function W. Since $\tilde{\mathbf{f}}$ is the optimal choice of filter frequency function is satisfies $\langle\, \hat{\mathbf{f}} \,|\, \hat{\mathbf{f}} \,\rangle = \langle\, \hat{\mathbf{f}} \mid \hat{\mathbf{f}}_{\text{ideal}} \,\rangle = \langle\, \hat{\mathbf{f}}_{\text{ideal}} \mid \hat{\mathbf{f}} \,\rangle$, see exercise 13.13. Thus:

$$\epsilon = \langle\, \hat{\mathbf{f}}_{\text{ideal}} \mid \hat{\mathbf{f}}_{\text{ideal}} \,\rangle - \langle\, \hat{\mathbf{f}} \mid \hat{\mathbf{f}}_{\text{ideal}} \,\rangle = \|\hat{\mathbf{f}}_{\text{ideal}}\|^2 - \hat{\mathbf{f}}^*_{\text{ideal}} \mathbf{W}\, \hat{\mathbf{f}} = \|\hat{\mathbf{f}}_{\text{ideal}}\|^2 - \hat{\mathbf{f}}^*_{\text{ideal}} \mathbf{W}\, \mathbf{B}\, \mathbf{f}.$$

We define a vector of the same size as $\mathbf{f}$: $\mathbf{f}_{\text{ideal}} = \mathbf{B}^{\star} \mathbf{W}\, \hat{\mathbf{f}}_{\text{ideal}}$:

$$\epsilon = \|\mathbf{f}_{\text{ideal}}\|^2 - \mathbf{f}^*_{\text{ideal}}\, \mathbf{f}.$$

The spatial mask removes basis vectors in $\mathbf{B}$, by setting elements in $\mathbf{f}$ equal to zero. How much the error ϵ increases when a coefficient is removed in $\mathbf{f}$ depends both on the size of that element in $\mathbf{f}$ and on the corresponding element in $\mathbf{f}_{\text{ideal}}$. It may be the case that the smallest element in $\mathbf{f}$ is not the one that cases the smallest increase in ϵ if the corresponding element in $\mathbf{f}_{\text{ideal}}$ is relatively large.

16.4. The filter coefficients can be seen as the Fourier coefficients of the periodic function $F(u)$ = the frequency function of the filter (*sic*). This means that if the spatial mask reduces the magnitude of the coefficients far from the center, i.e., corresponding to high frequencies in $F(u)$, then $F(u)$ has smoother appearance since it contains less of high frequencies.

17 Principal Component Analysis

17.1. We want to minimize

$$\epsilon = \mathsf{E}\|\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top\mathbf{v}\|^2, \quad \text{where} \quad \mathbf{B}^\top\mathbf{B} = \mathbf{I}$$

where E is the expectation operator over all signals/data $\mathbf{v}$. Expanding the squared norm gives

$$\begin{aligned}\|\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top\mathbf{v}\|^2 &= (\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top\mathbf{v})^\top(\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top\mathbf{v}) = \\ &= \mathbf{v}^\top\mathbf{v} - \mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v} - \mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v} + \mathbf{v}^\top\mathbf{B}\,\underbrace{\mathbf{B}^\top\mathbf{B}}_{=\mathbf{I}}\,\mathbf{B}^\top\mathbf{v} = \\ &= \|\mathbf{v}\|^2 - \ \mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v}\end{aligned}$$

Since we minimize ϵ over all possible $\mathbf{B}$ that satisfy $\mathbf{B}^\top\mathbf{B} = \mathbf{I}$, this is the same as minimizing

$$\mathsf{E}\left(\|\mathbf{v}\|^2 - \ \mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v}\right)$$

over these $\mathbf{B}$, but since the first term in this expression is independent of $\mathbf{B}$, this becomes equivalent to minimizing only the second term, or, *maximizing*

$$\epsilon_1 = \mathsf{E}\left(\mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v}\right)$$

17.2. In the case that B has only a single column $\mathbf{b}$, we can write $\mathbf{B} = \mathbf{b}$, and we now want to maximize

$$\epsilon_1 = \mathsf{E}\left(\mathbf{v}^\top\mathbf{b}\,\mathbf{b}^\top\mathbf{v}\right) = \mathsf{E}\left(\mathbf{b}^\top\mathbf{v}\,\mathbf{v}^\top\mathbf{b}\right) = \mathbf{b}^\top\mathsf{E}\left(\mathbf{v}\,\mathbf{v}^\top\right)\ \mathbf{b} = \mathbf{b}^\top\mathbf{C}\,\mathbf{b}$$

with the additional constraint

$$c = \mathbf{b}^\top\mathbf{b} = 1$$

In accordance with Lagrange's method the solution $\mathbf{b}$ must satisfy

$$\nabla\epsilon_1 = \lambda\nabla c$$

for some scalar multiplier (the Lagrange multiplier) λ. Inserting the above expressions for ϵ_1 and c gives

$$2\,\mathbf{C}\,\mathbf{b} = 2\lambda\mathbf{b} \quad \Rightarrow \quad \mathbf{C}\,\mathbf{b} = \lambda\,\mathbf{b}$$

This shows that $\mathbf{b}$ must be an eigenvector of $\mathbf{C}$, and λ is the corresponding eigenvalue. In fact, since $\|\mathbf{b}\| = 1$, it follows that $\mathbf{b}$ must be a normalized eigenvector of $\mathbf{C}$ with corresponding eigenvalue λ.

We insert this fact into the expression for ϵ_1 to see what its maximum value is

$$\epsilon_{1,max} = \mathbf{b}^\top\underbrace{\mathbf{C}\,\mathbf{b}}_{=\lambda\,\mathbf{b}} = \lambda\,\mathbf{b}^\top\mathbf{b} = \lambda$$

Since we want to maximize ϵ_1, we must choose λ as the largest eigenvalue of $\mathbf{C}$: $\lambda = \lambda_1$. To summarize: $\mathbf{b}$ is a normalized eigenvector of $\mathbf{C}$ corresponding to the largest eigenvalue λ_1.

17.3. The original problem is to minimize

$$\begin{aligned}\epsilon &= \mathsf{E}\left(\|\mathbf{v}\|^2 - \mathbf{v}^\top\mathbf{B}\,\mathbf{B}^\top\mathbf{v}\right) = \mathsf{E}\left(\mathbf{v}^\top\mathbf{v} - \mathbf{v}^\top\mathbf{b}\,\mathbf{b}^\top\mathbf{v}\right) = \mathsf{E}\left(\text{trace}(\mathbf{v}\mathbf{v}^\top) - \mathbf{b}^\top\mathbf{v}\,\mathbf{v}^\top\mathbf{b}\right) = \\ &= \text{trace}\mathbf{C} - \mathbf{b}^\top\mathbf{C}\,\mathbf{b} = \sum_{k=1}^{N}\lambda_k - \lambda_1 = \sum_{k=2}^{N}\lambda_k\end{aligned}$$

17.4. The error function in PCA is formulated as

$$\epsilon = \mathsf{E}\|\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top \mathbf{v}\|,$$

that is minimized over all choices of orthogonal basis matrix $\mathbf{B}$. Let $\mathbf{B}$ be the basis that minimizes ϵ, and let $\mathbf{B}' = \mathbf{B}\,\mathbf{Q}$ be another basis of the same subspace that is obtained by rotating the first basis within the subspace, using $\mathbf{Q} \in O(N)$. Then

$$\mathsf{E}\|\mathbf{v} - \mathbf{B}'\,\mathbf{B}'^\top \mathbf{v}\| = \mathsf{E}\|\mathbf{v} - \mathbf{B}\,\underbrace{\mathbf{Q}\,\mathbf{Q}^\top}_{=\mathbf{I}}\mathbf{B}^\top \mathbf{v}\| = \mathsf{E}\|\mathbf{v} - \mathbf{B}\,\mathbf{B}^\top \mathbf{v}\| = \epsilon$$

This means that basis $\mathbf{B}$ and basis $\mathbf{B}'$ both give the same value for ϵ.

17.5. XXX

17.6. XXX

17.7. XXX

18 Frames

18.1. Insert the definition of $\mathbf{F}$ directly into the left hand side, and use the standard properties of the scalar product:

$$\langle\, \mathbf{F}\,\mathbf{u} \mid \mathbf{v}\,\rangle = \langle\, \sum_{k=1}^{M} \langle\, \mathbf{u} \mid \mathbf{b}_k\,\rangle\, \mathbf{b}_k \mid \mathbf{v}\,\rangle = \sum_{k=1}^{M} \langle\, \mathbf{u} \mid \mathbf{b}_k\,\rangle\, \langle\, \mathbf{b}_k \mid \mathbf{v}\,\rangle = \sum_{k=1}^{M} \langle\, \mathbf{u} \mid \mathbf{b}_k\,\rangle\, \overline{\langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle} =$$

$$\langle\, \mathbf{u} \mid \langle\, \mathbf{v} \mid \sum_{k=1}^{M} \mathbf{b}_k\,\rangle\, \mathbf{b}_k\,\rangle = \langle\, \mathbf{u} \mid \mathbf{F}\,\mathbf{v}\,\rangle$$

18.3. The middle term in the frame condition can be expanded as

$$\sum_k |\langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle|^2 = \sum_k \langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle \langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle^* = \sum_k \langle\, \mathbf{v} \mid \langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle \mathbf{b}_k\,\rangle = \langle\, \mathbf{v} \mid \langle\, \mathbf{v} \mid \sum_k \mathbf{b}_k\,\rangle \mathbf{b}_k\,\rangle = \langle\, \mathbf{v} \mid \mathbf{F}\,\mathbf{v}\,\rangle$$

This means that we can formulate the frame condition as: there must exists constants A and B, where $0 < A \leq B < \infty$, such that for all $\mathbf{v} \in V$ it is the case that

$$A\,\|\mathbf{v}\|^2 \leq \langle\, \mathbf{v} \mid \mathbf{F}\,\mathbf{v}\,\rangle \leq B\,\|\mathbf{v}\|^2$$

18.4. Assuming $\mathbf{v} \neq \mathbf{0}$, the frame condition can be formulated as

$$A \leq \langle\, \frac{\mathbf{v}}{\|\mathbf{v}\|} \mid \mathbf{F}\, \frac{\mathbf{v}}{\|\mathbf{v}\|}\,\rangle \leq B$$

which means that it is sufficient to examine the condition only for a normalized vector $\hat{\mathbf{v}}$:

$$A \leq \langle\, \hat{\mathbf{v}} \mid \mathbf{F}\,\hat{\mathbf{v}}\,\rangle \leq B$$

This inequality is the same as

$$A \leq \langle\, \mathbf{F}\,\hat{\mathbf{v}} \mid \hat{\mathbf{v}}\,\rangle \leq B$$

and with the assumption that $\mathbf{G}_0 = \mathbf{I}$, this is the same as the condition

$$A \leq \hat{\mathbf{v}}^\top \mathbf{F}\,\hat{\mathbf{v}} \leq B$$

Maximizing $\hat{\mathbf{v}}^\top\mathbf{F}\hat{\mathbf{v}}$ over a normalized $\hat{\mathbf{v}}$ leads to a maximum that is reached for $\mathbf{v}$ that is a normalized eigenvector of $\mathbf{F}$ corresponding to the largest eigenvalue of $\mathbf{F}$ (see previous exercises). Hence, $B =$ the largest eigenvalue of $\mathbf{F}$. Similarly, A becomes the smallest eigenvalue of $\mathbf{F}$.

18.5. XXX

18.6. In this case we can set $\mathbf{v} = r(\cos\alpha, \sin\alpha)$. The center term in the frame condition then becomes:

$$\sum_{k=1}^{\infty} |\langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle|^2 = |\langle\, \mathbf{v} \mid \mathbf{b}_1\,\rangle|^2 + \sum_{k=2}^{\infty} |\langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle|^2 = r^2\cos^2\alpha + \sum_{k=2}^{\infty} (r/k)^2 \sin^2\alpha =$$

$$r^2\left[\cos^2\alpha + \sin^2\alpha \sum_{k=2}^{\infty}\frac{1}{k^2}\right] = r^2\left[\cos^2\alpha + (\frac{\pi^2}{6} - 1)\sin^2\alpha\right]$$

Since $\|\mathbf{v}\|^2 = r^2$, this leads to

$$(\frac{\pi^2}{6} - 1)\cdot\|\mathbf{v}\|^2 \leq \sum_{k=1}^{\infty} |\langle\, \mathbf{v} \mid \mathbf{b}_k\,\rangle|^2 \leq 1\cdot\|\mathbf{v}\|^2$$

which demonstrates that the frame condition is satisfied for the set $\mathbf{b}_k$. The lower frame bound is $(\frac{\pi^2}{6} - 1) \approx 0.64$ and the upper frame bound is 1.

18.7. Doing the same computations as in the previous exercise, we see that

$$\sum_{k=1}^{\infty} |\langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle|^2 = |\langle\, \mathbf{v} \mid \mathbf{b}_1 \,\rangle|^2 + \sum_{k=2}^{\infty} |\langle\, \mathbf{v} \mid \mathbf{b}_k \,\rangle|^2 = r^2 \cos^2 \alpha + \sum_{k=2}^{\infty} (r/k) \sin^2 \alpha =$$

$$r^2 \left[\cos^2 \alpha + \sin^2 \alpha \sum_{k=2}^{\infty} \frac{1}{k} \right]$$

Since the infinite sum does not converge to a finite value, we draw the conclusion that in this case the upper frame bound $B = \infty$ which implies that the set $\mathbf{b}_k$ fails to be a frame.

18.8. Direct insertion of known quantities gives

$$\mathbf{c}_0^\star \mathbf{c} = \mathbf{c}_0^\star \mathbf{B}^\star \mathbf{G}_0 \mathbf{v} = (\mathbf{B}\, \mathbf{c}_0)^\star \mathbf{G}_0 \mathbf{v} = \mathbf{0}^\star \mathbf{G}_0 \mathbf{v} = 0$$

Compute the squared norm of the general reconstructing coefficient $\mathbf{c} + \mathbf{c}_0$:

$$(\mathbf{c} + \mathbf{c}_0)^\star (\mathbf{c} + \mathbf{c}_0) = \mathbf{c}^\star \mathbf{c} + \mathbf{c}_0^\star \mathbf{c} + \mathbf{c}^\star \mathbf{c}_0 + \mathbf{c}_0^\star \mathbf{c}_0 = \|\mathbf{c}\|^2 + 0 + 0 + \|\mathbf{c}_0\|^2 = \|\mathbf{c}\|^2 + \|\mathbf{c}_0\|^2$$

This expression is minimized over all $\mathbf{c}_0$ for $\mathbf{c}_0 = \mathbf{0}$.

18.9. The frame operator $\mathbf{F}$ applied to $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ is given by

$$\mathbf{F}\, \mathbf{u} = \sum_{k=1}^{3} \langle\, \mathbf{u} \mid \mathbf{b}_k \,\rangle\, \mathbf{b}_k = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

This means that

$$\mathbf{F} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The eigenvalues of $\mathbf{F}$ are given as $\lambda_1 = 3$ and $\lambda_2 = 1$. This means that the frame bounds are $A = 1$ and $B = 3$, and it is not a tight frame.

18.10. The dual frame vectors are given as $\tilde{\mathbf{b}}_k = \mathbf{F}^{-1} \mathbf{b}_k$. With

$$\mathbf{F}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

this means

$$\tilde{\mathbf{b}}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \tilde{\mathbf{b}}_2 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \tilde{\mathbf{b}}_3 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

18.11. The reconstructing coefficients are found as the scalar product between $\mathbf{v}$ and the dual frame vectors:

$$c_1 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_1 \,\rangle = 1 \quad c_2 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_2 \,\rangle = 0 \quad c_3 = \langle\, \mathbf{v} \mid \tilde{\mathbf{b}}_3 \,\rangle = 1$$

Used in a linear combination with the frame vectors, the result is

$$c_1\, \mathbf{b}_1 + c_2\, \mathbf{b}_2 + c_3\, \mathbf{b}_3 = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{v}$$

18.12. With $\mathbf{B}$ as the matrix holding the frame vectors in its columns:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

we can add any vector in the null space of $\mathbf{B}$ to any set of reconstructing coefficients and still get a set of reconstructing coefficients. That null space is spanned by the single vector

$$\mathbf{c}_0 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

19 Sampling, over-sampling and reconstruction

19.1. In the case the expectation operator E represents the mean over all observed sample noise $n[\cdot]$.
Property A':

$$\mathsf{E}\{n_{rec}(t)\} = \mathsf{E}\left\{\sum_{k=-\infty}^{\infty} n[k]\,\mathrm{sinc}(t-k)\right\} = \sum_{k=-\infty}^{\infty} \underbrace{\mathsf{E}\,\{n[k]\}}_{=0}\,\mathrm{sinc}(t-k) = 0$$

Property B':

$$\begin{aligned}
\mathsf{E}\{n_{rec}(t_1)\,n_{rec}(t_2)\} &= \mathsf{E}\left\{\sum_{k=-\infty}^{\infty} n[k]\,\mathrm{sinc}(t_1-k)\sum_{l=-\infty}^{\infty} n[l]\,\mathrm{sinc}(t_2-l)\right\} = \\
&= \mathsf{E}\left\{\sum_{k\,l} n[k]\,n[l]\,\mathrm{sinc}(t_1-k)\,\mathrm{sinc}(t_2-l)\right\} = \\
&= \sum_{k\,l}\underbrace{\mathsf{E}\,\{n[k]\,n[l]\}}_{=\sigma^2\delta_{kl}}\,\mathrm{sinc}(t_1-k)\,\mathrm{sinc}(t_2-l) = \\
&= \sum_{k\,l}\sigma^2\delta_{kl}\,\mathrm{sinc}(t_1-k)\,\mathrm{sinc}(t_2-l) = /\text{Summation over } l/ = \\
&= \sigma^2\sum_{k}\mathrm{sinc}(t_1-k)\,\mathrm{sinc}(t_2-k)
\end{aligned}$$

This last expression can be interpreted as Poisson's summation formula that reconstruct the band-limited function $\mathrm{sinc}(t_1-k)$ that is sampled at integers $k \in \mathbb{Z}$. Thus:

$$\mathsf{E}\{n_{rec}(t_1)\,n_{rec}(t_2)\} = \sigma^2\mathrm{sinc}(t_1-t_2)$$

19.2. The new signal is

$$n(t) = \frac{n_1(t)+n_2(t)}{2}$$

where n_1 and n_2 are the two noise signals reconstructed from the two independent noise signals. The mean of n is given as

$$\bar{n} = \mathsf{E}\{n(t)\} = \mathsf{E}\{n_1(t)+n_2(t)\}/2 = (\underbrace{\mathsf{E}\{n_1(t)\}}_{=0}+\underbrace{\mathsf{E}\{n_2(t)\}}_{=0})/2 = 0$$

From this we compute the variance of n as

$$\begin{aligned}
\mathsf{E}\{(n(t)-\bar{n})^2\} = \mathsf{E}\{n(t)^2\} = \mathsf{E}\left\{\left(\frac{n_1(t)+n_2(t)}{2}\right)^2\right\} = \frac{1}{4}\left(\underbrace{\mathsf{E}\{n_1^2(t)\}}_{=\sigma^2}+2\,\mathsf{E}\{n_1(t)n_2(t)\}+\underbrace{\mathsf{E}\{n_2^2(t)\}}_{=\sigma^2}\right) = \\
= \frac{\sigma^2}{2}+\mathsf{E}\{n_1(t)n_2(t)\} = /n_1 \text{ and } n_2 \text{ are independent}/ = \frac{\sigma^2}{2}+\underbrace{\mathsf{E}\{n_1(t)\}}_{=0}\,\underbrace{\mathsf{E}\{n_1(t)\}}_{=0} = \frac{\sigma^2}{2}
\end{aligned}$$

19.3. It is sufficient to show that $\operatorname{sinc}(t)$ is orthogonal to $\operatorname{sinc}(t-k)$ for integers $k \neq 0$:

$$\langle \operatorname{sinc}(t) \,|\, \operatorname{sinc}(t-k) \rangle = \int_{-\infty}^{\infty} \operatorname{sinc}(t) \operatorname{sinc}(t-k)\, dt = /\text{Parseval}/ =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{rect}(\frac{u}{2\pi})\, e^{-iuk} \operatorname{rect}(\frac{u}{2\pi})\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iuk}\, dt = \delta[k]$$

This shows that the functions $\operatorname{sinc}(t-k)$ form an ON-set of vectors. According to the sampling theorem they must span the space of 2π-band-limited functions since any such function can be reconstructed as a linear combination of these functions together with samples from the function. The ON-property assures that the function set is linearly independent. In short: it forms an ON-basis of the space of 2π-band-limited functions.

19.4. Since the functions $\operatorname{sinc}(t-k), k \in \mathbb{Z}$ form an ON-basis for the space V of 2π-band-limited functions, it follows that for any $f \in V$:

$$f(t) = \sum_{k=-\infty}^{\infty} \langle f(\cdot) \,|\, \operatorname{sinc}(\cdot - k) \rangle \operatorname{sinc}(t-k)$$

where $\langle f(\cdot) \,|\, \operatorname{sinc}(\cdot - k) \rangle$ is the coordinate of f relative to basis function $\operatorname{sinc}(t-k)$. The sampling theorem, on the other hand, states that

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \operatorname{sinc}(t-k)$$

Since coordinates are unique, it follows that

$$f(k) = \langle f(\cdot) \,|\, \operatorname{sinc}(\cdot - k) \rangle$$

19.5. The set $\operatorname{sinc}(t-k/2), k \in \mathbb{Z}$ cannot be a basis since it is linearly dependent. For example, we have

$$\operatorname{sinc}(t - \frac{1}{2}) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}(k - \frac{1}{2}) \operatorname{sinc}(t-k)$$

as proven by the sampling theorem. The frame operator is in this case defined as

$$\mathbf{F}\, g = \sum_{k=-\infty}^{\infty} \langle g(t) \,|\, \operatorname{sinc}(t-k) \rangle \operatorname{sinc}(t-k) + \sum_{k=-\infty}^{\infty} \langle g(t) \,|\, \operatorname{sinc}(t-(k+1/2)) \rangle \operatorname{sinc}(t-(k+1/2))$$

where g is an arbitrary 2π-band-limited function. Using the result from the previous exercise, the frame operator can be written

$$\mathbf{F}\, g = g + g = 2\, g$$

which means that $\mathbf{F} = 2\,\mathbf{I}$. This frame operator has frame bounds $A = B = 2$, and is a tight frame.

19.6. Since the half unit spaced sinc-functions form a tight frame with frame bound $A = 2$, it follows that

$$g(t) = \frac{1}{A} \sum_{k=-\infty}^{\infty} \langle g(t) \,|\, \operatorname{sinc}(t-k/2) \rangle \operatorname{sinc}(t-k/2) = \frac{1}{2} \sum_{k=-\infty}^{\infty} f(t-k/2) \operatorname{sinc}(t-k/2)$$

20 Continuous Wavelet Transform

20.1. The squared norm of $\psi_{a,b}$ is given as

$$\|\psi_{a,b}\|^2 = \langle\, \psi_{a,b} \mid \psi_{a,b} \,\rangle = \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right) \right|^2 dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \left| \psi\left(\frac{t-b}{a}\right) \right|^2 dt =$$

$$= \Big/ \text{Variable transformation: } \tau = \frac{t-b}{a} \Big/ = \int_{-\infty}^{\infty} |\psi(\tau)|^2\, dt = \|\psi\|^2$$

20.2. We have

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|a|^2}\, W_f(a,b)\, \psi_{a,b}(t)\, da\, db = /\text{insert (1)}/ = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|a|^2} \left[\int_{-\infty}^{\infty} f(y)\, \overline{\psi_{a,b}}(y)\, dy\right] \psi_{a,b}(t)\, da\, db = /\text{Change order of integration}/ = \\ &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|a|^2}\, \overline{\psi}_{a,b}(y)\, \psi(t)\, da\, db\, dy = /\text{insert (2)}/ = \\ &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|a|^3}\, \overline{\psi}\left(\frac{y-b}{a}\right)\, \psi\left(\frac{t-b}{a}\right)\, da\, db\, dy \end{aligned}$$

20.3. We have

$$\begin{aligned} I(t,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|a|^3}\, \overline{\psi}\left(\frac{y-b}{a}\right)\, \psi\left(\frac{t-b}{a}\right)\, da\, db = /\text{Change order of integration}/ = \\ &= \int_{-\infty}^{\infty} \frac{1}{|a|^3} \int_{-\infty}^{\infty} \overline{\psi}\left(\frac{y-b}{a}\right)\, \psi\left(\frac{t-b}{a}\right)\, db\, da = /\text{Insert } p \text{ and } q \text{ from (6)}/ = \\ &= \int_{-\infty}^{\infty} \frac{1}{|a|^3} \int_{-\infty}^{\infty} p(b)\, \overline{q}(b)\, db\, da \end{aligned}$$

20.4. We have

$$J = \int_{-\infty}^{\infty} p(b)\, \overline{q}(b)\, db = /\text{Parseval}/ = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(u)\, \overline{Q}(u)\, du$$

where P and Q are the Fourier transforms of p and q, respectively. Both p and q are seen as functions of a single variable, here denoted "b". Since both p and q are simple variable transformations of ψ, we get

$$P(u) = |a|e^{iut}\Psi(-au) \quad Q(u) = |a|e^{iuy}\Psi(-au).$$

Inserted into J this leads to

$$J = \frac{|a|^2}{2\pi} \int_{-\infty}^{\infty} e^{iu(t-y)} |\Psi(-au)|^2\, du.$$

20.5. Inserting J back into $I(t,y)$ gives

$$\begin{aligned} I(t,y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|a|} \int_{-\infty}^{\infty} e^{iu(t-y)}\, |\Psi(-au)|^2\, du\, da = /\text{Change order of integration}/ = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(t-y)} \int_{-\infty}^{\infty} \frac{1}{|a|}\, |\Psi(-au)|^2\, da\, du \end{aligned}$$

20.6. We have

$$K = \int_{-\infty}^{\infty} \frac{1}{|a|} \, |\Psi(-au)|^2 \, da = \text{/Variable transformation: } v = -au/ = \int_{-\infty}^{\infty} \frac{|\Psi(v)|^2}{|v|} \, dv$$

Notice that here we replace $-au$ with v, and $|a|$ with $|u|/|v|$, but also da with $dv/|u|$. This last replacement works if we also keep the integration limits, from $-\infty$ to ∞. In all these expressions is u considered as a constant parameters.

20.7. Putting the constant K back into the expression for $I(t, y)$ gives

$$I(t, y) = \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{iu(t-y)} \, du$$

This integral (together with $\frac{1}{2\pi}$) can be seen as the inverse Fourier transform of the frequency function "1", which is the delta function δ, evaluated in the time domain at $t - y$:

$$I(t, y) = K \, \delta(t - y)$$

20.8. We can now express g as

$$g(t) = \int_{-\infty}^{\infty} f(y) \, K \, \delta(t - y) \, dy = K \, f(t)$$

20.9. From (8), which should be valid for general functions f, follows that the inverse wavelet transform exists only if $K \neq 0$ and $K \neq \infty$. By definition $K \geq 0$, so the inverse transform exists only if $0 < K < \infty$.

20.10. XXX

20.11. (a) $-\frac{b}{|a|^{3/2}} \, e^{-\frac{1}{2} \left(\frac{b}{a}\right)^2}$

(b) $e^{-\frac{1}{2} \left(\frac{b}{a}\right)^2} / \sqrt{|a|}$

(c) $\sqrt{|a|} \left[e^{\frac{(1-2b)^2}{8a^2}} - e^{\frac{(1+2b)^2}{8a^2}} \right]$

(d) $-\sqrt{2\pi} \, |a|^{3/2} e^{-\frac{a^2}{2}} \sin b$

20.12. XXX

20.13. XXX

21 Filterbanks

21.1. The effect of down-sampling followed by up-sampling is to set every second sample to zero:

$$x'[k] = x[k] \cdot \frac{1 + (-1)^k}{2} \qquad \text{Here all } \textit{odd} \text{ samples are set to zero}$$

Multiplication of functions in the signal domain corresponds to 2π-circular convolution in the frequency domain

$$\begin{aligned} X'(u) &= \frac{1}{2\pi} X(u) * \left(\frac{\pi(2\,\delta(u) + \delta(u+\pi) + \delta(u-\pi))}{2} \right) = \\ &= \frac{X(u) + \frac{1}{2}X(u+\pi) + \frac{1}{2}X(u-\pi)}{2} = /X \text{ is } 2\pi\text{-periodic}/ = \frac{X(u) + X(u+\pi)}{2} \end{aligned}$$

21.2. In the frequency domain, we get

$$U_0(u) = S(u) \cdot H_0(u) \quad V_0(u) = S(u) \cdot G_0(u)$$

where S is the Fourier transform of the input signal s. Furthermore

$$\begin{aligned} U_1(u) &= \frac{1}{2}\left(U_0(u) + U_0(u+\pi)\right) = \frac{1}{2}\left(S(u)\,H_0(u) + S(u+\pi)\,H_0(u+\pi)\right) \\ V_1(u) &= \frac{1}{2}\left(V_0(u) + V_0(u+\pi)\right) = \frac{1}{2}\left(S(u)\,G_0(u) + S(u+\pi)\,G_0(u+\pi)\right) \end{aligned}$$

and

$$\begin{aligned} S'(u) &= U_1(u)H_1(u) + V_1(u)G_1(u) = \\ &= \frac{1}{2}\left(S(u)\,H_0(u) + S(u+\pi)\,H_0(u+\pi)\right) H_1(u) + \frac{1}{2}\left(S(u)\,G_0(u) + S(u+\pi)\,G_0(u+\pi)\right) G_1(u) = \\ &= \frac{1}{2}\underbrace{\left(H_0(u)H_1(u) + G_0(u)G_1(u)\right)}_{=2} S(u) + \frac{1}{2}\underbrace{\left(H_0(u)H_1(u+\pi) + G_0(u)G_1(u+\pi)\right)}_{=0} S(u+\pi) \end{aligned}$$

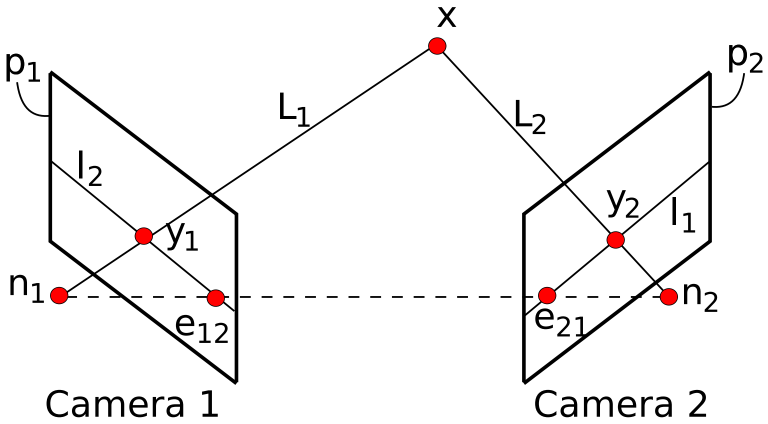
We require that $S' = S$ for all choices of S, leading to the two constraints

$$\begin{aligned} H_0(u)H_1(u) + G_0(u)G_1(u) &= 2 \\ H_0(u)H_1(u+\pi) + G_0(u)G_1(u+\pi) &= 0 \end{aligned}$$

23 Stereo Geometry

23.1 The set of all 3D points that project onto the image point $\mathbf{x}$ form a 3D line (or ray) that goes through the camera center $\mathbf{n}$, and the 3D-point $\tilde{\mathbf{X}} = \mathbf{C}^{+}\mathbf{x}$. The projection of a 3D line in an image is in general a 2D line.

23.2 See figure below.



23.3 For two image points $\mathbf{y}_1$, $\mathbf{y}_2$ in image 1 and 2 respectively, the epipolar constraint reads $\mathbf{y}_1^T\mathbf{F}\mathbf{y}_2 = 0$, if $\mathbf{y}_1$, $\mathbf{y}_2$ are given in homogeneous coordinates.

23.4 All epipolar lines should go through this point, see 1. Denote the point by $\mathbf{e}$. This point should lie on the line $\mathbf{l} = \mathbf{F}\mathbf{x}\ \forall\mathbf{x}$. This means $\mathbf{e}^T\mathbf{l} = 0 = \mathbf{e}^T\mathbf{F}\mathbf{x}\ \forall\mathbf{x}$. For this to be true we must have $\mathbf{e}^T\mathbf{F} = \mathbf{0}$, i.e. $\mathbf{e}$ is a left null vector of $\mathbf{F}$.

23.5 XXX

23.6 XXX

23.7 XXX

23.8 XXX

23.9 XXX

23.10 XXX

23.11 XXX

23.12 XXX

23.13 XXX

24 Triangulation

24.1 XXX

24.2 XXX

24.3 XXX

24.4 XXX

25 Rectification

25.1 XXX

25.2 XXX

25.3 XXX

25.4 XXX

25.5 XXX

25.6 XXX

25.7 XXX