TSBB15
Computer Vision

Lecture 4
Motion estimation and optical flow

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Motion

In many applications it is the case that

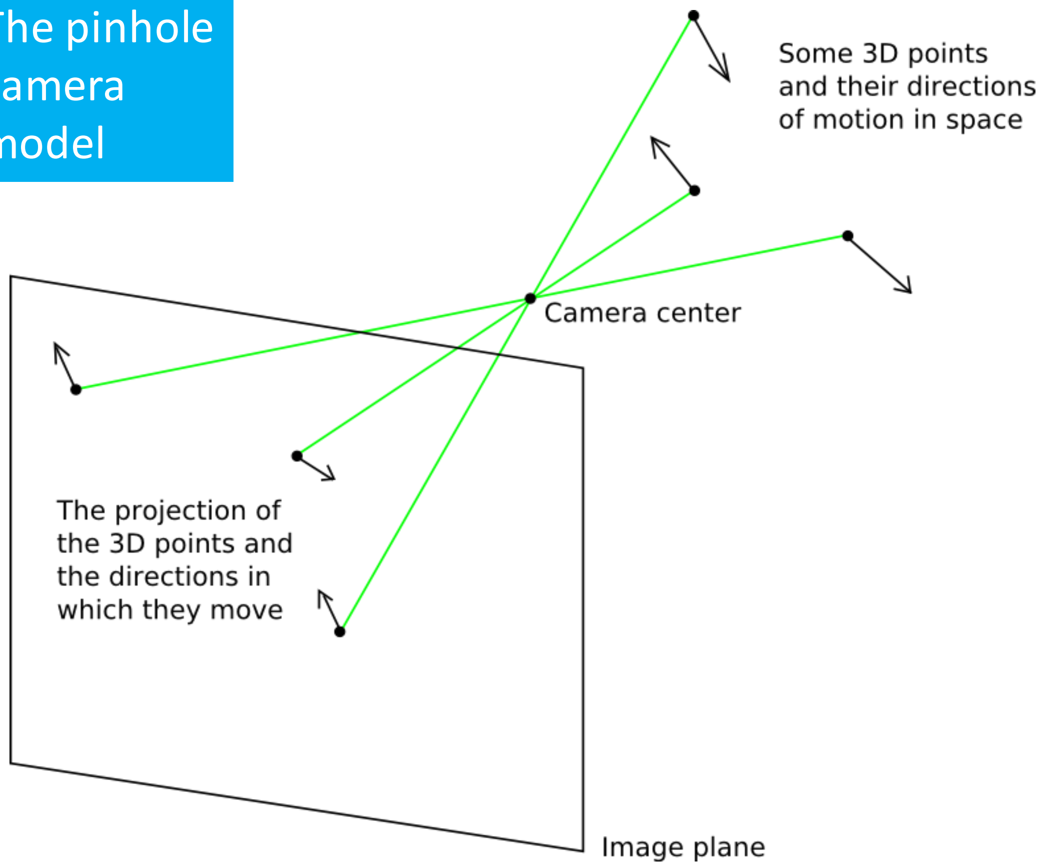
- the scene depicted in the image is dynamic
  - moving objects
  - deformable objects
- or the camera is moving relative to the scene
- in general: both cases

Motion

- From the camera's (viewer's) perspective these two cases are indistinguishable
  - Unless a high-level interpretation of the scene is available
- However, we can describe how points in the scene move relative to some reference frame, e.g., as defined by the camera

The motion field

The pinhole camera model



The *motion field* is the projection of the 3D motion onto the image plane

It can be represented as a vector valued function of the image coordinate

m(**x**)

The motion field

- If we can measure the motion field $\mathbf{m}(\mathbf{x})$ it is possible to infer
  - how points and objects are moving relative the camera, or
  - how the camera is moving relative to the scene (*ego-motion* estimation)

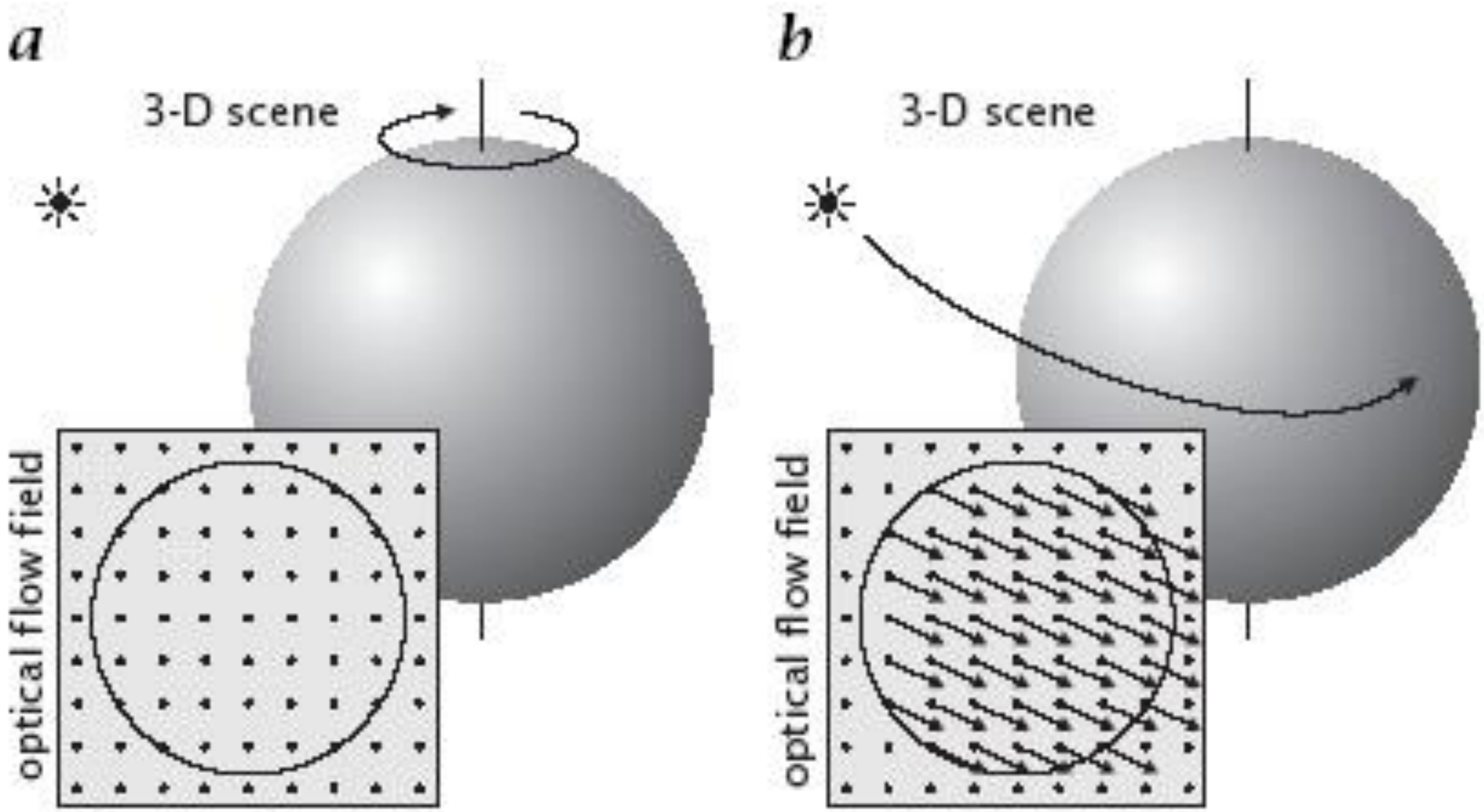
The motion field

- In practice, we cannot measure $\mathbf{m}(\mathbf{x})$ directly
- However, we can measure how the image intensity moves/varies over time
  - Optical flow

Will be formally defined shortly

- But there is no direct relation between the optical flow and the motion field
  - 3D motion may not always generate temporal variations in the image
    - 3D points that move along the projection lines have constant positions in the image
  - Temporal variations in the image may not always correspond to 3D motion

Physical vs visual motion



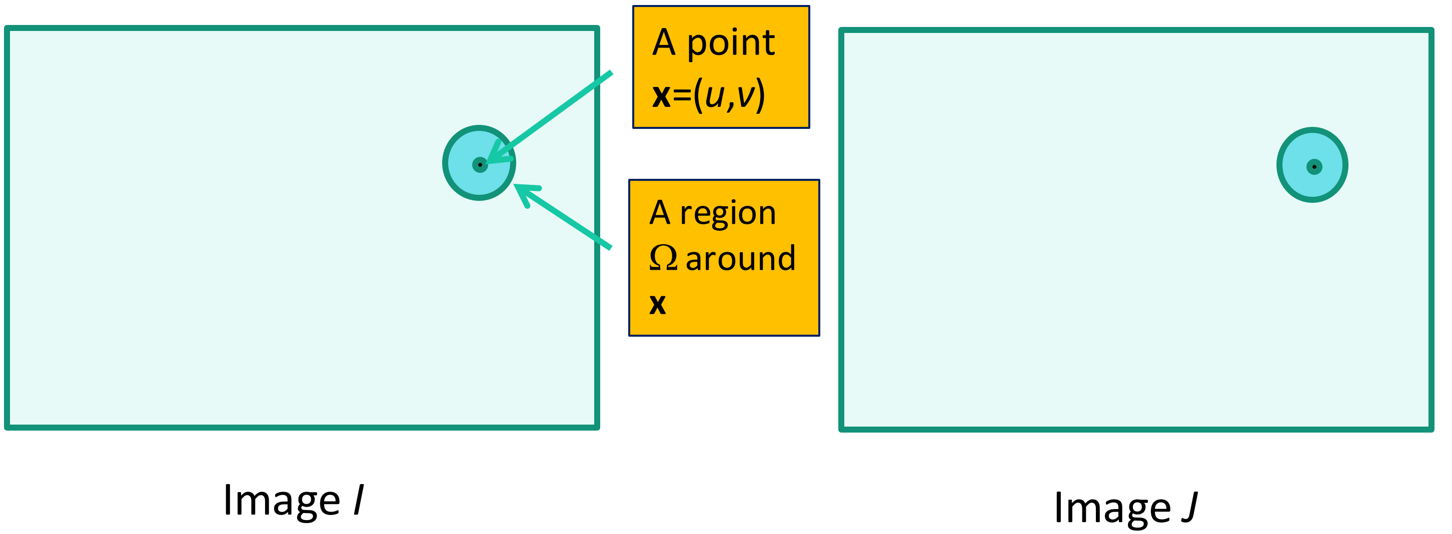
From Jähne & Haussecker

Displacement estimation

- One approach to motion estimation considers **two images** of the same scene, e.g.
  - Taken at two different time points, same camera position
    - Images from a video sequence, e.g., two consecutive images. Displacement is an estimate of the motion field $\mathbf{m}(\mathbf{x})$
  - Taken from two different position, possibly at the same time point
    - Stereo images. Displacement is an estimate of depth in the scene (assuming a stationary scene)

Example (from *Middlebury*)

Mathematical model



- **Assumption**:

  $J(\mathbf{x}) = I(\mathbf{x} + \mathbf{d}) \quad \text{for all } \mathbf{x} \in \Omega$

- Pixel values are constant, but displaced by $\mathbf{d}$

  Depends on position of **x**

- How can we determine $\mathbf{d}$ for each point $\mathbf{x}$?

Estimation of **d**

- **d**, at point **x**, can be estimated by forming a cost function, based on the constancy of the pixel values:

$$\epsilon = \int_{\Omega_0} w(\mathbf{y}) \left(I(\mathbf{x}+\mathbf{y}+\mathbf{d}) - J(\mathbf{x}+\mathbf{y}) \right)^2 \, d\mathbf{y}$$

A region of the origin, same size as Ω

A weighting function, e.g., a Gaussian, of same size as Ω

- The minimizer of ϵ is an estimate of **d** at **x**, which we then use as an estimate of $\mathbf{m}(\mathbf{x})$

Estimation of d

- As an estimate of $\mathbf{m}(\mathbf{x})$, $\mathbf{d}(\mathbf{x})$ is referred to as ***optic flow*** (or optical flow)
- Finding the minimizer of ϵ is a non-linear estimation problem
  – Computationally complex problem
- It can be simplified by a linearization of I

Linearization of *I*

- At each point **x**+**y**, the dependency on **d** in the intensity function *I* can be expressed as a Taylor expansion:

$$\boxed{\nabla I(\mathbf{x}+\mathbf{y}) = \begin{pmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{pmatrix} = \text{Image gradient at } \mathbf{x}+\mathbf{y}}$$

$$I(\mathbf{x}+\mathbf{y}+\mathbf{d}) = I(\mathbf{x}+\mathbf{y}) + \nabla I(\mathbf{x}+\mathbf{y}) \cdot \mathbf{d}$$

- **Assumption**: higher order terms in **d** can be neglected

Linear estimation of **d**

With this linearlization of I at hand:

$$\epsilon = \int_{\Omega_0} w(\mathbf{y}) \left(I(\mathbf{x}+\mathbf{y}) - J(\mathbf{x}+\mathbf{y}) + \underline{\nabla I(\mathbf{x}+\mathbf{y}) \cdot \mathbf{d}}\right)^2 \, d\mathbf{y}$$

Equation (A)

$$\boxed{\frac{\partial I}{\partial u} v_1 + \frac{\partial I}{\partial v} v_2}$$

- We want to find the minimum of ϵ with respect to the elements of $\mathbf{d} = (v_1, v_2)$
- Find **d** where $\begin{pmatrix} \frac{\partial \epsilon}{\partial v_1} \\ \frac{\partial \epsilon}{\partial v_2} \end{pmatrix} = \mathbf{0}$

Determining **d**

$$\begin{pmatrix} \frac{\partial \epsilon}{\partial v_1} \\ \frac{\partial \epsilon}{\partial v_2} \end{pmatrix} = \begin{pmatrix} 2\int_{\Omega_0} w(\mathbf{y})\left(I(\mathbf{x}+\mathbf{y}) - J(\mathbf{x}+\mathbf{y}) + \nabla I(\mathbf{x}+\mathbf{y})\cdot\mathbf{d}\right)\frac{\partial I}{\partial u}\,d\mathbf{y} \\ 2\int_{\Omega_0} w(\mathbf{y})\left(I(\mathbf{x}+\mathbf{y}) - J(\mathbf{x}+\mathbf{y}) + \nabla I(\mathbf{x}+\mathbf{y})\cdot\mathbf{d}\right)\frac{\partial I}{\partial v}\,d\mathbf{y} \end{pmatrix}$$

$$\Downarrow$$

$$\int_{\Omega_0} w(\mathbf{y}) \begin{pmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{pmatrix} \left(I(\mathbf{x}+\mathbf{y}) - J(\mathbf{x}+\mathbf{y}) + \nabla^{\mathrm{T}} I(\mathbf{x}+\mathbf{y})\mathbf{d}\right)\,d\mathbf{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Lucas-Kanade equation

Assumption: **d** is constant within Ω,
i.e., **d** is independent of **y**

$$\Downarrow$$

$$\underbrace{\int_{\Omega_0} w(\mathbf{y})\nabla I(\mathbf{x}+\mathbf{y})\nabla^{\mathrm{T}} I(\mathbf{x}+\mathbf{y})\, d\mathbf{y}\, \mathbf{d}}_{=\mathbf{T}(\mathbf{x})} = \underbrace{\int_{\Omega_0} w(\mathbf{y})\nabla I(\mathbf{x}+\mathbf{y})\left(J(\mathbf{x}+\mathbf{y}) - I(\mathbf{x}+\mathbf{y})\right)\, d\mathbf{y}}_{=\mathbf{s}(\mathbf{x})}$$

The structure tensor

$$\boxed{\mathbf{T}\,\mathbf{d} = \mathbf{s}}$$

This is ***the Lucas-Kanade equation (LK-equation)***.
One equation per pixel in the image (gives one **d** per pixel)

Determining **d**

- In principle, **d** can be determined from the LK-equation as

$$\boxed{\mathbf{d} = \mathbf{T}^{-1}\,\mathbf{s}}$$

- Only works if **T** is not singular, i.e.,
  I in Ω **must not be i1D**
- Lucas & Kanade: *An Iterative Image Registration Technique with an Application to Stereo Vision*, IUW, 1981

Alternative derivation of LK

- The LK-equation derived here is based on finding the local displacement between two images
- An alternative derivation is provided by the brightness constancy principle

Brightness constancy

- Think of the intensity function I as explicitly depending on the 3 variables (u, v, t)
- Basic assumption:
  - If we observe intensity I at (u, v, t), this intensity **remains constant over time**, but it may change position as a function of time
- This is referred to as: ***brightness constancy***

Mathematical formulation

Means: the total derivative of I w.r.t. t is = 0

$$\frac{dI}{dt} = 0$$

Expand in partial derivatives of I:

$$\frac{\partial I}{\partial t}\frac{dt}{dt} + \frac{\partial I}{\partial u}\frac{du}{dt} + \frac{\partial I}{\partial v}\frac{dv}{dt} = 0$$

Mathematical formulation

Cont.

$$\frac{\partial I}{\partial t}\underbrace{\frac{dt}{dt}}_{=1}+\frac{\partial I}{\partial u}\underbrace{\frac{du}{dt}}_{=v_1}+\frac{\partial I}{\partial v}\underbrace{\frac{dv}{dt}}_{=v_2}=0$$

- $\mathbf{v} = (v_1, v_2)$ is the velocity vector of the intensity I at (u, v, t)
- $\mathbf{v}$ is a function of (u, v, t), $\mathbf{v} = \mathbf{v}(\mathbf{x})$
- Local estimate of the motion field $\mathbf{m}(\mathbf{x})$

BCCE / Optic flow equation

Cont. $$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial u} v_1 + \frac{\partial I}{\partial v} v_2 = 0$$

Alternative formulation: $$\frac{\partial I}{\partial t} + \nabla I \cdot \mathbf{v} = 0$$

- This is the ***Brightness Constancy Constraint Equation (BCCE)***
- A.k.a. the optic (optical) flow equation

BCCE

- Is a differential equation
- It assumes that we can determine/estimate the temporal derivative of I at (u, v, t)
  - In practice, it must be estimated in terms of finite differences
  - Compare to the two-image derivation of the LK-eq
- BCCE is one equation per pixel (and time)
  - But it has 2 unknowns: (v_1, v_2)
  - Cannot be solved at the pixel level

Determining **v**

- At a pixel $\mathbf{x} = (u, v)$, at time t, we can formulate a cost function

$$\epsilon = \int_{\Omega_0} w(\mathbf{y}) \left(\frac{\partial I}{\partial t} + \nabla I(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} \right)^2 d\mathbf{y}$$

- Assumes that **v** is constant within Ω
- This cost function is very similar to the one used for the 2-image case, Equation (A), slide 14

LK-equation, again...

- Minimizing ϵ, therefore, implies finding $\mathbf{v}$ such that

$$\boxed{\mathbf{T}\,\mathbf{v} = \mathbf{s}}$$

Continuous time LK-eq

- Where

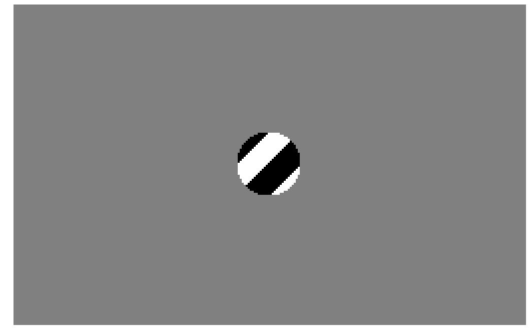
$$\mathbf{T}(\mathbf{x}) = \int_{\Omega_0} w(\mathbf{y}) \nabla I(\mathbf{x}+\mathbf{y}) \nabla^{\mathrm{T}} I(\mathbf{x}+\mathbf{y})\, d\mathbf{y}$$

$$\mathbf{s}(\mathbf{x}) = -\int_{\Omega_0} w(\mathbf{y}) \frac{\partial I}{\partial t} \nabla I(\mathbf{x}+\mathbf{y})\, d\mathbf{y}$$

The aperture problem

- Regardless of how the LK-eq has been derived, it cannot be solved robustly for pixels where I in Ω is i1D
- Even approximately i1D may cause problems
- This is related to the so-called aperture problem:
  - ***In a i1D region we cannot determine the local displacement/velocity along a line/edge***

The aperture problem



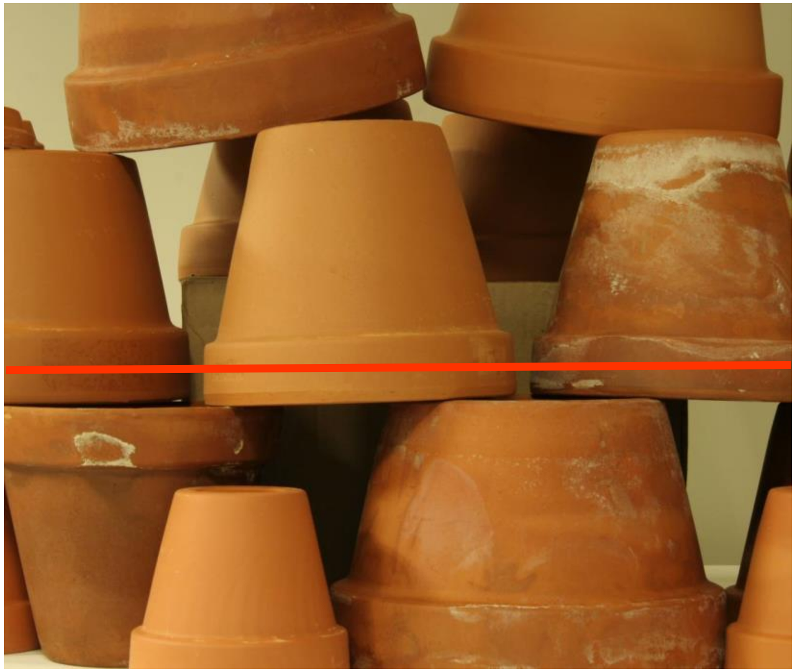
- Is the pattern in the circle moving down, right, or right-down?
- Since the pattern is i1D, its velocity cannot be completely determined
- We can, however, determine a unique *normal velocity*
  - *How?*

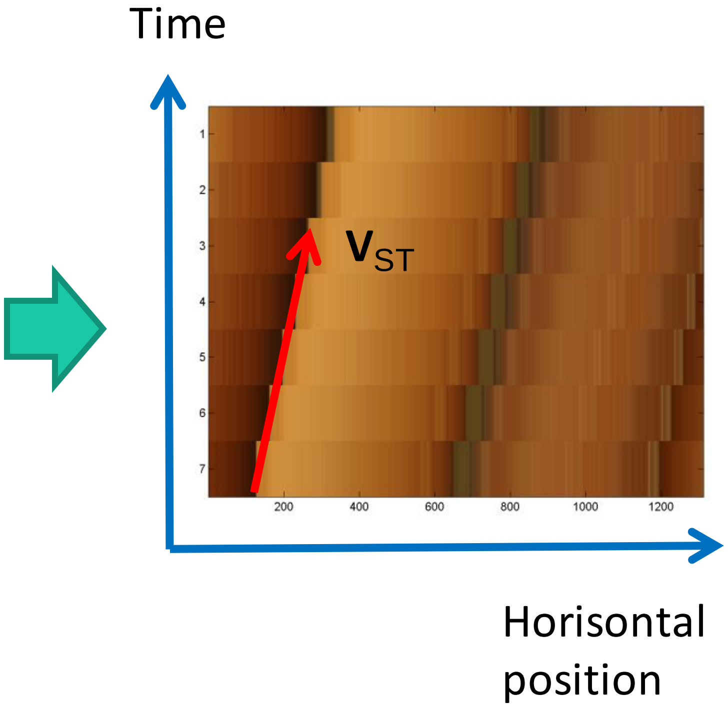
BCCE revisited

- **A consequence of BCCE:**
  In the 3D spatio-temporal volume,
  I must be constant in a direction given by
  $\mathbf{v}_{\mathrm{ST}} = (v_1, v_2, 1)$

- This implies that $\nabla_{\mathrm{ST}} I$, the 3D spatio-temporal gradient of I, is orthogonal to $\mathbf{v}_{\mathrm{ST}}$

Example




Time
1
2
3
4
5
6
7
V_{ST}
200
400
600
800
1000
1200
Horisontal position

A new cost function

- We define a new cost function $\varepsilon_{\mathrm{ST}}$ as

$$\epsilon_{\mathrm{ST}} = \int_{\Omega_0} w(\mathbf{y}) \left(\hat{\mathbf{v}}_{\mathrm{ST}}^{\mathrm{T}} \nabla_{\mathrm{ST}} I \right)^2 \, d\mathbf{y}$$

  where

$$\hat{\mathbf{v}}_{\mathrm{ST}} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \quad \|\hat{\mathbf{v}}_{\mathrm{ST}}\| = 1\,, \quad \nabla_{\mathrm{ST}} I = \begin{pmatrix} \frac{\partial I}{\partial x_1} \\ \frac{\partial I}{\partial x_2} \\ \frac{\partial I}{\partial x_3} \end{pmatrix}$$

Spatio-temporal motion vector

- $\hat{\mathbf{v}}_{\mathsf{ST}}$ (and $\mathbf{v}_{\mathrm{ST}}$) is called the *spatio-temporal motion vector* (it is 3-dimensional)
- $\nabla_{\mathrm{ST}} I$ is the spatio-temporal gradient of I (also 3-dimensional)
- We will minimize $\varepsilon_{\mathrm{ST}}$ over $\hat{\mathbf{v}}_{\mathsf{ST}}$, with the additional constraint

$$\|\hat{\mathbf{v}}_{\mathsf{ST}}\| = 1$$

- This is a *total least squares* formulation of how to determine $\mathbf{v}(\mathbf{x})$

Finding the minimum of $\varepsilon_{\mathrm{ST}}$

- The constraint can be expressed as

$$c = \|\hat{\mathbf{v}}_{\mathrm{ST}}\|^2 = r_1^2 + r_2^2 + r_3^2 = 1$$

- The solution is given by $\hat{\mathbf{v}}_{\mathrm{ST}} = (r_1, r_2, r_3)$ that satisfies

$$\frac{\partial}{\partial r_k}\,\varepsilon = \lambda\,\frac{\partial}{\partial r_k}\,c$$

for $k = 1, 2, 3$ (why?)

Lagrange's method for minimisation with constraints

The 3D structure tensor revisited

- These 3 equations can be rewritten as

$$\left[\int_{\Omega} w(\mathbf{x})\, \nabla_{ST} I \, \nabla_{ST}^{T} I \, d\mathbf{x}\right] \hat{\mathbf{v}}_{ST} = \lambda\, \hat{\mathbf{v}}_{ST}$$

(why?)

- Note that the expression inside the bracket is a 3D structure tensor!

The 3D structure tensor revisited

- We rewrite this as

$$\mathbf{T}_{\mathsf{3D}}\,\hat{\mathbf{v}}_{ST} = \lambda\,\hat{\mathbf{v}}_{ST}$$

- This means that the $\hat{\mathrm{v}}_{\mathsf{ST}}$ which minimizes ε must be an eigenvector of $\mathbf{T}_{\mathrm{3D}}$
- It should also be normalized: $\|\hat{\mathrm{v}}_{\mathsf{ST}}\| = 1$
- The eigenvector that minimizes ε is the one of smallest eigenvalue (why?)

The 3D structure tensor revisited

- Once $\hat{\mathbf{v}}_{\mathrm{ST}} = (r_1, r_2, r_3)$ has been determined we can find $\mathbf{v}_{\mathrm{ST}}$ that is
  - Parallel to $\hat{\mathbf{v}}_{\mathrm{ST}}$
  - Has its last component = 1
- The first two components of $\mathbf{v}_{\mathrm{ST}}$ are the motion vector $\mathbf{v} = (v_1, v_2)$

$$v_1 = \frac{r_1}{r_3} \qquad v_2 = \frac{r_2}{r_3}$$

Summary

- We now have 2 alternatives to local motion estimation based on BCCE:

1. least squares minimization (based on $\mathbf{T}_{2D}$ and $\mathbf{s}$)
2. total least squares minimization (based on $\mathbf{T}_{3D}$)

Summary: Least squares minimization

- Minimize

$$\varepsilon_{ST} = \int_{\Omega} w(\mathbf{x})\,[\mathbf{v}_{ST} \cdot \nabla_3 I]^2 \; d\mathbf{x}$$

  where $\mathbf{v}_{\mathrm{ST}} = (v_1, v_2, 1)$ over the motion components $\mathbf{v} = (v_1, v_2)$
- Find $\mathbf{v}$ by solving $\mathbf{T}_{2\mathrm{D}}\,\mathbf{v} = \mathbf{s}$
- We can see $\mathbf{v}_{\mathrm{ST}}$ as a homogeneous representation of $\mathbf{v}$

Summary: Total least squares minimization

- Minimize
$$\varepsilon_{ST} = \int_{\Omega} w(\mathbf{x}) \left[\hat{\mathbf{v}}_{ST} \cdot \nabla_3 I\right]^2 \, d\mathbf{x}$$
over all components of $\hat{\mathbf{v}}_{\mathsf{ST}} = (r_1, r_2, r_3)$ and with the constraint $\|\hat{\mathbf{v}}_{\mathsf{ST}}\| = 1$
- Find $\hat{\mathbf{v}}_{\mathsf{ST}}$ as the eigenvector of smallest eigenvalue with respect to $\mathbf{T}_{3\mathrm{D}}$
- Find $\mathbf{v}$ from $\hat{\mathbf{v}}_{\mathsf{ST}}$ as $v_1 = \frac{r_1}{r_3} \quad v_2 = \frac{r_2}{r_3}$

The 3D tensor

- In the 3D case, we compute a structure tensor $\mathbf{T}_{3D}$, a symmetric 3×3 matrix, that can be decomposed as (the spectral theorem)

$$\mathrm{T}_{3\mathrm{D}} = \lambda_1 \, \hat{\mathrm{e}}_1 \, \hat{\mathrm{e}}_1^T + \lambda_2 \, \hat{\mathrm{e}}_2 \, \hat{\mathrm{e}}_2^T + \lambda_3 \, \hat{\mathrm{e}}_3 \, \hat{\mathrm{e}}_3^T$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ are the eigenvalues of $\mathbf{T}_{3D}$ and $\hat{\mathbf{e}}_k$ are the corresponding eigenvectors (an orthonormal set)

The 3D structure tensor

- In general (*not only in the case of motion*) we can distinguish between three cases of the local 3D signal
  - The signal is constant on parallel planes (i1D)
  - The signal is constant on parallel lines (i2D)
  - The signal is isotropic
- Remember that **T** is formed as

$$\mathbf{T}(\mathbf{x}) = \int_{\Omega_0} w(\mathbf{y}) \nabla I(\mathbf{x}+\mathbf{y}) \nabla^{\mathrm{T}} I(\mathbf{x}+\mathbf{y})\, d\mathbf{y}$$

The signal is constant on parallel planes (Lasagna)

- (Case 1) The 3D signal is i1D
  - The gradient $\nabla_3 I$ is always parallel to the normal vector of the planes

$$\mathrm{T} = \lambda_1 \, \hat{\mathrm{e}}_1 \, \hat{\mathrm{e}}_1^T$$

  - **T** has rank 1
  - $\hat{\mathbf{e}}_1$ is a normal vector to the planes
  - A moving 2D line generates a 3D signal that is i1D $\Rightarrow$ **T** has rank 1

The signal is constant on parallel planes

- In this case, the Fourier transform of I is concentrated along a line through the origin, in the direction of $\hat{\mathbf{e}}_1$

The signal is constant on parallel lines (Spaghetti)

- (Case 2) The 3D signal is intrinsic 2D (i2D)
  - The gradient $\nabla_3 I$ is always perpendicular to the direction $\hat{\mathbf{e}}_3$ of the lines

$$\mathbf{T} = \lambda_1 \, \hat{\mathrm{e}}_1 \, \hat{\mathrm{e}}_1^T + \lambda_2 \, \hat{\mathrm{e}}_2 \, \hat{\mathrm{e}}_2^T$$

  - $\hat{\mathbf{e}}_3$ is an eigenvector of eigenvalue 0 relative to $\mathbf{T}$
  - $\mathbf{T}$ has rank 2
  - A moving point generates a 3D signal that is i2D $\Rightarrow$ $\mathbf{T}$ has rank 2

The signal is constant on parallel lines

- In this case, the Fourier transform of I is concentrated to a plane through the origin, that has $\hat{\mathbf{e}}_3$ as its normal vector
- In other words, the plane is spanned by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$

The signal is isotropic (Dumpling)

- (Case 3) The signal varies uniformly in all directions
  - The gradient $\nabla_3 I$ is not restricted to some subspace

$$\mathbf{T} = \lambda_1\, \hat{\mathrm{e}}_1\, \hat{\mathrm{e}}_1^T + \lambda_2\, \hat{\mathrm{e}}_2\, \hat{\mathrm{e}}_2^T + \lambda_3\, \hat{\mathrm{e}}_3\, \hat{\mathrm{e}}_3^T$$

where λ_1, λ_2 and λ_3 all are $\neq 0$.

  - **T** has rank 3
  - Not consistent with the BCCE

The signal is isotropic

- In the isotropic case, variations in all directions are uniformly distributed
- Implies that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$
- We can write $\mathbf{T} = \lambda\ \mathbf{I}$ ($\mathbf{I}$ is the identity tensor)
- The Fourier transform of the signal extends into all 3 dimensions

Confidence measures

- As confidence measures for the three cases we can use, *for example*:

$$c_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1}$$ Case 1

$$c_2 = \frac{\lambda_2 - \lambda_3}{\lambda_1}$$ Case 2

$$c_3 = \frac{\lambda_3}{\lambda_1}$$ Case 3

Confidence measures

- They satisfy $c_1 + c_2 + c_3 = 1$.
- Furthermore
  - i1D-signal $\Rightarrow$ $\mathbf{T}$ has rank 1 $\Rightarrow$ $\lambda_1 > 0, \lambda_2 = \lambda_3 = 0 \Rightarrow c_1 = 1, c_2 = c_3 = 0$.
  - i2D-signal $\Rightarrow$ $\mathbf{T}$ has rank 2 $\Rightarrow$ $\lambda_1 \geq \lambda_2 > 0, \lambda_3 = 0 \Rightarrow c_2 \neq 0, c_3 = 0$.
  - Isotropic signal $\Rightarrow$ $\mathbf{T}$ has rank 3 $\Rightarrow c_3 \neq 0$.

Decomposing **T**

- Based on these confidence measures, **T** can be decomposed as

$$\begin{aligned}\mathbf{T} &= \lambda_1\,\hat{\mathbf{e}}_1\,\hat{\mathbf{e}}_1^T + \lambda_2\,\hat{\mathbf{e}}_2\,\hat{\mathbf{e}}_2^T + \lambda_3\,\hat{\mathbf{e}}_3\,\hat{\mathbf{e}}_3^T \\ &= (\lambda_1 - \lambda_2)\,\hat{\mathbf{e}}_1\,\hat{\mathbf{e}}_1^T + \\ &\quad +(\lambda_2 - \lambda_3)\,(\hat{\mathbf{e}}_1\,\hat{\mathbf{e}}_1^T + \hat{\mathbf{e}}_2\,\hat{\mathbf{e}}_2^T) + \\ &\quad +\lambda_3\,(\hat{\mathbf{e}}_1\,\hat{\mathbf{e}}_1^T + \hat{\mathbf{e}}_2\,\hat{\mathbf{e}}_2^T + \hat{\mathbf{e}}_3\,\hat{\mathbf{e}}_3^T) \\ &= \lambda_1\,[c_1\,\mathbf{T}_{\mathsf{rang1}} + c_2\,\mathbf{T}_{\mathsf{rang2}} + c_3\,\mathbf{I}]\end{aligned}$$

Summary

- Given a local picture of the signal:
  - The directions along which the signal is constant correspond to the null space of $\mathbf{T}$
  - $\mathbf{T}$ has a range that is orthogonal to this null space
  - In the Fourier domain: the energy is concentrated to the range of $\mathbf{T}$

Summary

- The rank of $\mathbf{T}$ equals the dimension of its range
- The range represent the dimensions in the Fourier domain where there is energy
- We can define confidence measures (in various ways) that indicate which rank or case that $\mathbf{T}$ represents
- In general, $\mathbf{T}$ can be a combination of the different cases

Computation of the motion vector (rank 2)

- At each point (x_1, x_2, t) we can estimate the local 3D structure tensor $\mathbf{T}$
- If $\mathbf{T}$ has rank 2 it corresponds to a non-i1D signal in the 2D image
- Since $\mathbf{T}$ has rank 2 we can ”uniquely” determine an eigenvector of smallest eigenvalue:

$$\hat{\mathbf{v}}_{\mathrm{ST}} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}$$

Computation of the motion vector (rank 2)

- From the previous derivations we know that

$$\hat{\mathbf{v}}_{\mathsf{ST}} \sim \mathbf{v}_{\mathsf{ST}} = (v_1 \; v_2 \; 1)$$

- Consequently, we can compute the motion components as

$$v_1 = \frac{r_1}{r_3} \qquad v_2 = \frac{r_2}{r_3}$$

Computation of the motion vector (rank 1)

- If $\mathbf{T}$ has rank 1 it means that the corresponding 2D-signal is i1D
  - A moving line or edge
- The null space of $\mathbf{T}$ is 2-dimensional
- We cannot uniquely determine $\mathbf{v}_{\mathrm{ST}}$, and therefore $\mathbf{v}$ cannot be uniquely determined
- Related to the aperture problem

Computation of the motion vector (rank 1)

- However, in this case we can determine the *normal motion* of the 2D-signal

- Let $\mathbf{p}=(p_1, p_2, p_3)$ be an eigenvector of largest eigenvalue relative to $\mathbf{T}$

Computation of the motion vector (rank 1)

- The spatio-temporal normal motion vector $\mathbf{v}_{\text{ST}}$ must satisfy

$$\mathbf{p}^T \mathbf{v}_{ST} = 0$$

1

$$p_1 \, v_1 + p_2 \, v_2 + p_3 = 0$$

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \kappa \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

2

- (why?)

Computation of the motion vector (rank 1)

- From these two relations, the normal motion is given as

$$\mathbf{v}_{\mathrm{norm}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{p_3}{p_1^2+p_2^2} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Computation of the motion vector (rank 3)

- Finally, if **T** has rank 3 this implies that the local signal does not satisfy the conditions expressed in BCCE. (why?)

A strategy for motion estimation

- Compute the 3D tensor $\mathbf{T}_3$
- Determine its eigenvalues
- Classify the tensor into each of the three cases, based on some confidence measures (how?)
- If rank 1: compute the normal motion
- If rank 2: compute the "true" motion
- If rank 3: no motion can be determined